Taking Place Value Seriously
Exercises

The following exercises are intended as a companion to the essay, "Taking Place Value Seriously". The numbering refers to the sections of the essay. Thus, problem 2.4 relates to §2 of the essay, and problem 4.2 relates to §4.

The problems vary in difficulty and purpose. Many are routine exercises which should be easily doable by a reader who understands the ideas discussed in the essay. Others are considerably more challenging, and call on skills unrelated to the essay. Still, these harder problems have a core which connects directly to the themes of the essay. Some problems would be suitable for tests examining a basic grasp of the concepts in the essay, others would be suitable for class discussion, others for projects, and a few for challenges to the strong students. There has been no attempt to rate the exercises for difficulty, or to comment on individual problems.

2.1: Add mentally:

a) $3 + 5 + 7$ b) $13 + 25 + 37$ c) $4 + 27 + 36$ d) $24 + 13 + 33$
e) $98 + 47$ f) $997 + 154$ g) $18 + 27 + 13 + 42$

2.2: Find the missing number.

a) $4 + 8 = ? + 7$
b) $4 + 8 = ? + 7$
c) $4 + 8 = 7 + ?$

2.3: Find the missing number.

a) $54 + 48 = ? + 47$
b) $54 + 48 = 55 + ?$
c) $54 + 48 = 47 + ?$

2.4: Find the missing number.

a) $254 + 648 = ? + 650$
b) $254 + 648 = 256 + ?$
c) $254 + 648 = 650 + ?$

2.5) a) Given that $817 + 444 = 1261$, find $816 + 445$.

b) Given that $817 + 444 = 1261$, find $818 + 442$.

c) Given that $817 + 444 = 1261$, find $818 + 444$.

d) Given that $817 + 444 = 1261$, find $816 + 443$.

e) Given that $817 + 444 = 1261$, find $844 + 417$.

f) Given that $817 + 444 = 1261$, find $845 + 416$.

2.6: Add:

a) $374 + 628$ b) $3074 + 6028$

c) 30704+60208 d) $3704 + 6208$

2.7: Add:

a) 3,721 + 654 + 86

b) 3,651 + 724 + 86

c) 3,684 + 726 + 51

d) 3,786 + 651 + 24

Explain.

2.8: Add:

a) $45 + 21$ b) $45 + 23$

c) $45 + 25$ d) $45 + 27$.

2.9: Subtract:

a) $66 - 45$ b) $68 - 45$

c) $70 - 45$ d) $92 - 45$

2.10: Add:

a) $45 + 21$ b) $43 + 23$

c) $41 + 25$ d) $39 + 27$

e) $37 + 29$

2.11: Subtract:

a) $66 - 21$ b) $66 - 23$

c) $66 - 25$ d) $66 - 27$

e) $66 - 29$

2.12: Add:

a) $524 + 473$ b) $524 + 475$

c) $524 + 477$ d) $524 + 479$

2.13: Subtract:

a) 997 - 473 b) 999 - 475

c) 1001 - 477 d) 1003 - 479

2.14: Find the missing number.

a) $254 + 648 = ? + 248$

b) $254 + 648 = 658 + ?$

c) $254 + 648 = ? + 258$

2.15: Perform the following multiplications mentally:

a) $25 \times 13 \times 4$ b) 5×38

c) 55×40 d) $2 \times 36 \times 125$

2.16: Compute the following expressions:

a) $13 \times 8 + 13 \times 2 + 6 \times 27 + 6 \times 23$

b) $21 \times 17 + 21 \times 13 - 6 \times 24 - 6 \times 6$

3.1: For the following pairs of numbers V and v, find the smallest fraction of the form $\frac{1}{n}$ (that is, $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, etc.) which is larger than the relative error with which v approximates V. That is, find the largest integer n such that $\left| \frac{V-v}{V} \right| \leq \frac{1}{n}$.

a) $V = 20$, $v = 29$ b) $V = 40$, $v = 49$

c) $V = 800$, $v = 899$ d) $V = 6000$, $v = 6999$

e) $V = 1100$, $v = 1199$ f) $V = 24000$, $v = 24999$.

3.2. On the basis of the examples in exercise 3.1, do you have any predictions about relative error?

3.3. The whole number v approximates the whole number V with a relative error of less than $\frac{1}{1000}$. Also, the order of magnitude of v is larger than the order of magnitude of V. What are the smallest possible values for v and V?

3.4: Order the following numbers expressed in scientific notation.

a) 3.1416×10^0 b) 1.414×10^2

c) 1.5×10^2 d) 9.8704×10^1 e) 1.4137×10^2

3.5: What is the percentage error in approximating 1 mile/hour by $\frac{3}{2}$ foot/sec?

3.6. a) An inch equals 2.54 centimeters. What is the percentage error in replacing 2.54 by 2.5?

b) What is the percentage error in approximating 1 foot by 30 cm?

c) Approximately what is your height in centimeters?

d) What is the percentage error in approximating 1 meter by 1.1 yards?

e) What is the percentage error in approximating 1 yard by .9 meters?

f) What is the percentage error in approximating 1 km by $\frac{5}{8}$ miles?

3.7: A cube with double the volume of a cube of sidelength s has sidelength $2^{\frac{1}{3}}s$. What is the percentage error in approximating $2^{\frac{1}{3}}$ by $\frac{5}{4}$? What is the percentage error in the volume?

3.8: A cereal bowl has the shape of a truncated circular cone, with top radius equal to twice the bottom radius. (In fact, this is a fairly good approximation to the shape of many cereal bowls.) Suppose you want to have 1% milk on your cereal, but you only have 2% milk and skim milk. You want to make a 1% mixture by filling the bowl half full of 2% milk, and then filling it with skim milk.

a) How far up the bowl should you fill with 2% milk? (That is, what fraction of the depth of the bowl should be the level of the 2% milk?)

b) How badly off would you be if you filled it to 2/3 of its depth? That is, what would be the percentage error made by approximating the correct depth by 2/3? Is 2/3 too high or too low?

c) What about 5/8? $\frac{13}{20}$?

3.9: What is the relative error in approximating π by 3? By $\frac{22}{7}$? By $\frac{355}{113}$?

3.10: What is the relative error in replacing $\frac{a}{b}$ with $\frac{a}{b+1}$?

3.11: **Refined Decimal Estimation Theorem**

a) Argue that: If we have two decimal whole numbers V and v, both of magnitude m, whose decimal components of magnitude ℓ or larger are equal, then we can write $V = c10^{\ell} + d$ and $v = c10^{\ell} + e$, where c has magnitude $m - \ell$ and d and e have magnitude at most $\ell - 1$.

b) Show that the converse is also true. In these terms, the BDET says that

$$|V - v| = |d - e| < 10^{\ell}$$

and

$$\frac{|V - v|}{V} = \frac{|d - e|}{c10^{\ell} + d} \leq \frac{1}{c} \leq 10^{m-\ell}.$$

The final statement is all we can say if all we assume is that c is some number of magnitude $m - \ell$. However, if we assume we know what c is, then we get the **Refined Decimal Estimation Theorem:**

If $V = c10^{\ell} + d$, where c has magnitude $m - \ell$, and d has magnitude less than ℓ, and v has all its decimal components of magnitude ℓ or greater the same as V, then

$$\frac{|V - v|}{V} \leq \frac{1}{c}.$$

4.1: a) Suppose you are given a random number in the 30s and a random number in the 50s. That is, you have numbers of the form $30 + d$ and $50 + d'$, where d and d' are digits, but you have no idea what they are - they could be anything from 0 to 9, all equally likely. What is your best guess for the sum of the two numbers? What is the smallest error you could allow, and have at least a 50-50 chance of having your guess be wrong by this error or less?

b) What if, instead, you had a number in the 80s and a number in the 90s?

c) What if you had three numbers, one in the 30s, one in the 50s, and one in the 80s. What would be your best guess, and what error would you have to allow to expect at least 50% chance of being within that error?

4.2: Find the contribution of the 6 largest products of decimal components to the product $(314, 159)^2$, How many decimal places agree with the full product? How many decimal places of agreement do you get if you if you take the sum of the 10 largest products?

4.3: a) Repeat 4.2 with the product 539×1859. What are the relative errors of approximation of the sums to the full product?

b) Repeat with 5329×18769.

4.4: Order the following sums from smallest to largest.

a) $784 + 436$ b) $786 + 435$ c) $736 + 474$ d) $734 + 586$

4.5: a) If v approximates V with relative error less than an amount a, and u approximates U with relative error less than a, and both V and U are positive, then does $u + v$ approximate $V + U$ with relative error less than a?

b) What can you say about the relative error with which $u - v$ approximates $U - V$?

4.6: The Celsius and Fahrenheit temperature scales are related by the equation $F = (\frac{9}{5})C + 32$. Here C is a temperature in degrees Celsius, and F is the corresponding number of degrees in the Fahrenheit scale. Consider the following scheme for approximately converting a Celsius temperature C to a Fahrenheit temperature F':

1) Double C.

2) From the result, subtract its 10s digit.

3) Add 32. The result is F'

a) What is the largest possible value of $F - F'$?

b) What is the smallest possible value of $F - F'$?

c) Graph F and F' as functions of C on the same graph. Is F' a good approximation?

d) What happens if for step 3), we add 30 rather than 32? Which scheme do you favor?

5.1: a) Compute 3004×2011. b) Compute $(3x+4)(2x+11)$
c) What is the relation between these two computations?

5.2: Compute: a) $(x+1)(x^2+1)$ b) 11×101 c) $(x+1)(x^2+1)(x^4+1)$
d) $11 \times 101 \times 10001$

5.3: a) Show that $x^3+1=(x+1)(x^2-x+1)$.
How is this related to the factorization $1001 = 7 \times 11 \times 13$?
b) Show that

$$\begin{aligned} x^5+1 &= (x+1)(x^4-x^3+x^2-x+1) \\ &= (x+1)(x^2+\frac{1+\sqrt{5}}{2}x+1)(x^2+\frac{1-\sqrt{5}}{2}x+1) \end{aligned}$$

Use the first equation to factor 100001.

5.4: Compute: a)11×9 b) 19×21 c) 29×31.
Predict 39×41.

5.5. Compute: a)15^2 b) 25^2 c)35^2 d)45^2 e) Predict 55^2.

5.6: Compute a) $(1\frac{1}{2})^2$ b) $(2\frac{1}{2})^2$ c) $(3\frac{1}{2})^2$ d) $(4\frac{1}{2})^2$. e) Predict $(5\frac{1}{2})^2$

The next two exercises assume some knowledge of Pascal's Triangle.

5.7: Compute a) 11^2 b) 11^3 c) 11^4 d) 11^5.

5.8: Compute a) 101^4 b) 101^5 c) 101^6 d) 101^7 e)101^8 f)101^9.

5.9: A strategy for factoring a largish whole number n is the following:
i) Find the smallest number m such that $m^2 \geq n$.
ii) For $k = 0, 1, 2, 3, \ldots$, compute $(m+k)^2 - n$. If for some k, it is a perfect square, say $(m+k)^2 - n = \ell^2$, then $n = (m+k+\ell)(m+k-\ell)$.
Use this strategy to factor:
a) 2491 b) 10001 c) 6319 d) 3239.