1. Problems for Pencil and Paper

The foundation of mathematics learning is the mastery of whole number arithmetic and the place-value system. (See the discussion of Principle 1 in the Lead Essay.) A similar mastery of fractions and decimals is a key goal for subsequent learning. An important aspect of this mastery is the ability to do arithmetic calculations accurately by hand, without a calculator. For computational fluency, students should be able to add and multiply single-digit numbers automatically and also do the corresponding subtraction and division problems very rapidly. Their ability to do these calculations—which are the foundation of all arithmetic computations—automatically and with complete accuracy should be developed fully in the early grades. They should also be able to multiply numbers by $10, 100$, etc. very rapidly. In addition, students should be able do the following operations correctly using only pencil and paper:

- add a set of multi-digit numbers, such as half a dozen 3-digit numbers
- subtract one multi-digit number from another
- mutliply two multi-digit numbers, such as a 4-digit number by a 2-digit number
- divide two multi-digit numbers, such as divide a 4-digit number by a 2-digit number
- combine understanding of place value with addition, subtraction, multiplication, and division (e.g. go from 36×12 to 36×120 and 36×1.2)
- translate between improper fractions and mixed numbers
- add, subtract, multiply, and divide fractions
- find the decimal equivalent of a given fraction and the fraction equivalent of a given decimal
- add, subtract, multiply, and divide decimals
- work with percentages, and convert fluently between percentages, decimals, and fractions
- add, subtract, multiply and divide numbers written in scientific notation, and convert between scientific notation and standard notation.

Indeed, as explained in the Lead Essay, we believe that a mastery of these pencil-and-paper computations is crucial to a solid understanding of numbers and arithmetic. This level of competency also enables students to understand the individual steps in complex computations. Such computational fluency is also useful in many real-world situations, from financial transactions to grocery shopping.

We emphasize that the ability to do arithmetic with fractions by hand is important. It is necessary for success in algebra; a skill with decimals but not fractions (for example, being able to handle operations with fractions solely by means of a calculator) is *not* adequate. Indeed, fluency in the arithmetic of fractions is crucial background for developing fluency in the algebra of fractions. For fractions appear in algebra in the form of algebraic expressions that cannot be treated as numbers on a calculator. Besides this, fractions play a key role in understanding statistics and probability (and conversion of fractions to decimals makes its almost impossible to trace an error or to easily assess how the answer will change if a problem is changed slightly).

In the following problems we illustrate these skills. We also include several problems in algebra and basic trigonometry of a similar flavor.

1:

$$234 + 582 = \qquad 901 - 74 =$$
$$582 \times 7 = \qquad 392 \div 7 =$$

Discussion: Prior to doing these calculations, students must be able to add and multiply one digit numbers *automatically* (that is, without pencil and paper, and quickly, without the need to compute) with complete accuracy. They should also be able to do the inverse problems (subtraction, division, such as $16 - 9$, $56 \div 8$) automatically. The ability to do these basic calculations automatically is fundamental for all that follows.

2:

$$\begin{array}{r} 357 \\ +\ 842 \\ +\ 2,392 \\ \hline \end{array}$$

3: Calculate 37×79. Then calculate $37 \times .79$.

4: Calculate

$$23\overline{)805} \qquad \text{and also} \qquad 2.3\overline{)8.05}$$

5: Convert each of the following to a whole number, a proper fraction in lowest terms or a mixed number with fractional part in lowest terms.

$$\frac{7}{4} \qquad 25\% \qquad 23.24 \qquad \frac{632}{27} \qquad .\overline{9}$$

$$62.5\% \qquad \frac{63}{784} \qquad 0.0412 \qquad 0.03125\% \qquad .3\overline{25}$$

Discussion: In the problem above, for convenience we have placed somewhat different types of number representations together. For students, including these ten calculations as parts of a single problem might significantly increase the difficulty.

6: Which of the following fractions equals 0.8: 3/4, 4/5, 5/4, 2/3, 3/2, 4/3 ?

7: Write 84 and 350 each as a product of prime numbers. Then find their greatest common divisor.

Discussion: Factoring a number into primes is a useful tool in finding greatest common divisors and least common multiples, and in determining when one number is a multiple of another.

8: Write the answers to the following problems as fractions, proper or improper, using the number one in the denominator in case of a whole number answer.

$$\frac{2}{3}+\frac{5}{6}= \qquad \frac{5}{6}-\frac{2}{3}= \qquad \frac{2}{3}\times\frac{5}{6}= \qquad \frac{5}{6}\div\frac{2}{3}=$$

$$\frac{2}{3} \text{ of } 12\frac{1}{2}\% = \qquad 3.225+1.725= \qquad 4.75\div(5/7)=$$

$$17\% \text{ of } 17 = \qquad \frac{5}{6}\times\left(\frac{3}{4}+\frac{1}{3}\right)= \qquad \left(\frac{5}{6}\times\frac{3}{4}\right)+\frac{1}{3}=$$

9: Write the answers to the following problems as fractions, proper or improper, with natural numbers (that is, positive integers) as denominators.

$$^{-}7+{}^{-}4= \qquad ^{-}7-{}^{-}4= \qquad ^{-}7\times{}^{-}4= \qquad ^{-}7\div{}^{-}4=$$

$$\frac{^{-}5}{3}+\frac{7}{2}= \qquad ^{-}3\,\tfrac{4}{5}\times 2\,\tfrac{1}{7}= \qquad ^{-}2.75\div{}^{-}0.25= \qquad 4-({}^{-}3-7)=$$

$$(4-{}^{-}3)-7= \qquad 4\div({}^{-}5\times 2)= \qquad (4\div{}^{-}5)\times 2= \qquad ^{-}4\times({}^{-}6-17)=$$

10: Express the answers to the following calculations as fractions, whether proper or improper.

$$\frac{13}{35} - \frac{5}{21} = \qquad \frac{{}^{-}19}{28} + \frac{47}{70} = \qquad \frac{22}{15} - \frac{{}^{-}4}{21} - \frac{6}{35} =$$

11: Write each of the following expressions as a fraction having denominator 36.

$$\frac{2/3}{3/2} \qquad \frac{5/2}{6} \qquad \frac{26}{72} \qquad \frac{14/(4/5)}{45}$$

Discussion: Placing fractions over a common denominator enables one to easily order them on the number line, and is also useful for some calculations.

12: Give your answers to the following calculations as either integers or as decimals with at most three significant figures.

$$5^4 = \qquad ({}^{-}5)^4 = \qquad 4^5 = \qquad 3^{{}^{-}5} = \qquad (7/5)^2 = \qquad ({}^{-}1.7)^3 =$$

13: Write each answer or approximate answer as a power of 10 times a number having one nonzero digit to the left of the decimal point and two digits to the right of the decimal point.

$$(3.71 \times 10^3) + (2.12 \times 10^2) = \qquad (5.20 \times 10^4) \times \ (2.30 \times 10^{{}^{-}3}) =$$
$$(3.54 \times 10^0) \div (7.00 \times 10^{17}) = \qquad (3.1 \times 10^4)^3 =$$

14: Solve each equation for x.

$$\frac{6}{25} + \frac{6}{25} + \frac{6}{25} = \frac{x}{75} \qquad \frac{x}{2} = \frac{3x}{4} - 1 \qquad \frac{2}{x} + 7 = 12$$
$$.01 + .6x = .05 \qquad x^2 + 3x + 2 = 0.$$

15: Write each of the following as sums and/or differences of monomials.

$$(3x^2 - 2)(x + 5) = \qquad 7x[(3x - 2) + 5(x - 6)] = \qquad (2x^2 + 5x - 7)(x^2 + 6) =$$

Discussion: Asking students to simplify is sometimes ambiguous because, for instance, a factored form is simplest for many purposes.

16: Write the following expressions in factored form in such a way that in any factor in which x appears it only appears to the first power and its coefficient equals 1.

$$x^2 - 9 = \qquad 4x^2 - 36 = \qquad x^2 + 6x + 8 = \qquad 3x^2 + 12x + 12 =$$
$$\sqrt{5}\,x^2 - 9\sqrt{5} = \qquad x^2 - 13 = \qquad 2x^2 - 8y^2 = \qquad \sqrt{6}\,x^2 - \sqrt{96} =$$

17: Simplify each of the following expressions for those values of x for which the denominator does not equal 0.

$$\frac{x^2-9}{x-3} \qquad \frac{x^2-9}{x+3} \qquad \frac{x^2+5x+6}{x+3} \qquad \frac{3x^2-48}{6x+24}$$

18:

$$\sin 30^\circ = \qquad \tan(\pi/4) = \qquad \cos 45^\circ = \qquad \arcsin(\sqrt{3}\,/2) =$$

Discussion: Some might prefer that these values of the trigonometric functions be memorized. We have put them here because someone who does not have them in memory can (and should be able to) do a short calculation to obtain the correct values.

The following problems are somewhat more technical, but illustrate skills which high school students should master in order to be adequately prepared for a college major in a discipline that makes use of quantitative methods. Since high school students who do not anticipate such a major (or even college attendance) may later change, it is advisable to encourage as many as possible to achieve this mastery.

19: Complete the square to write each of these expressions as $ap(x)^2 + b$ where $p(x)$ is a polynomial and a, b are numbers.

$$x^2+3x-4 \qquad -2x^2+8x-3 \qquad x^6+4x^3+7$$

Then sketch the graphs of these functions.

20:

$$\begin{aligned}(x^4+x^3-x-1) \div (x^2-1) &= \\ [(x^2-1)/(x^2+1)] \div [(x+1)/(x^2+1)] &= \\ \frac{-2a}{a^2-1} + \frac{1}{(a-1)^2} &= \end{aligned}$$

21: Rationalize the denominator and write the new numerator in a form not requiring parentheses. Then start over, rationalizing the numerator and writing the new denominator without parentheses.

$$\frac{\sqrt{x+4}+9}{\sqrt{x+4}-2} = \qquad \frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \qquad \frac{x^{1/3}-5}{x^{2/3}+5x^{1/3}+25} =$$

22: Find the coordinates of the point where the lines $2x+3y=4$ and $3x+8y=5$ intersect.

23:

$$\frac{1}{3}\log_b 64 = \qquad \log_3 81 = \qquad -\log_b(1/c) = \qquad \log_{10}(1/1000) =$$

24: Give simple equivalent expressions involving tan .

$$\cot(\tfrac{\pi}{2}-\theta) = \qquad \cot(\theta-\tfrac{\pi}{2}) = \qquad \frac{2\tan\theta}{1-\tan^2\theta} = \qquad \tan(\pi+\theta) =$$