3. Logical Reasoning in Mathematics

Many state standards emphasize the importance of reasoning. We agree—disciplined mathematical reasoning is crucial to understanding and to properly using mathematics. (See the Lead Essay, Principles for School Mathematics 2–4, and the discussion of these principles given there.) Below we shall present problems that require such reasoning.

Before doing so, we offer a few remarks. First, though 'reasoning' is widely mentioned in state standards, it is apparent that this word is used to describe many different things. For example, students who are asked

> *You want to purchase a bookcase to hold 37 books. If each shelf can hold 7 books, how many shelves should the bookcase you purchase have?*

need to compute that 37 divided by 7 is 5 with a remainder of 2, and reason from this that to hold all the books one needs 6 shelves (and not just 5!). In almost all applications of mathematics the formal procedures (in this case division with remainder) must be supplemented by an understanding of the context and by reasoning about this context. This type of contextual or situational reasoning is certainly important, and questions such as the one above have a solid place in the curriculum and on state tests. However, we wish to highlight here that *the ability to use and to understand mathematics requires logical reasoning that goes far beyond this.* By mathematical reasoning or logical reasoning we mean—and we believe that state standards should include in a significant way—precise deductive reasoning.

Deductive reasoning skills are crucial in mathematics (as well as in many other walks of life). However, achieving a high level of student competence in deductive reasoning will require great care on the part of teachers. Indeed, the level of precision necessary to communicate mathematics is quite high, and it is easy for the generalist to overlook this. 'Three times higher than' does not mean the same thing as 'three times as high as' . The phrases 'six divided by three', 'six divides three', and 'six divided into three' all have different mathematical meanings. It is difficult to use deductive reasoning effectively without attention to this precision. All involved should be encouraged to cultivate careful reading, listening, presenting, and writing skills in the domain of mathematics.

The problems below all involve deductive reasoning, but they are of several types. First there are problems that require an understanding of quantifiers ('for some', 'for all'), negation ('not all', 'some are not', etc.), and other mathematical language. Students need to understand this language and to evaluate whether a given statement involving quantifiers is true. They also need to learn that a statement that is only true sometimes is mathematically false (e.g. the statement 'the product of two irrationals is irrational' is false). Second are problems that require understanding of 'if-then' deductive reasoning itself. Such problems come in a wide variety of levels and a wide range of sophistication, including multi-step implications. It is useful for working with 'if-then' statements to know that a statement and its contrapositive are logically equivalent and to be able to formulate the contrapositive of a given statement. However, this is only one, limited, part of working with 'if-then' statements. It is the ability to carry out multi-step deductive reasoning and the ability to detect incorrect implications and explain why they are wrong that are the primary goals here. At the highest level are problems that couple this reasoning with algebra or geometry.

Problems involving quantifiers and other mathematical language

1: Decide which of the following statements are true and which are false. Explain your answer.

(a) Some whole numbers are integers.

(b) Some whole numbers are not integers.

(c) Some integers are whole numbers.

(d) Some integers are not whole numbers.

(e) Not all integers are whole numbers.

(f) All integers are whole numbers.

(g) Different fractions necessarily represent different rational numbers.

(h) Between any two rational numbers there is at least one other rational number.

(i) A system of two simultaneous linear homogeneous equations in two unknowns necessarily has at least one solution.

Discussion: Statements (a)–(f) involve three things: understanding the quantifiers 'some' and 'all', understanding that 'not' is attached to the noun that follows it (as opposed to negating the entire sentence), and understanding the concepts 'integer' and 'whole number'. The fact that 'not' sometimes negates an entire sentence and other times negates only an aspect of the sentence may cause some students difficulty (a difficulty that also occurs in ordinary communication). However, with explanations from the teacher, students can become proficient at analyzing and explaining such statements in the early grades.

Statement (g) requires understanding the phrase 'necessarily represent' and understanding the difference between a statement sometimes being true and always being true. The mathematical fact involved is one that must be mastered in order to work with fractions.

The problem of deciding whether statement (h) is true or false and explaining the reason is a somewhat sophisticated problem about a basic concept. The logic aspect of the problem is centered around the use of the phrase 'at least' (a phrase that is especially difficult for those who are just learning English) and around the requirement that the answer be explained.

The last statement involves standard mathematics terminology, some mathematical knowledge, and a correct understanding of the phrase 'at least'.

2: Decide which of the following statements are true and which are false. Explain your answer.

(a) All equilateral triangles are isosceles.

(b) Some isosceles triangles are right triangles.

(c) Some right triangles are isosceles.

(d) Some equilateral triangles are right triangles.

(e) Some isosceles triangles are right triangles and equilateral.

(f) No isosceles triangle is equilateral.

(g) Not all isosceles triangles are equilateral or right triangles.

Discussion: These are similar to problem 1, but based on geometry, in this case on the definition of various kinds of triangles. For part (g), note that in mathematics 'or' means 'one or the other or both'.

Problems 1 and 2 are meant to be representative; other variations which concern the same aspect of mathematical reasoning in the context of other aspects of math knowledge are readily constructed.

Problems involving 'if-then' statements and deductive reasoning

3: Which of the following statements is logically equivalent to "If it is Saturday, then I am not in school."?

(1) If I am not in school, then it is Saturday.

(2) If it is not Saturday, then I am not in school.

(3) If I am in school, then it is not Saturday.

(4) If it is Saturday, then I am in school.

Discussion: This is Problem 8 on Regents High School Examination Mathematics A of the State of New York for June 17, 2003. It requires a student to know what 'logically equivalent' means (a high school student should know the meaning of this phrase). The student must also be able to work with 'if', 'then', and 'not'. A nice feature of this problem is that it is written in the present tense, thus avoiding any concerns about causality.

One might contrast this with the problem "What is the inverse of the statement: 'If John cuts the grass, then he earns $8.00' ?" This tests only the definition of the 'inverse' of an 'if-then' statement, a definition which is considerably less important than the contrapositive (or the converse). It does not test deductive reasoning at a high level.

4: Which of the following statements has a true converse?:

(a) If two triangles are congruent, then they have the same perimeter.

(b) If two squares are congruent, then they have the same area.

(c) If two rectangles have the same perimeter, then they have the same area.

(d) If the area of a triangle is less than 1, then each of its sides has length less than 1.

Discussion: These problems illustrate that a statement and its converse may both be true, both be false, or one may be true and the other false. It is a common mistake to confuse the truth of a statement with the truth of its converse, both in mathematics and in the larger world. The ability to detect such potentially false reasoning is a basic life skill.

5: Some kings have beards. All kings wear red. All men who wear red are tall. Based solely on this information, which of the following statements are true? (a) Some men with beards are tall. (b) All tall men are kings. (c) Not all small men with beards wearing red are kings.

Discussion: This is a familiar type of problem and we do not list the possibilities exhaustively. Such problems should be mastered.

6: A,B,C,D, and E are friends, no two of whom have the same height. B is taller than C, A is not the tallest, E is taller than B, D is taller than exactly two of the others, and D is taller than B. What is the order of the heights of these friends from tallest to shortest?

Discussion: Once again, this is a familiar type of deductive reasoning problem. The point is that there is one answer and it can be arrived at deductively.

7: Joshua and Joel together have the same number of scarves as Sarah and Samantha together. Also, Joshua has more scarves than Sarah. What can you conclude?

Discussion: This problem is suitable for the elementary level, but it may be more useful for a classroom discussion or homework than for a test. One can also ask the related problem in an algebraic format: If $a + b = c + d$ and $a < c$, what can you conclude? Similarly, some of the algebraic statements in problem 8 below can be made into word problems.

8: Decide which of the following statements about real numbers are true and which are false. If the statements are true, give an argument which shows this. If the statements are false, give a counterexample.

(a) If $a < b$ and $c < d$ then $ac < bd$.

(b) If $a^2 < b^2$ then $a < b$.

(c) If $a^3 < b^3$ then $a < b$.

(d) If $(x + y)^2 = x^2 + y^2$ then $x = 0$ or $y = 0$.

(e) If $x^2 = y^2$ then $x = y$.

(f) $x^3 = y^3$ if and only if $x = y$.

Discussion: Statement (a) is a good example of an erroneous implication one might write in an algebra class. Plausible-looking is not the same thing as true! Statement (c) requires understanding a property of the function $f(x) = x^3$. Statement (d) is true and the student should be able to give a complete proof by the time they complete a first course concerning algebra. Statement (e) is the converse of a true statement. However, it is false. Just as in the geometric context of problem 4, it is important for algebra that students understand clearly that the converse of a true if-then statement need not be true. Statement (f) uses the phrase 'if and only if'.

9: Malcom says that

$$\frac{8}{11} > \frac{7}{10}$$

because $8 > 7$ and $11 > 10$. Even though it is true that $\frac{8}{11} > \frac{7}{10}$, is Malcom's reasoning correct? If Malcolm's reasoning is correct, explain why clearly; if Malcolm's reasoning is not correct, give Malcolm an example that shows why not. [This problem comes directly from *Mathematics for Elementary Teachers* by S. Beckmann, Pearson Addison Wesley 2005, pg. 93, problem 15.]

Discussion: In this case the statement $8/11 > 7/10$ is true but the reasoning offered to justify it is not correct. Indeed, the justification offered is based on the false statement: 'If $0 < a < b$ and $0 < c < d$, then $\frac{a}{c} < \frac{b}{d}$.' It is important that students learn that such reasoning, even when used to justify something true, is not mathematically valid. Note that though the algebraic form of the false statement requires sophisticated algebra skills to understand (and debunk), the problem as stated is suitable for middle school students.

In conclusion, we echo the Lead Essay in emphasizing that the mathematical reasoning skills whose development is promoted here should also play a role in the way that mathematics is presented in the curriculum. To illustrate this, we offer two problems that could form bases for classroom discussions that involve mathematical reasoning.

10: Your best friend knows that 'two wrongs don't make a right'. This friend can not believe that for numbers, a negative times a negative is a positive. Explain to your friend why the product of two negative numbers must always be positive.

Discussion: This basic fact about multiplication is sometimes presented to students as a "rule" without full justification, but it can be explained. The explanation requires an application of the distributive law and is somewhat subtle.

11: If the sum of the digits of an integer, written in base 10, is divisible by 3, then the number itself is divisible by 3. Explain why this is true for integers with two or three digits.

Discussion: This is a sophisticated problem concerning the reason why an 'if-then' statement is true. The solution requires an understanding of the base 10 number system and the knowledge that 10 and 100 each have remainder 1 upon division by 3. Once students have understood this divisibility test for two and three digit numbers one could ask them to explain it in general.