Ratios, Rates, Percents and Proportion

The first serious applications of student's growing skills with numbers, and particularly fractions, appear in the area of ratios, proportions, and percents. These include constant velocity and multiple rate problems, determining the height of a vertical pole given the length of its shadow and the length of the shadow at the same time of a nearby pole of known height, and many other types of problems that are interesting to students and provide crucial foundations for more advanced mathematics.

Unfortunately, when the difficulties that students often have with fractions are combined with the confusion surrounding ratios, proportions, and percents, it seems that the majority of U.S. students have severe difficulties at this point. On the other hand, students in most of the high achieving countries solve very sophisticated problems in these areas from about grade three on. We will look at some third grade problems later, but to give some idea of the level of these problems in high achieving countries, here is a sixth grade problem from a Japanese exam:

> *The 132 meter long train travels at 87 kilometers per hour and the 118 meter long train travels at 93 kilometers per hour. Both trains are traveling in the same direction on parallel tracks. How many seconds does it take from the time the front of the locomotive for the faster train reaches the end of the slower train to the time that the end of the faster train reaches the front of the locomotive on the slower one?* ANS: $\frac{1}{24}$ *hour = 150 seconds.*

A large fraction of the U.S. elementary teaching corps also find this material challenging, and do not convey it effectively to students. The seriousness and energy with which the country upgrades the teaching corps so that this critically important material can be taught well to all students will provide a good litmus test as to our sincerity about improving U.S. mathematics education. In this section, we summarize the key points regarding ratio and proportion, focusing on the *teaching issues* that are core to a reasonable treatment, the crucial *definitions* that students need to see, the *reasoning* that underlies the material, and the kinds of *problems* students should be able to solve. Our discussion is based on the practices in countries where all students achieve significantly more in mathematics than is the case in the United States, so that we are confident it is realistic. On the other hand, it reflects expectations that are considerably higher than we currently hold in this country.

It is important that all of these topics are seen by students as closely related, in fact aspects of just a very few basic concepts. Consequently, we present them is this way here. One further remark should be made. This article is addressed to a mathematicially sophisticated audience, and uses appropriate algebraic symbolism and techniques throughout. This might give the impression that it suggests that such algebraic symbolism be introduced to cover these topics in K - 8. In fact, this is not the case. What is being recommended for the classroom has no algebra in the early grades and involves only minimal algebra later on.

Ratios:

Students first need to know what a ratio is: the ratio of the quantity A to the quantity B is the quotient $\frac{A}{B}$. Thus ratio is an alternative language for talking about division[1]. If A and B are quantities of different types, then the ratio $\frac{A}{B}$ retains the units of A (in the numerator) and B (in the denominator), and in this case, it is called a *rate*. We will discuss rates specifically later on. It is worth noting that when quotients are treated as ratios, the basic operations of arithmetic are not always appropriate. Thus the ratio of $A + A'$ to $B + B'$ is not the sum of the separate ratios, but rather $\frac{A+A'}{B+B'}$. However, the ratio of AA' to BB' *is* the product of the separate ratios.

Although ratios may first be encountered as quotients of whole numbers, eventually students will have to deal with situations where A and B are themselves fractions. (Indeed, eventually A and B may have to be fairly general real numbers, as with the ratio of the [length of the] diagonal of a square to the [length of the] side, or the ratio of the circumference of a circle to its diameter. However, we will restrict A and B to be rational numbers, i.e., quotients of integers in the discussion here.) Then the ratio $\frac{A}{B}$ is a quotient of fractions. Thus, to be successful with ratios, students will need to be comfortable with rational arithmetic. They should understand that if A and B are fractions, then $\frac{A}{B}$ is again a fraction; and specifically, that if $A = \frac{c}{d}$ and $B = \frac{e}{f}$, then

$$\frac{A}{B} = \frac{\frac{c}{d}}{\frac{e}{f}} = \frac{cf}{de}.$$

Similarly, they should be able to add ratios, (if this makes sense for a given problem) and know that the formula

$$\frac{A}{B} + \frac{C}{D} = \frac{AD + BC}{BD}$$

is valid when A, B, C and D are fractions, not just when they are whole numbers. (However, in this case, the expression is not a quotient of whole numbers, but of fractions.) Additionally, they should understand the property of equality

$$\frac{A}{B} = \frac{C}{D} \qquad \text{is equivalent to } AD = BC.$$

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Ratios appear as early as third grade in some state standards and in a number of the successful foreign programs.

1 Strictly speaking, ratios are homogeneous coordinates for points in projective space and the common notation for ratios, $a{:}b$ or $a{:}b{:}c\cdots$, indicates this, but the definition above is sufficient for school mathematics and will be used exclusively in the sequel.

2 Cross multiplication is subject to many misinterpretations and has to be handled with care. For example if one has $\frac{a}{b} = \frac{c}{d} + e$, it often happens that students will replace this expression by $ad{=}bc{+}e$.

When taking ratios, the order of the two numbers matters, and the ratio of B to A which is $\frac{B}{A}$ is the reciprocal of the ratio of A to B. Ratios are almost always fractions, but one usually contrives to make the ratio come out to be a whole number for third grade. Thus one may have a problem such as:

If 15 *items cost $4.50, what is the unit cost?*

Students should understand that the unit cost is the ratio of the total cost to the number of units. So, in this case, if 15 items cost $4.50, then the unit cost is $\frac{4.5}{15}$ dollars or $\frac{450}{15} = 30$ cents. Thus, the first contact with ratios can come before students have studied fractions.

Once students have learned about fractions, ratios should be revisited and the above procedure explained. For example, if 15 items cost 4.5 dollars, then the cost of a single item would be (the value of) one part when four and a half dollars, (i.e., 450 cents), is divided into 15 equal parts. By the division interpretation of a fraction, the size of one part when 450 is divided into 15 equal parts is exactly $\frac{450}{15}$. So it is 30 cents. In general, if n items cost x dollars, then the cost of one item is $\frac{x}{n}$ dollars for exactly the same reason.

By grade six, it would be reasonable for students to consider problems such as the following: if 2.5 pounds of beef costs $22.25, what is the cost of beef per pound? The unit cost is the ratio $\frac{22.25}{2.5}$, and one recognizes this as a quotient of two fractions, $\frac{2225}{100}$ divided by $\frac{25}{10}$.

Dimensions and Unit Conversions:

Ratios of units of measurement of the same kind – feet to inches, meters to kilometers, kilometers to miles, ounces to gallons, hours to seconds – are involved whenever a quantity measured using one unit must be expressed in terms of a different unit. Such ratios are called *unit conversions*, and the units involved are called *dimensions*. Unit conversions usually first appear in the context of money.

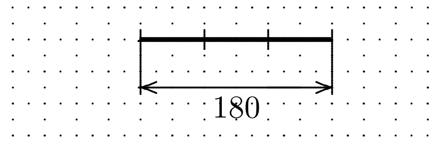
Dimensions sometimes appear by grade three in state standards, and sometimes appear even earlier in successful foreign programs. Here is an example of a third grade California geometry standard that deals with dimensions and unit conversions.

> *Carry out simple unit conversions within a system of measurement (e.g., centimeters and meters, hours and minutes).*

It is worth noting that, as indicated in the discussion above, dimensions and unit conversions have already appeared in the study of money. Students should get considerable practice with ratios, and with unit conversions. But care should be taken that proportional relationships not be introduced until students are comfortable with the basic concepts of ratio and unit conversion.

Many problems of the following type (taken from a Russian third grade textbook) are appropriate at this grade level:

1. *A train traveled for* 3 *hours and covered a total of* 180*km. Each hour it traveled the same distance. How many kilometers did the train cover each hour?*



2. *In* 10 *minutes a plane flew* 150 *km. covering the same distance each minute. How many kilometers did it fly each minute?*

Students should see many such problems, and become accustomed to the reasoning used in solving them. For example, in problem 1, since the train travels the same distance each hour, the distance it travels in an hour (no matter what it is) when repeated three times will fill up 180 km. Therefore the distance it travels in an hour is $180 \div 3$, because the meaning of "dividing by 3" is finding that number m so that $3 \times m = 180$. So the answer is 60 km. Problem 2 is similar: the distance the plane flies in a minute, when repeated 10 times, would fill up all of 150 km. So this distance is $150 \div 10 = 15$ km.

Percents:

Technically, percent is a *function* that takes numbers to new numbers. However, this is difficult for students to grasp before the algebra course. Consequently, it is best to simplify here. We suggest the following definition. A *percent* is by definition a complex fraction whose denominator is 100. If the percent is $\frac{x}{100}$ for some fraction x, then it is customary to call it x *percent* and denote it by $x\%$. Thus percents are simply a special kind of complex fraction, in the same way that finite decimals are a special kind of fraction. Percent is used frequently for expressing ratios and proportional relationships, as with the interest on a loan for a given time period.

Percents first appear around grade 5 in a number of states. For example, standards similar to

Use fractions and percentages to compare data sets of different sizes.

appear in a number of state standards at the fifth grade level.

Students should recognize that percents are special ratios, where the denominator (or b in the ratio of a to b) is 100. When students see a ratio in the form $\frac{a}{100}$ even if a is a fraction and not a whole number, they should understand that, according to the definition, the ratio a to 100 is the same as a percent, written as $a\%$. Once they understand this – and it is strongly advised that this be presented to students as a definition – then they should be able to sort out problems like the following:

What percent of 20 *is 7? (What percent is* 7 *of* 20*?)*

We follow the definition, and so must put the ratio $\frac{7}{20}$ into the form $\frac{a}{100}$. Since $100 = 5\times 20$, equivalent fractions gives $\frac{7}{20} = \frac{5\times 7}{5\times 20} = \frac{35}{100} = 35\%$. At the fifth grade level, students can also do something slightly more complicated:

What percent of 125 *is 24? (What percent is* 24 *of* 125*?)*

We have to express $\frac{24}{125}$ as $\frac{a}{100}$. In this case, no integer multiple of 125 is equal to 100, but $2 \times 125 = 250$ and it should immediately come to mind that $4 \times 250 = 1000$, which is almost as good as 100. So by equivalent fractions again: $\frac{24}{125} = \frac{8\times 24}{8\times 125} = \frac{192}{1000} = 19.2\%$. Note that this is also a fraction divided by a fraction.

More complicated problems of this type, such as what percent of 17 is 4, should be postponed to the point where student skills with fractions are sufficiently advanced.

Rates:

As noted at the outset, rates are ratios with units attached. It is vital to respect the units when doing arithmetic with rates. A product AB has units which are the product of the units of A and the units of B. But there is a subtlety in handling the unit conversions that has to be justified for students.

We need to address the issue of units explicitly for teachers and students in the lower grades. It seems likely that the discussion needs to be very detailed and careful. For example, we know that 1 yard = 3 feet. In terms of the number line, this means if the unit 1 is one *foot*, then we call the number 3 on this number line *1 yard*. The equality 1 yd. = 3 ft. is usually written also as $1 = 3\frac{\text{ft.}}{\text{yd.}}$. Observe that 1 sq. yd., written also as 1 yd.^{2}, is by definition the area of a square whose side has length 1 yd. Simliarly, 1 ft.^{2} is by definition the area of the square with a side of length equal to 1 ft. Since 1 yard is 3 feet, the square with a side of length 1 yard is *paved* by $3 \times 3 = 9$ squares each with a side of length equal to 1 ft., in the sense that these 9 identical squares fill up the big square and overlap each other at most on the edges. Therefore the area of the big square is 9 ft.^{2}, i.e., 1 yd.^{2} = 9 ft.^{2}. Sometimes this is written as $1 = 9\frac{\text{ft.}^2}{\text{yd.}^2}$.

How many feet are in 2.3 yd.? Recall that the meaning of 2.3 yd. is the number on the number line whose unit is 1 yd. So

$$2.3 \text{ yd.} = 2 \text{ yd.} + 0.3 \text{ yd.} = 2 \text{ yd.} + \frac{3}{10} \text{ yd.}$$

Now $\frac{3}{10}$ yd. is the length of 3 parts when 1 yard is divided into 10 parts of equal length (by definition of $\frac{3}{10}$), and therefore $\frac{3}{10}$ yd. $= \frac{3}{10} \times 1$ yd. $= \frac{3}{10} \times 3$ ft., using the interpretation of fraction multiplication. Of course 2 yd. $= 2 \times 3$ ft.. Altogether,

$$2.3 \text{ yd.} = \{(2 \times 3) + (\frac{3}{10} \times 3)\} \text{ ft.} = (2.3 \times 3) \text{ ft.}$$

Exactly the same reasoning shows that

$$y \text{ yd.} = 3y \text{ ft.}$$

which is of course the same as

$$y \text{ ft.} = \frac{y}{3} \text{ yd.}$$

These are the so-called *conversion formulas* for yards and feet, and the seemingly over-pedantic explanation of these formulas above is meant to correct the usual presentation of mnemonic devices for such conversion using so-called *dimension analysis*. For example, since have agreed to write $1 = \frac{3 \text{ ft.}}{1 \text{ yd.}}$, the dimenson analysis would have

$$2.3 \text{ yd.} = (2.3 \times 1) \text{ yd.} = (2.3 \times \frac{3 \text{ ft.}}{1 \text{ yd.}}) \text{ yd.}$$

Upon "cancelling" the *yd.* from top and bottom, we get

$$2.3 \text{ yd.} = (2.3 \times 3) \text{ ft.}$$

The reason for the pedantic explanation above is precisely to justify why this entirely mechanical process also yields the correct answer.

In like manner, 14.2 yd.^{2} is

$$14.2 \times 9 \text{ ft.}^2 = 127.8 \text{ ft.}^2$$

As another illustration, the common notion of speed has units miles/hour, or *mph*. Suppose a car travels (in whatever fashion) a total distance of $30,000$ ft. in 5 minutes, then by definition, the *average speed* of the car *in this time interval* is the total distance traveled in the time interval divided by the length of the time interval, i.e., the average speed in this particular case is

$$\frac{30,000 \text{ ft.}}{5 \text{ min.}} = 6,000 \text{ ft./min.}$$

Suppose we want to express this average speed as mph. Then, because $1 \text{ m.} = 5280 \text{ ft.}$, the conversion formula in this case gives

$$30,000 \text{ ft.} = \frac{30,000}{5280} \text{ m.} = 5\frac{15}{22} \text{ m.}$$

Also $5 \text{ min.} = \frac{5}{60} \text{ hr.}$, so

$$\frac{30,000 \text{ ft.}}{5 \text{ min.}} = \frac{5\frac{15}{22} \text{ m.}}{\frac{5}{60} \text{ hr.}} = \frac{125}{22} \times \frac{60}{5} \text{ mph} = 68\frac{2}{11} \text{ mph}$$

Or, one could have done it by considering

$$1 \text{ ft./min.} = \frac{\frac{1}{5280}\text{m.}}{\frac{1}{60} \text{ hr.}} = \frac{1}{88} \text{ mph},$$

so that

$$60,000 \text{ ft./min.} = 60,000 \times \frac{1}{88} \text{ mph} = \frac{750}{11} \text{ mph} = 68\frac{2}{11} \text{ mph}$$

Again, once the mathematical reasoning is understood, one can see why the method of dimension analysis applied to this case is valid:

$$\frac{60,000 \text{ ft.}}{1 \text{ min.}} = \frac{60,000 \text{ ft.}}{1 \text{ min.}} \times \frac{1 \text{ m.}}{5280 \text{ ft.}} \times \frac{60 \text{ min.}}{1\text{hr.}}$$

which (after "cancelling" the *ft.* and *min.* from top and bottom) works out to be $68\frac{2}{11}$ mph again.

When rates are added, it is essential that they be expressed in the same units. Some of the most challenging rate problems involve addition of rates. For example,

> *If John can paint each of three rooms in 1 hour, and Shauna can paint each room in 2 hours, then how long will it take them working together, to paint all 3 rooms?*

As long as units are carefully attended to, the arithmetic of rates follows the familiar rules.

A key example of rates is the general concept of motion *with constant velocity* or *constant speed*, which is significantly more sophisticated than average speed. For this kind of motion, the basic fact is that *distance traveled is equal to velocity multiplied by the time-duration* - usually abbreviated to "distance is velocity multiplied by time" or the self-explanatory formula

$$d = v \cdot t.$$

This formula is not straightforward. Justifying it requires a careful argument while assuming that it is evident places a heavy burden on students for whom it is often not obvious. The meaning of "constant speed" is that equal distances are traversed in equal times. To go from this definition to the above formula requires a several-step argument which is needed in several places in K-12 mathematics, but is usually ignored. A very brief sketch can be given as follows. Suppose that v is the distance traveled in unit time. Then by breaking up a time segment of whole number length n into n unit intervals, and comparing travel times, we see the formula is valid for all whole number times. Next, by dividing up the unit interval into m equal intervals of length $\frac{1}{m}$, one sees it must also be true for time intervals of the form $\frac{1}{m}$. Combining this with the preceding argument, it follows for all rational times. This is sufficient for most purposes. How and when to present this argument to students needs study. A relatively informal discussion of the issues this argument addresses can be illuminating, and can serve to make the formula reasonable. In any case, the formula should be stated clearly as the correct description of motion at constant speed.

One can proceed as follows with beginning students. In case the time t is a whole number (5 hours, 12 minutes, etc.), this formula is easy to verify. If the velocity v is 55 miles an hour, then the distance traveled after 2 hours is $55 + 55 = 2 \times 55$ miles, after 3 hours is $55+55+55 = 3\times 55$, after 4 hours is $55+55+55+55 = 4\times 55$ miles, etc. After n hours (with

n an integer), the distance traveled d is then $55+55+\cdots+55$ (n times) $= n\times 55 = 55n$. Since $v = 55$ and $t = n$, the formula is correct in this case. Clearly the velocity can be any v instead of 55 and the reasoning remains unchanged. So the formula is correct in general when t is a whole number.

If t is not a whole number, say $t = 6\frac{2}{5}$ hours and v is 55 miles an hour, then *the meaning of constant velocity is that the distance traveled in fractional hours such as* $\frac{2}{5}$ *is exactly* $\frac{2}{5} \times 55$ *miles.*

It is important to explain this meaning of "constant velocity" to students. We now verify the formula in this case too. During the first 6 hours, the distance traveled is, as we have seen, 6×55 miles. In the remaining $\frac{2}{5}$ of an hour, we are given that the distance traveled is $\frac{2}{5} \times 55$ miles. Thus the total distance traveled is $(6 \times 55) + (\frac{2}{5} \times 55)$ miles which, by the distributive law, is

$$(6 + \frac{2}{5}) \times 55$$

hours, which is our $t \cdot v$.

As a simple illustration of the formula consider the problem

A passenger traveled 120 km by bus. The speed of the bus was a constant 45 *km per hour. How long did the passenger travel by bus?*

Thus $d = 120$ km and $v = 45$ km per hour. According to the formula, $120 = 45 \times t$, where t is the total time duration of the passenger in the bus. Multiply both sides by $\frac{1}{45}$ and we get $\frac{120}{45} = t$ and so $t = 2\frac{30}{45} = 2\frac{2}{3}$ hours, or 2 hours and 40 minutes.

The other issue of substance is that many ratio problems are accessible without extensive symbolic computations or "setting up proportions." Consider the problem:

A train travels with constant velocity and gets from Town A to Town B in $4\frac{2}{3}$ *hours. These two towns are 224 miles apart. At the same velocity, how long would it take the train to cover 300 miles?*

From the data, the velocity is $\frac{224}{4\frac{2}{3}}$ miles per hour, or 48 miles per hour. Therefore to travel 300 miles, it would take $\frac{300}{48} = 6\frac{1}{4}$ hours, or 6 hours and 15 minutes. Another example of this kind is:

I spent \$36 to purchase 9 cans of Peefle. How much do I have to spend to purchase 16 cans?

The price per can is $\frac{36}{9} = 4$ dollars, so to buy 16 cans, I would have to pay $16 \times 4 = 64$ dollars. We emphasize once again that no "setting up a proportion" is necessary.

Percentage increase and decrease problems: Percentage increase, percentage decrease problems and related problems are quite tricky and somewhat non-intuitive. The fact that an $x\%$ increase (multiplying by $(1 + \frac{x}{100})$), followed by the same percentage decrease, (multiplying by $(1 - \frac{x}{100})$), is *not* going to get back to where you started since $(1 + x)(1 - x) = 1 - x^2$, is extremely mystifying to students. Part of the problem is that when this topic appears, students generally lack the algebraic skills to go through the argument. A second difficulty is that students often lack skill and practice in dividing fractions.

Students should convince themselves, via direct calculation, that a 20% increase, followed by a 20% decrease *does not* get one back to where one started. For example, a 20% increase of \$100 gives \$120, but a 20% decrease of \$120 yields $120 - (20\% \times 120) = 120 - 24 = 96$ dollars, which is less than \$100. By looking at further examples, say 10% and 15%, they should come to understand that an n-percent increase followed by the same percent decrease will always give *less* than what they started with, and they should be able to give a heuristic argument to justify this.

Later, in the first algebra course, it should be effective to apply the result $(a+b)(a-b) = a^2-b^2$ to explain exactly what the deviation in percentage increase, decrease problems is.

Proportions:

A *proportion* is usually defined as an equality of ratios: four *ordered* numbers A, B, C and D define a proportion if

$$\frac{A}{B} = \frac{C}{D}.$$

Proportions are a major source of difficulty in the K-8 curriculum. Part of the trouble may come from the fact that they involve division of fractions, which is one of the parts of arithmetic least understood by students. Beyond that, however, lies another difficulty. Just as ratios are a way of talking about division without using the term, proportions are a way of talking about linear functions without really mentioning them. That is, very often when one is talking about proportions, one has in mind two quantities which can take different values, but the ratio of the two quantities is a fixed number (or rate). Such quantities are said to vary in *direct proportion*. The formula expressing the relationship between the two varying quantities (call them A and B) is the linear equation $A = kB$, where k is a fixed number (or rate), called the *constant of proportionality*. Examples include the distance traveled in a given time at a constant speed ($d = rt$), the cost of n identical items ($C = pn$), the number of parts of some sort in identical objects, i.e., number of legs on c chickens ($L = 2c$), and so forth.

Proportions arise from this equation by taking two values A_1 and A_2 of A and two corresponding values B_1 and B_2 for B. The equations $A_1 = kB_1$ and $A_2 = kB_2$ imply that

$$\frac{A_1}{B_1} = k = \frac{A_2}{B_2}.$$

This just says that A_1, B_1, A_2 and B_2 form a proportion. The fact that proportions are treated in the curriculum before it is commonly thought appropriate to discuss variables and linear equations means that the heart of what is going on cannot be directly confronted. The difficulty is compounded by discussions in textbooks of solving problems about proportions by "setting up a proportion", such as $\frac{A_1}{B_1} = \frac{A_2}{B_2}$ above, without being able to explain what it takes to "set it up" or why the procedure is correct. However, if students can learn about proportions in a mathematically correct way, as we are suggesting here, this knowledge can be a tremendous aid when they come to an algebra course.

Basic proportional relationships appear as early as grade three. For example, standards of the following kind:

> *Extend and recognize a linear pattern by its rules (e.g., the number of legs on a given number of horses may be calculated by counting by 4s or by multiplying the number of horses by 4).*

However, standards of this kind should just be regarded as preparation for proportions.

Typically, proportions occur in the form that four quantities a, b, A and B give a proportion, and three of the four are known. It is then required to find the fourth. Thus, if A is not known, but a, b, and B are given, then $A = \frac{aB}{b}$. The crux of the matter, and it is one that consistently puzzles students, is how to set up the correct proportions to solve problems. We will address this issue below.

Proportions can be *motivated* via examples like relating n and the number of wheels on n bicycles, and noting that the ratios are the same for each n. (Similarly for counting the total number of legs on n chickens, and the cost of n items when the unit cost is known.) But it is very important that we do not stop at this point. Notice that if one bicycle has 2 wheels, two would have 2×2 wheels and three would have 2×3 wheels and, in general, n bicycles will have $2n$ wheels. So we have an association of $n \mapsto 2n$. We recognize this as a linear function of n. Put another way, we have a relationship

$$\frac{\text{number of bicycles}}{\text{number of wheels}} = \frac{1}{2}$$

So we have the constancy of a certain ratio, and this is the key to proportions.

Because bicycle wheel problems are too simple (one can guess the answer without doing any mathematics), let us illustrate with a different kind of problem:

> If a building at 5:00 PM has a shadow that is 75 feet long, while, at the same time, a vertical pole that is 6 feet long makes a shadow that is 11 feet long, then how high is the building?

What is implicitly assumed here, and what *must* be made explicit when teaching this material, is the fact that

$$\frac{\text{height of object}}{\text{length of shadow}} = \text{constant}$$

(This comes from considerations of similar triangles which will be taken up in a high school course.) Now we can do the problem. Let B be the height of the building. Then we know $\frac{B}{75}$ is equal to this constant while for the pole, the ratio $\frac{6}{11}$ is also equal to the same constant. Therefore the two numbers $\frac{B}{75}$ and $\frac{6}{11}$ are equal, i.e.,

$$\frac{B}{75} = \frac{6}{11}$$

The solution of the problem is now straightforward: multiply both sides by 75 to get $B = 75 \times \frac{6}{11} = 40\frac{10}{11}$ feet. (Or cross-multiply and get $11B = 6 \times 75$, so $B = \frac{450}{11} = 40\frac{10}{11}$ feet.) The equation displayed just above is traditionally said to be obtained by "setting up a proportion". Whatever it is called, the key thing for a student to know is that there is no guesswork involved. Once we know the constancy of the ratio of the height-of-object to length-of-shadow, the displayed equation is automatic and then getting the solution becomes routine.

To recapitulate: once we get the equality of two ratios (i.e. set up the correct proportion) in the form $\frac{a}{b} = \frac{x}{c}$, where a, b, c are known and x is the number we want to know, we can cross-multiply to get to $bx = ac$, and then divide by b to get $x = \frac{ac}{b}$. Alternatively, simply multiplying by c in the original equation will give the same result. It is useful for students to try multiple methods for solving these equations in order to support the understanding that correct methods of isolating the variable and solving the equation all give the same result.

Problems of motion at constant speed provide a good introduction to proportions. From our discussion of such problems we know that the distance d traveled in a given time t at constant speed v is determined by the equation $\frac{d}{t} = v$; that is, the ratio of distance to duration of travel is the constant v no matter what time interval t is chosen. Here as usual, we understand that all the numbers involved are fractions. Thus $\frac{d}{t}$ is a quotient of fractions. The following examples serve to illustrate the situation.

> **Example 1** *Suppose a train travels at constant speed. If it takes 2 hours and 40 minutes to go from town A to town B which are 160 miles apart, how long will it take to go from town C to town D which are 225 miles apart?*

Because we know the meaning of constant speed, we know that if x is the time it takes to go from C to D, then $225/x$ is the speed of the train, v, just as $160/2\frac{40}{60}$ is also the speed of the train v. Therefore

$$\frac{225}{x} = \frac{160}{2\frac{40}{60}}$$

as both are equal to v. Cross-multiply to get $160x = 225 \times 2\frac{2}{3}$, so that $x = \frac{225}{160} \times \frac{8}{3} = \frac{15}{4} = 3\frac{3}{4}$ hours, i.e., 3 hours and 45 minutes.

Note that we did not "set up a correct proportion". We merely observed what "constant speed" means and made use of it.

> **Example 2** *Water is coming out of a faucet at a constant rate. If it takes 3 minutes to fill up a container with a capacity of 15.5 gallons, how long will it take for it to fill a tub of 25 gallons?*

As is the case with all problems in mathematics, all the terms must be understood. Here, the key to the problem is to understand what is meant by *constant rate.* Students should know it means that if we measure the amount of water, say w gallons, coming out of the faucet during a time interval of t minutes, then $\frac{w}{t}$ is always a fixed quantity, say r, no matter what time interval is chosen. Note that r has units of $\frac{\text{gallon}}{\text{minute}}$. (The justification for $\frac{w}{t} = r$ is identical to the justification for the formula discussed above for constant speed travel.) Hence, if it takes x minutes to fill the tub, then both $\frac{25}{x}$ and $\frac{15.5}{3}$ are equal to r, and therefore

$$\frac{25}{x} = \frac{15.5}{3}.$$

(Notice that both are quotients of fractions because 15.5 is really $\frac{155}{10}$ and x is expected to be a fraction and not a whole number.) Cross-multiply to get $15.5x = 75$ so $x = \frac{75}{15.5} = 4\frac{26}{31}$ minutes.

As before, the displayed equation above was not "set up as a proportion". Rather, it is a statement of the constancy of the rate of water flow. In general, we want students to come to understand that *all claims about two things being proportional amount to saying that certain quotients are constants.*

Direct and Inverse Variation:

The final topics that we mention are

(1) Direct variation: Two quantities that change in such a way that their ratio remains constant are said to be be *directly proportional* or to *vary directly.* Thus this is simply further vocabulary for proportion

(2) Inverse variation: Two quantities that change in such a way that their product is constant are said to vary *inversely.*

Direct variation is the school math vocabulary for saying that one quantity is a linear function (without constant term!) of another quantity. It applies to constant speed motion: $\frac{d}{t} = v$, or $d = vt$; to constant flow problems: $\frac{w}{t} = r$, or $w = rt$; to cost-quantity relationships: $C = pN$ (where N is the number purchased of some item, C is the total cost of the order, and p is the unit price); and to many other interesting and important situations.

Inverse variation occurs in problems like the following taken from a sixth grade Russian text:

> *Three workers can perform a certain task in six hours. How much time will two workers need to perform this same task if all workers work at the same speed?*

If x denotes the unknown time, the relevant equations are $3 \times 6 = T$, which is given, and $T = 2 \times x$, which is posited. Combining them gives $x = 9$ hours.

Both direct and inverse variation are captured by the same equation, namely $c = ab$. The difference between them is which terms vary and which stay constant. If c and b vary, and a remains fixed, then c varies directly with b. On the other hand, if a and b vary, and c remains fixed, then a and b vary inversely. Note that for a given relationship of the form $c = ab$, either of these situations may obtain, depending on the circumstances. Thus, in the Russian problem above, we had a fixed job, and a variable source of labor, with consequent variation in the time needed. Time and rate of labor for a given job are inversely related. On the other hand, the same equation might be used to calculate the amount of time needed to do a certain quantity of work, with fixed labor input. Then one would be in a direct variation situation: amount of work done would vary directly with time allotted.

There is substantial confusion about these concepts in the United States at this time. To illustrate the problem we offer this example of an actual fifth grade state standard referring to direct and inverse variation in which everything is incorrect:

> *Identify and describe relationships between two quantities that vary directly (e.g., length of a square and its area), and inversely (e.g., number of children to the size of a piece of pizza).*

(Of course, while the difficulties with the first example are severe, the difficulties with the second are easily fixed.)

It should be realized that problems like the "three worker" question above involve very important techniques and problem solving skills. But the tension between vocabulary and the core concepts that the vocabulary is supposed to summarize is a very basic issue in instruction. There is a tendency for K - 8 teachers in this country to teach mathematics as though it were vocabulary and not much more. Students are often taught to look for key words in problems and, when they see them, do certain things that they learn by rote. Moreover, as the last standard indicates, sometimes the actual meanings of these words become so distorted that students are unable to understand the underlying concepts. Many students give up at these points.

In the case of direct and inverse variation, it would seem better to entirely suppress the vocabulary and simply discuss the equation

$$c = ab,$$

particularly the special cases where a is a constant or c is constant.