5. Lengthy Calculations

Many real-world problems in mathematics are computationally difficult. However, the problems actually presented to students are usually restricted in type or carefully contrived to be easily worked. Our hope as teachers is that if students can do the simple problems then they can do longer problems that use the same methods or ideas. This problem set focuses on such longer problems. Though they may be too long for a test, it would be valuable to include a few explicit "long problems" in the curriculum. For instance, when learning multiplication of multi-digit integers most practice problems have 2-3 digits. To this could be added a few multiplications of two 4-digit numbers. We suggest that these be explicitly labeled "long problems" and not scattered in with the others. Further they should be presented only after similar short problems have been well mastered so there is no qualitative reason for students to have trouble with them. If short-problem skills do generalize then students should get significant pride and satisfaction from discovering this.

Let us remark that some state standards give the impression that problems become appropriate for higher grade levels as the number of digits increases. To us a more important indicator of difficulty is structure. For instance, addition of fractions is easiest to understand and perform when all the denominators are the same, somewhat more difficult when the denominators are not the same but one of the denominators is also the least common denominator, and still more difficult when neither of the preceding conditions holds. When long problems do not involve new elements of structure, we believe that students who have mastered short problems should be encouraged to do long problems on the same topic. Thus we believe that students who obtain a good command of the multiplication of two multi-digit numbers in a given school year, for example, should be encouraged to do long problems on this topic without waiting for an additional year or more to go by. The same principle applies at all levels of K-12 education, including high school algebra and beyond.

The problems in this section are meant to be solved, once again, with pencil-and-paper. Admittedly, at a later stage of development some of these long problems might be turned over to a calculator, but we believe that there is genuine value in being able to do such problems by hand.

For the more elementary problems, we do not write explicit examples but simply describe the kind of calculation to be done.

1: Add two 10-digit numbers. Add a 10-digit number and an 8-digit number. Add four 5-digit numbers.

Discussion: At the lower grades, some students might find their lack of knowledge of the names of large numbers a psychological roadblock. After solving such a problem, students should realize that they can do such a calculation without the names. Of course, it is also important that students learn what the names of 10-digit numbers are.

2: Subtract one 10-digit number from another. Subtract an 8-digit number from a 10-digit number.

3: Multiply two 4-digit numbers. Multiply two 9 or 10-digit numbers ending in strings of 5 or 6 zeroes (e.g. $4,201,000,000 \times 176,000,000$). Multiply two 9 or 10-digit numbers ending in strings with 4 or 5 zeroes and one non-zero digit (e.g. $4,201,000,007 \times 176,000,060$)

4: Divide a 7-digit number by a 3-digit number. Divide a 10-digit number ending in 5 zeroes by a 5-digit number ending in 3 zeroes. Give the answers as quotients with remainders.

5: Without using scientific notation, calculate

$$0.0000000067 \times 5000000\,.$$

Then do the calculation again by first writing each of given numbers in scientific notation.

6: Write $5/17$ as a repeating decimal.

Discussion: When the students do the long division, they can see why two 1's in succession do not necessarily indicate that the repeating pattern has started. Also note that the period is 16 and that many calculators do not display this many digits.

7: Write $.378\overline{2474}$ as a fraction.

8: Add a stack of decimal numbers, a short stack of fractions and a short stack of mixed numbers.

The above problems concern computations with numbers. We next give some problems in the area of algebra in order to emphasize that, as noted above, fluency in algebraic computations also ought to include working several lengthy problems. Note that problems 10 and 11 involve geometry as well as algebra. Problems 11–14 below may be more suitable for an Algebra 2 course.

9: Write both

$$(x+y)^6 \qquad \text{and} \qquad (u-2v)^6$$

as sums of monomials. Then check the formulas by inserting $x = y = 1$ and $u = v = 1$. (Note: this is only a partial check of the formulas. Another, arguably better, check is $x = u = 10$, $y = v = 1$.)

10: Find where the line meets the circle:

$$y = 3x \qquad \text{and} \qquad x^2 + y^2 = 36.$$

11: Solve the 3 simultaneous equations:

$$x - y + z = 0 \qquad y - 2z = 0 \qquad x^2 + y^2 + z^2 = 16.$$

Discussion: This problem is algebraic but it answers a geometric problem as well. The first two equations are planes, and they intersect in a line. Solving the 3 equations simultaneously gives the points that are on the intersection of this line with the sphere $x^2 + y^2 + z^2 = 16$.

12: Write $(a + b + 2c - d)^3$ as a sum of monomials.

Discussion: One way to do this is to apply the standard formula $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$ with $x = a + b + 2c$ and $y = -d$. The computation then reduces to several simpler problems.

13: Solve for w, x, y, z:

$$\begin{array}{rrrrcr} w & +x & +y & +z & = & 10 \\ w & & +3y & -2z & = & 2 \\ -3w & +x & -9y & +7z & = & 0 \\ -2w & -3x & -6y & -7z & = & -18. \end{array}$$

Discussion: The method of choice here is the Gaussian Elimination process and not Cramer's Rule.

14: Find the value of the function $V(x, y, z) = 8xyz$ if x, y, z, are positive numbers and they satisfy the equations

$$\begin{aligned} x^2 + \frac{y^2}{4} + \frac{z^2}{9} &= 1 \\ 8yz + 2\lambda x &= 0 \\ 8zx + 2\lambda \frac{y}{4} &= 0 \\ 8xy + 2\lambda \frac{z}{9} &= 0 \end{aligned}$$

for some quantity λ.

Discussion: This calculation arises in calculus [1]. When students encounter such a problem in college, the discussion is on obtaining the above equations. Once they do so, it is assumed that students have the algebra skills to solve them. Indeed, this is regarded as the easy step; if such a step actually causes students to struggle then it distracts them from the central calculus concepts involved.

[1] in finding the box of largest volume with each edge parallel to one of the coordinate axes that is contained in the ellipsoid $x^2 + y^2/4 + z^2/9 = 1$