2. Patterns and Sequences

Mathematics is crucial in science and the social sciences because it allows one to express precisely the relationship between different quantities. The study of such relationships is the heart of algebra. It is natural to want to introduce children to mathematical notions such as variables, relations, and functions as soon as possible, and this is often done through the study of patterns. For example, the doubling rule beginning with the number 1 gives the sequence

$$1, 2, 4, 8, \ldots,$$

and in later mathematical studies this leads to a discussion of the function $y = 2^x$ and to the notion of exponential growth.

It is important in discussing and in testing patterns that the basic idea of mathematics as a rigorous subject involving disciplined reasoning not be lost. (See the Lead Essay, Principles for School Mathematics 3 and 4.) For example, if students are to investigate the sum of the first n odd numbers

$$\begin{aligned} 1 &= 1 \\ 1+3 &= 4 \\ 1+3+5 &= 9 \\ 1+3+5+7 &= 16 \end{aligned}$$

and notice that the answer is the perfect square n^2, it is important that they learn that this is a mathematical fact which is not established for all n simply by checking some cases, but which can be (and will be!) established at an appropriate later time in their studies. (See the discussion of this topic in the NCTM Standards for Algebra, Grades 3-5.)

The difference between noticing a pattern and showing conclusively that the observed pattern continues is often overlooked. To explain why this is not automatic, consider the sequence

$$1, 2, 4, ?.$$

One way to continue the sequence is to assign the value $? = 8$; this can be arrived at by the doubling rule, for example. However, if one simply asks to continue this pattern then in fact *any number can be assigned as the value of* ?. That is, there is no single mathematically valid answer here which can be arrived at by mathematical reasoning. To emphasize this, let us mention a few different ways of continuing the sequence which also arise from natural mathematical constructions:

$$\text{(i)} \qquad 124/999 = .\overline{124} = .124124124\ldots$$

$$\text{(ii)} \qquad 138/1111 = .\overline{1242} = .124212421242\ldots.$$

(iii) The next term is ? = 7. This arises from a geometric problem: count the number of regions into which lines in "general position" divide the plane. For instance, if there were no lines, there would be 1 region—namely, the entire plane; if there were one line, the two sides of that line would make 2 regions; two intersecting lines would divide the plane into 4 regions (here "general position" means not parallel). The next term (the number of regions obtained from three non-concurrent pairwise intersecting lines) is 7.

The point here is that though context may allow one to select a particular next term which is most likely to fit a given collection of data or to apply to a given specific situation, it is simply incorrect to say that there is a single next answer to the problem in the absence of additional information.

This leads us to a crucial principle:

Test and homework problems involving patterns and sequences should not rely on unstated assumptions, nor should they imply that there is a unique answer when this is not mathematically justified. In particular, a given sequence of numbers with no other information stated has infinitely many continuations.

Adhering to this principle is crucial if students are to be fairly tested, are to develop proper notions of mathematical reasoning, and if they are to find mathematics presented as a series of logical steps rather than rules without justification.

In fact, claiming that an observed pattern generalizes without justification not only misleads students about the role of mathematical reasoning in understanding and using mathematics, it can also lead to false conclusions. Students who look at the first five terms of the sequences $(n+2)^2$ and 2^n (namely $9, 16, 25, 36, 49$ and $2, 4, 8, 16, 32$, respectively) might expect that the term in a given position of first sequence is always larger than the term in the corresponding position of the second sequence, but this is not true from $n = 6$ onwards. If students are asked to consider a circle with several points selected on its circumference and draw all the chords with these as endpoints, then they will notice that these chords divide the circle into a number of regions. For small numbers of points $n = 1, 2, 3, 4, 5$ they can observe that the number of regions is $1, 2, 4, 8, 16$ (respectively) and they may think that the number is always 2^{n-1}. But this is not true; the number of regions when $n = 6$ is 31 and not 32. Similarly in real-world applications, extrapolation from data is important but context is crucial; if the share price of a certain stock at the end of March, April, and May is \$10, \$20, and \$30 (respectively), it would be rash to expect the price to be \$40 at the end of June simply on the basis of this information.

With this as background we illustrate problems that meet and do not meet the basic criterion above. We label the latter harmful since they in fact deliver a false impression of mathematics itself.

Harmful problems:

1: What is the next term in the sequence $1, 2, 4$?

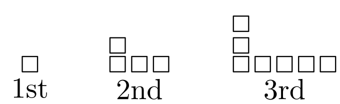
2: Give the continuation of the pattern which begins $3, 1, 4, 1, 5$.

Discussion: The objections above apply. In fact, as noted above, there are infinitely many answers, and with the information given there is no mathematical reason to prefer any one of them.

3: The Fair Meadows County Library charges a fine of $0.50 for a book that is two days late and $0.75 if it is three days late. What is the late fee for a book that is returned one day late? Two weeks late? One hundred days late?

Discussion: This problem cannot be solved with the information given. What if the library has a one-day grace period? What if it has a system of fines that is not linear, such as a cut-off at the cost of the book plus a service charge? The point here is that someone working on this problem is required to make an assumption that is not stated in the problem.

4: Each arrangement in the pattern below is made up of tiles. How many tiles are there in the fifth arrangement in this pattern?



Discussion: Once again the "pattern" is not described mathematically. There is no way to answer the question without guessing the particular continuation that its author had in mind. In fact there are infinitely many other continuations, each of which is consistent with the information given in the problem. The question could be changed to address this problem by stating that the successive arrangements are obtained by adding boxes to the top of the leftmost edge and to the right of the bottom edge and nowhere else and that the numbers of tiles on these edges are described by arithmetic progressions.

We also discuss two other problems that would satisfy, albeit minimally, the criterion stated above, but which do not seem of great value to us:

5: State a rule which gives a sequence beginning $3, 1, 4, 1, 5$, and use your rule to find the next term in the sequence.

Discussion: Though well-intentioned, this is somewhat misleading in that the rule which states by caveat that the sequence is 3,1,4,1,5,1000, continuing and repeating forever is mathematically just as valid as any other rule (even one more ostensibly natural such as the digits of π). So there is almost no mathematical content in the question.

6: (For elementary school students) Find 5 patterns among the entries of Pascal's triangle:

$$\begin{array}{c} 1 \\ 1\ \ 1 \\ 1\ \ 2\ \ 1 \\ 1\ \ 3\ \ 3\ \ 1 \\ \vdots \end{array}$$

Discussion: This problem makes no hidden assumptions. The issue is that there is no great gain in mathematical knowledge for elementary students who may notice that entries of the rows add up to powers of 2, etc. Prior to work done counting combinations, such investigations do little to contribute either to the mastery of core topics or to the development of rich mathematical reasoning skills, and seem to us of limited value.

Helpful problems:

7: Which of the following rules gives rise to a sequence beginning $1, 2, 4$? For each of the rules that does so, find the next term predicted by that rule. (7-a) the doubling rule: the next term is twice the preceding one. (7-b) add one to the first number to get the second, two to the second number to get the third, and in general add N to the N-th number to get the $(N+1)$-st. (7-c) triple any number in the sequence and subtract 1 from this to get the next number. (7-d) add 1 to the numbers in odd positions in the list and add 2 to the numbers in even positions in the list in order to get the next. (7-e) square any number in the sequence and add 1 to the result to get the next number.

8: Fill in the next three terms of the following sequence in which the rule for all but the first two terms is that each term is 1 more than the sum of the preceding two terms: $1, 2, 4, \_\_, \_\_, \_\_.$

Discussion: Note that the difficulty in understanding the rule from prose is nontrivial and this is the most difficult aspect of this problem. But this is the kind of reading that is very important in mathematics.

9: Notebooks are sold at a fixed price per notebook. At a special sale, if a customer purchases from 2 up to 6 notebooks, the fixed price is reduced by 5 cents times the number of notebooks purchased. If 3 notebooks cost \$2.25, how much would 1 notebook cost? How many notebooks would you need to purchase to save 1/3 of the original price on each notebook?

10: The Fair Meadows County Library charges a fine of a fixed amount per day for each day a book is overdue. If the fine is \$0.50 for a book that is two days late and \$0.75 if it is three days late. what is the late fee for a book that is returned one day late? Two weeks late? One hundred days late?

We note that a more systematic study of sequences may be undertaken using algebraic symbolism. This leads to the definition and discussion of arithmetic and geometric progressions, to the notion of recursion which is tied to computer science, and to proof by mathematical induction. The algebraic symbolism used in recursions makes it easier to write questions for advanced middle school or high school students than for K-6 students. Here are five such questions with different levels of difficulty.

11:

(a) What are the first five terms of the arithmetic progression which begins 2, 6? What is the 20th term? (For advanced high school students: What is the sum of the first 20 terms?)

(b) What are the first five terms of the geometric progression which begins 2, 6? What is the 20th term? (For advanced high school students: What is the sum of the first 20 terms?)

12: Suppose that $a_1 = 1$, and for $n > 1$, that $a_n = 2a_{n-1}$. So,

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2a_1 = 2 \times 1 = 2 \\ a_3 &= 2a_2 = 2 \times 2 = 4\,. \end{aligned}$$

Find a_4, a_7, and a_{14}.

13: It is given that $a_n = 2a_{n-1}$ and that $a_6 = 12$. Find a_1, a_3, and a_9. Then find a closed (i.e. explicit) formula for a_n for arbitrary n, and prove your formula.

Discussion: The preferred solution for the first part is to divide 12 by 2^{6-1}, but it could also be solved by successively dividing by 2 five times. This part also reinforces the arithmetic fluency called for in many state grade 3-7 standards.

14: It is given that $b_1 = 1$ and that for $m \geq 1$, that

$$b_{m+1} = m + b_m\,.$$

Find $b_7 - b_5$.

Discussion: This problem is related to the geometric problem described in part (iii) of the discussion of sequences which begin $1, 2, 4$.

15: A sequence begins with $c_0 = 1$, $c_1 = 2$, and continues for $m \geq 0$ via the recursion

$$c_{m+2} = 1 + c_{m+1} + c_m \, .$$

Find c_6.

Discussion: It happens that the sequence in this problem and the one in problem (13) both begin as $1, 2, 4, 7$. But the next term in the preceding problem is 11 while the next term here is 12. Once again, we see an illustration of the principle that there is no unique way to continue a finite sequence of numbers in the absence of additional information.

As a concluding remark, we note that patterns do have a place in mathematics, *when combined with mathematical reasoning*. Since it may be easy to detect a pattern and there is a temptation to let that be a substitute for careful mathematical reasoning, it is important that teachers present the detection of possible patterns as only the first step in a more detailed investigation, *and not be satisfied with that step alone*. As an example, one could ask students to investigate whether or not $n^2 - n + 5$ is prime as n ranges over the whole numbers. Though this expression is prime for $n = 0, 1, 2, 3, 4$, in fact, it is composite for $n = 5$, and also for $n = 10, 15, 20$. Students who notice this could then *prove* that it is composite for all integers n that are mutliples of 5 except $n = 0$. Noticing that the expression is also composite for $n = 6, 11, 16, 21$ leads to another provable fact: values of n that are one more than a multiple of 5 yield composite numbers with the single exception that $n = 1$ yields the prime number 5. (The proof involves rewriting the polynomial as $n(n - 1) + 5$.) The process of detecting patterns *and then using careful mathematical reasoning to establish them* is a useful technique in mathematical knowledge-building.