Gradient Percolation and related questions

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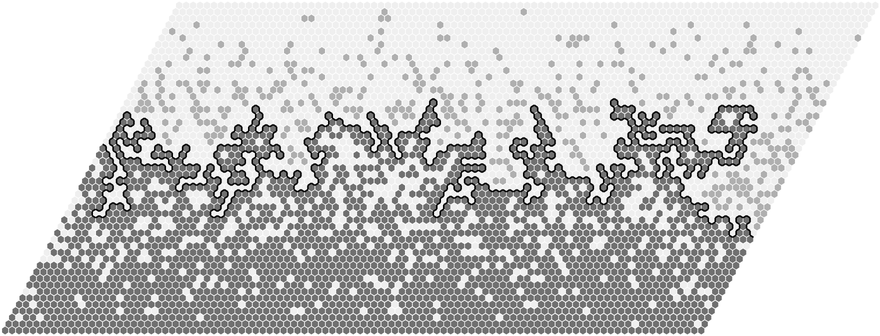


Introduction

Gradient Percolation is a model of inhomogeneous percolation introduced by physicists (J.F. Gouyet, M. Rosso, B. Sapoval) in 1985. The "front" of percolation which appears is a simple example of random interface (e.g. an interface created by welding two pieces of metal). It can be used to model phenomena of diffusion or corrosion.

Main source of inspiration: *Self-Organized Percolation Power Laws with and without Fractal Geometry in the Etching of Random Solids* (A. Desolneux, B. Sapoval et A. Baldassarri, 2004).

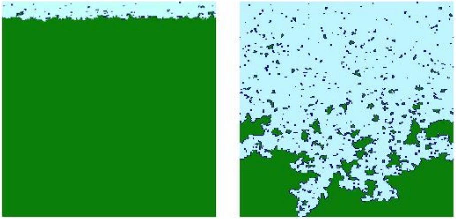
Introduction

It provides an example where a *fractal geometry* spontaneously appears. It is a case where one experimentally observes the fractal dimension ≈ 1.75 and various critical exponents (amplitude of the front, length. . .) that seemed related to those of standard percolation.

We will explain how one can prove these empirical observations, based on the recent works by G. Lawler, O. Schramm, W. Werner and S. Smirnov, that provide a very precise description of percolation near the critical point in 2 dimensions.

Introduction

Some of its characteristics being probably universal, its study is important to understand more complex models. It can for instance be viewed as the approximation of a more “dynamical” model, where random resistances are assigned to each site of a material (*Etching Gradient Percolation*).



(Fig B. Sapoval)

Introduction

Theoretical importance:

- *spontaneous* appearance of the percolation phase transition
- revealing of some critical exponents of percolation
- universality of the observed behavior (?)

Practical importance:

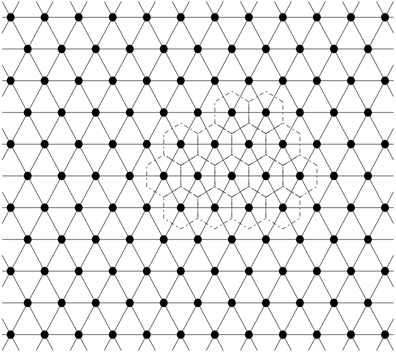
- efficient way of estimating p_c (Sapoval, Ziff. . .)

1. Standard percolation background
   - Framework
   - Main properties
   - Near-critical percolation: what is known
2. Gradient Percolation
   - Setting
   - Main properties
   - Behavior of some macroscopic quantities
   - Estimating p_c
3. Off-critical percolation: some properties

Standard percolation background

Site percolation

We consider the *triangular lattice*:



Remark

Why the triangular lattice ?

We restrict to the triangular lattice, since at present, this is the only one for which the existence and the value of critical exponents have been proved.

However, the results presented here are likely to remain true on other lattices, like the square lattice $\mathbb{Z}^2$.

Existence of a phase transition at $p = 1/2$

Percolation features a *phase transition*, at $p = 1/2$ on the triangular lattice:

- If $p < 1/2$: a.s. no infinite cluster (*sub-critical* regime).
- If $p > 1/2$: a.s. a *unique* infinite cluster (*super-critical* regime).

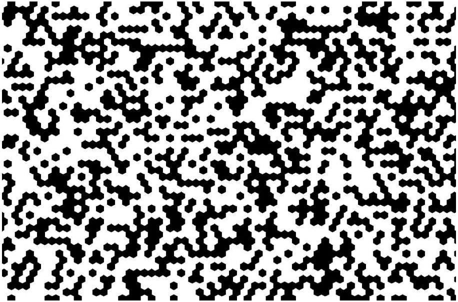
If $p = 1/2$: *critical* regime, a.s. no infinite cluster.

Exponential decay

In *sub-critical* regime ($p < 1/2$), there exists a constant $C(p)$ such that

$$\mathbb{P}_p(0 \rightsquigarrow \partial S_n) \leq e^{-C(p)n}$$

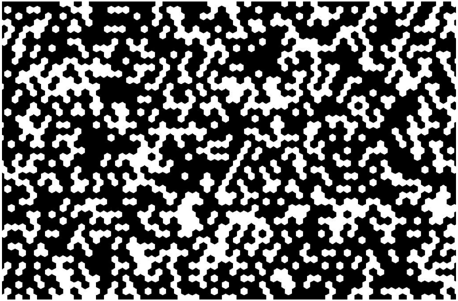
$\Rightarrow$ fast "decorrelation" of distant points (speed *depends on p* !).



Exponential decay

In *super-critical* regime ($p > 1/2$), we have similarly

$$\mathbb{P}_p(0 \rightsquigarrow \partial S_n | 0 \nrightarrow \infty) \leq e^{-C(p)n}$$



Critical regime

At the critical point $p = 1/2$:

Critical regime

The Russo-Seymour-Welsh theorem is a key tool for studying critical percolation:

Theorem (Russo-Seymour-Welsh)

For each $k \geq 1$, there exists $\delta_k > 0$ such that

$$\mathbb{P}_{1/2}(\textit{crossing } [0, kn] \times [0, n] \textit{ from left to right}) \geq \delta_k$$

Near-critical percolation: what is known

Two main ingredients:

(1) Study of critical percolation

(2) Scaling techniques

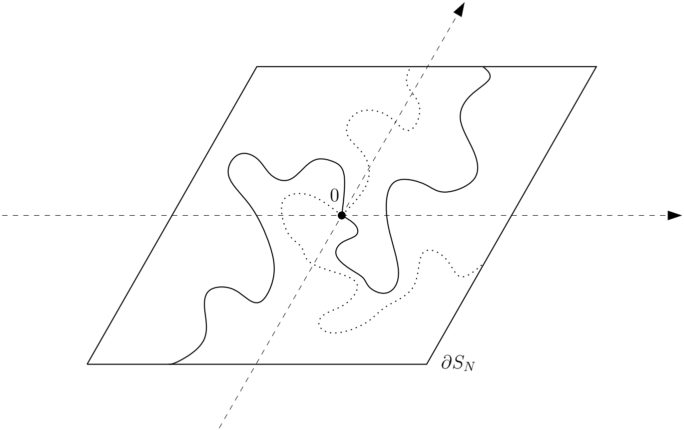
$\Rightarrow$ Description of percolation near the critical point.

1st ingredient: study of critical percolation

A precise description of critical percolation was made possible by the introduction of SLE processes in 1999 by O. Schramm, and its subsequent study by G. Lawler, O. Schramm et W. Werner.

Another important step: conformal invariance of critical percolation in the scaling limit (S. Smirnov - 2001), that allows to go from discrete to continuum.

We will use in particular the “arm-events”:



Their probability decays like a power law, described by the "j-arm exponents":

Theorem (Lawler, Schramm, Werner, Smirnov)

We have

$$\mathbb{P}_{1/2}(0 \rightsquigarrow \partial S_n) \approx n^{-5/48}$$

and for each $j \geq 2$, for every fixed (non constant) sequence of "colors" for the j arms,

$$\mathbb{P}_{1/2}(j \text{ arms } \partial S_j \rightsquigarrow \partial S_n) \approx n^{-(j^2-1)/12}$$

2nd ingredient: scaling techniques (H. Kesten)

For some $\epsilon_0 > 0$ fixed once for all (sufficiently small), we define a *characteristic length* ($\mathcal{C}_H$ denotes existence of a left-right crossing):

Definition

$$L(p, \epsilon_0) = \begin{cases} \min\{n \;\; s.t. \;\; \mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) \leq \epsilon_0\} & \textit{if } p < 1/2 \\ \min\{n \;\; s.t. \;\; \mathbb{P}_p(\mathcal{C}^*_H([0,n] \times [0,n])) \leq \epsilon_0\} & \textit{if } p > 1/2 \end{cases}$$

"It still looks like critical percolation"

For instance, the RSW theorem remains true for parallelograms of size $\leq L(p)$, and the probability to observe a path $0 \rightsquigarrow \partial S_n$ remains of the same order of magnitude.

For Gradient Percolation, we will need

Lemma

We have

$$\mathbb{P}_p(\text{a black and a white arm } 0 \rightsquigarrow \partial S_n)$$

$$\asymp \mathbb{P}_{1/2}(\text{a black and a white arm } 0 \rightsquigarrow \partial S_n)$$

uniformly in p, $n \leq L(p)$.

"We quickly become sub-critical"

We have the following lemma, showing exponential decay with respect to $L(p)$ (control of speed for variable p):

Lemma

There exists a constant $C > 0$ such that for each n, each $p < 1/2$,

$$\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) \leq Ce^{-n/L(p)}$$

This lemma implies in particular if $p > 1/2$:

$$\mathbb{P}_p[0 \rightsquigarrow \infty] \asymp \mathbb{P}_p[0 \rightsquigarrow \partial S_{L(p)}]$$

To sum up, $L(p)$ is at the same time:

- a scale on which everything looks like critical percolation.
- a scale at which connectivity properties start to change drastically.

Consequences for the characteristic functions

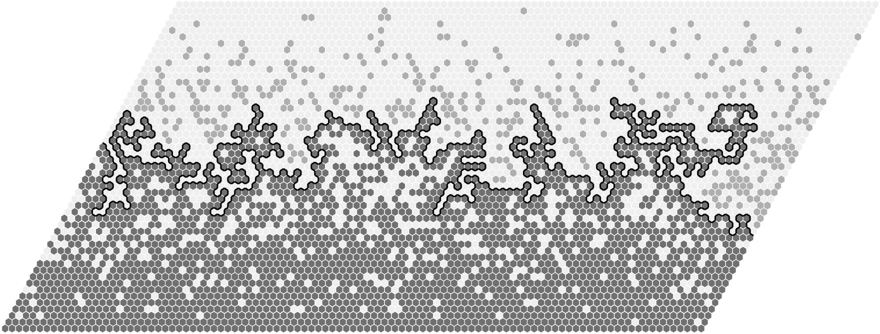
These ingredients allow to obtain the critical exponents of standard percolation, associated to the characteristic functions used to describe macroscopically the model (ξ, χ, θ ...). For Gradient Percolation, we will use only the exponent for the length L:

$$L(p) \approx |p - 1/2|^{-4/3}$$

But for example, this implies that the density $\theta(p)$ of the infinite cluster satisfies $(5/36 = (-5/48) \times (-4/3))$:

$$\theta(p) \approx (p - 1/2)^{5/36} \qquad (p \to 1/2^+)$$

Gradient Percolation

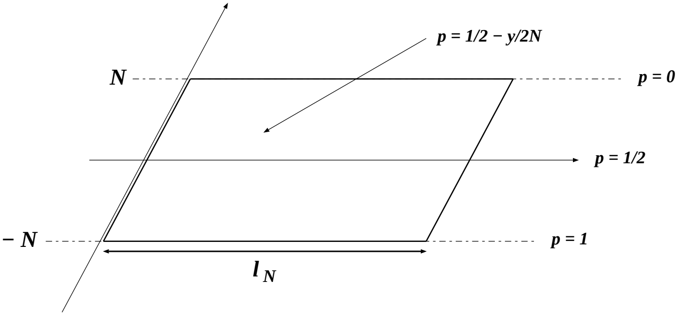


Ref: P.N., *Critical Exponents of Planar Gradient Percolation*

We consider a strip $\mathcal{S}_N = [0, \ell_N] \times [-N, N]$, of finite width $2N$, in which the percolation parameter decreases linearly in y:

$$p(z) = 1/2 - y/2N$$

We denote by $\mathbb{P}$ the associated probability measure.



With this convention, all the sites on the lower boundary are occupied ($p = 1$), all the sites on the upper boundary are vacant ($p = 0$).

$\Rightarrow$ Two different regions appear:

- At the bottom of $\mathcal{S}_N$, the parameter is close to 1, we are in a super-critical region and most occupied sites are connected to the bottom: "big" cluster of occupied sites.
- At the top of $\mathcal{S}_N$, the parameter is close to 0, we are in a sub-critical region and most vacant sites are connected to the top (by vacant paths): "big" cluster of vacant sites.

The characteristic phenomenon of this model is the appearance of a unique “front”, an interface touching simultaneously these two clusters.

Hypothesis on ℓ_N: we will assume that for two constants $\epsilon, \gamma > 0$,

$$N^{4/7+\epsilon} \leq \ell_N \leq N^{\gamma}$$

Thus $\ell_N = N$ is OK.

Heuristics

The critical behavior of this model remains localized in a "critical strip" around $p = 1/2$, a strip in which we can consider percolation as almost critical.

We get away from the critical line $p = 1/2$: the characteristic length associated to the percolation parameter decreases $\Longrightarrow$ at some point, it gets of the same order as the distance from the critical line. This distance is the width σ_N of the critical strip

$$\sigma_N = L(1/2 \pm \sigma_N/2N)$$

The vertical fluctuations of the front are of order σ_N.

The exponent for $L(p)$ implies that $\sigma_N \approx N^{4/7}$.

Heuristics

Hence, we expect:

- $\longrightarrow$ **uniqueness** of the front.
- $\longrightarrow$ **decorrelation** of points at horizontal distance $\gg \sigma_N$.
- $\longrightarrow$ **width** of the front of the order of σ_N.

Uniqueness

There exists with probability very close to 1 a unique front, that we denote by $\mathcal{F}_N$:

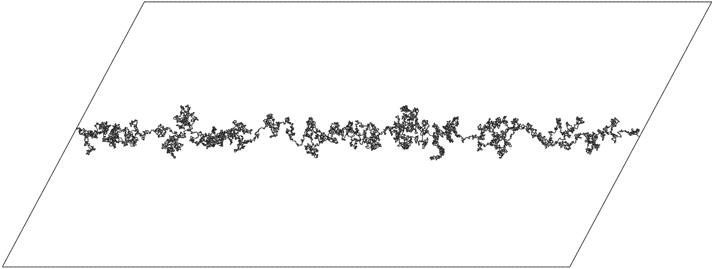
Lemma (N.)

There exists $\delta' > 0$ such that for each N sufficiently large,

$$\mathbb{P}(\textit{the boundaries of the two "big" clusters coincide}) \geq 1 - e^{-N^{\delta'}}$$

Consequence: a site x is on the front *iff* there exist two arms, one occupied to the bottom of $\mathcal{S}_N$, and one vacant to the top. Moreover, this is a **local** property (depending on a neighborhood of x of size $\approx \sigma_N$) $\Rightarrow$ **decorrelation** of points at horizontal distance $\gg \sigma_N$.

Width of the front



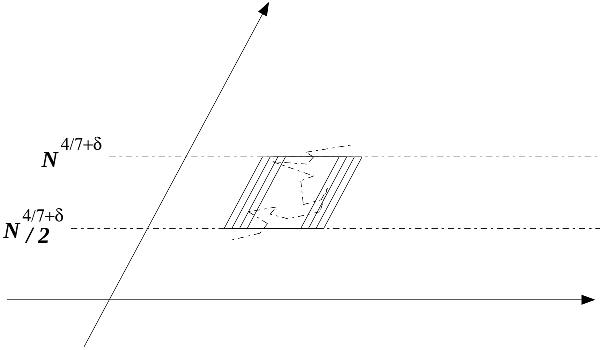
Width of the front

The scale $\sigma_N \approx N^{4/7}$ is actually the order of magnitude of the vertical fluctuations:

Theorem (N.)

- *For each $\delta > 0$, there exists $\delta' > 0$ such that for N sufficiently large,*
$$\mathbb{P}(\mathcal{F}_N \subseteq [\pm N^{4/7-\delta}]) \leq e^{-N^{\delta'}} \tag{1}$$
- *For each $\delta > 0$, there exists $\delta' > 0$ such that for N sufficiently large,*
$$\mathbb{P}(\mathcal{F}_N \nsubseteq [\pm N^{4/7+\delta}]) \leq e^{-N^{\delta'}} \tag{2}$$

Width of the front



Representative example: length of the front

To estimate a quantity related to the front:

(1) Only the edges in the critical strip must be counted.

(2) For these edges, being on the front is equivalent to the existence of two arms of length σ_N:

$$\mathbb{P}(e \in \mathcal{F}_n) \approx (\sigma_N)^{-1/4} \approx N^{-1/7}$$

(2 arm exponent: 1/4).

Representative example: length of the front

Thus, for the length T_N of the front:

Proposition (N.)

For each $\delta > 0$, we have for N sufficiently large:

$$N^{3/7-\delta}\ell_N \leq \mathbb{E}[T_N] \leq N^{3/7+\delta}\ell_N$$

For $\ell_N = N$, this gives $\mathbb{E}[T_N] \approx N^{10/7}$.

Noteworthy property: In a box of size σ_N, approximately N points are located on the front.

Variance of T_N

The decorrelation of points at horizontal distance $\gg N^{4/7}$ implies:

Theorem (N.)

If for some $\epsilon > 0$, $\ell_N \geq N^{4/7+\epsilon}$, *then*

$$\frac{T_N}{\mathbb{E}[T_N]} \longrightarrow 1 \quad \text{in } L^2\text{, when } N \to \infty$$

$\Rightarrow$ Concentration of T_N around $\mathbb{E}[T_N] \approx N^{3/7}\ell_N$

Outer boundaries of the front

We can introduce the lower and upper boundaries of the front: we denote by U_N^+ and U_N^- their respective lengths. The proof of the results on the length T_N can easily be adapted, and the 3-arm exponent (equal to $2/3$) gives:

$$U_N^\pm \approx N^{4/21} \ell_N$$

Estimating p_c

We introduce the mean height:

$$Y_N = \frac{1}{T_N} \sum_e y_e \mathbb{I}_{e \in \mathcal{F}_N}$$

and we normalize it:

$$\tilde{Y}_N = \frac{1}{2} + \frac{Y_N}{2N}$$

For symmetry reasons, $\mathbb{E}[\tilde{Y}_N] = 1/2$, and the decorrelation property implies that:

$$\mathrm{Var}(\tilde{Y}_N) \leq \frac{1}{N^{2/7-\delta}\ell_N}$$

Estimating p_c

But on other lattices, like $\mathbb{Z}^2$?
We still have $L(p) \leq |p - p_c|^{-A}$. $\Rightarrow$ The front still converges toward p_c.

The results presented here come from the exponents of standard percolation. $\Rightarrow$ For *universality* reasons, we can think that the critical exponents remain the same on other lattices, like the square lattice.

Estimating p_c

Question: Behavior of $\tilde{Y}_N$ when we lose symmetry ? We probably still have (decorrelation) if ℓ_N is sufficiently large ($\ell_N = N^2$ for example):

$$\tilde{Y}_N \approx \mathbb{E}[\tilde{Y}_N]$$

and also (localization):

$$p_c - N^{-3/7} \leq \mathbb{E}[\tilde{Y}_N] \leq p_c + N^{-3/7}$$

But we can hope for a much better bound (in $1/N$ for instance).

Quadratic deviation

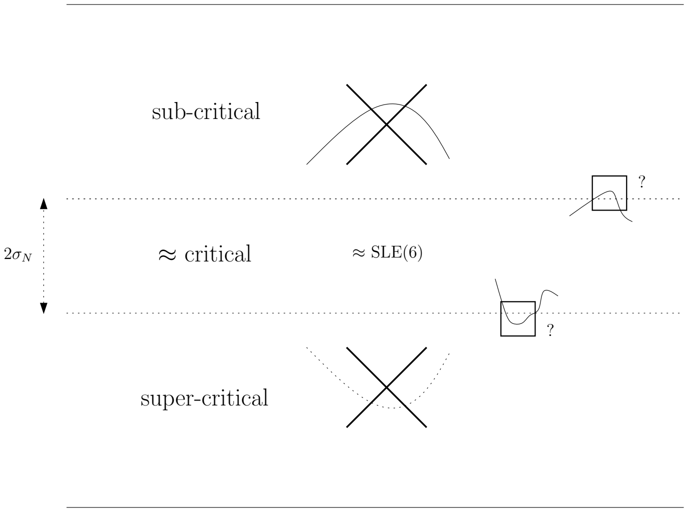
We also have

$$Y_N^{(2)} = \frac{1}{T_N} \sum_e y_e^2 \mathbb{I}_{e \in \mathcal{F}_N} \approx N^{8/7}$$

By using the decorrelation, we deduce from it that the mean quadratic deviation satisfies:

$$\sigma'_N = \sqrt{\frac{1}{T_N} \sum_e (y_e - Y_N)^2 \mathbb{I}_{e \in \mathcal{F}_N}} \approx N^{4/7}$$

Summary


sub-critical
?
$2\sigma_N$
$\approx$ critical
$\approx$ SLE(6)
?
super-critical

Discrete Asymmetry

Using standard arguments due to M. Aizenman and A. Burchard, we can show the existence of *scaling limits*, that one would like to relate to SLE(6). **But:**

Proposition

Consider a box of size σ_N centered on the line $y = -2\sigma_N$: it contains $\approx N$ sites of the front, but

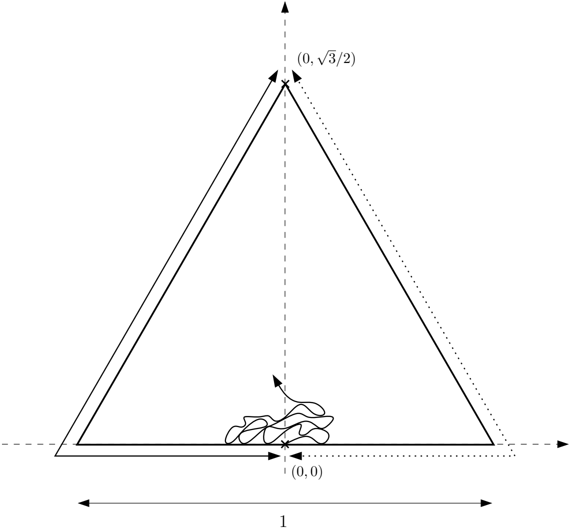
$$\#\textit{black sites} - \#\textit{white sites} \approx N^{4/7} \gg \sqrt{N}.$$

And in fact, in the scaling limit, the law of the curve will be singular with respect to SLE(6). This is related to *off-critical* percolation.

Off-critical percolation: some properties

(joint work with Wendelin Werner)

Setting



Critical case

Take mesh size $\delta_n \to 0$, T^{δ_n} a discrete approximation of this triangle T, and perform a percolation of parameter $p = 1/2$ with the specified boundary conditions.

Theorem (Smirnov, Camia-Newman)

The interface γ^{δ_n} *converges in law to a chordal SLE(6) from* $(0,0)$ *to* $(0, \sqrt{3}/2)$.

Near-critical case

Take now a parameter $p_n = p(\delta_n)$ so that the associated characteristic length is of order $1/\delta_n$ (*finite-size scaling*):

$$p(\delta_n) = \inf\{p > 1/2 \text{ s.t. } L(p) < 1/\delta_n\}$$

In this way, the crossing probabilities at macroscopic scale are different from the critical case, but do not degenerate either.

Theorem (N., Werner)

The sequence of interfaces γ^{δ_n} possesses subsequential limits, and the law of any such limit γ is singular with respect to that of SLE(6).

Remarks

If one zooms in (after taking the scaling limit), the behavior is closer and closer to critical percolation. The crossing probabilities converge to the critical ones.

Analogy: N-step random walk with drift $C/\sqrt{N}$. Note that the law of Brownian motion with drift is absolutely continuous with respect to the law of Brownian Motion.

Sketch of proof

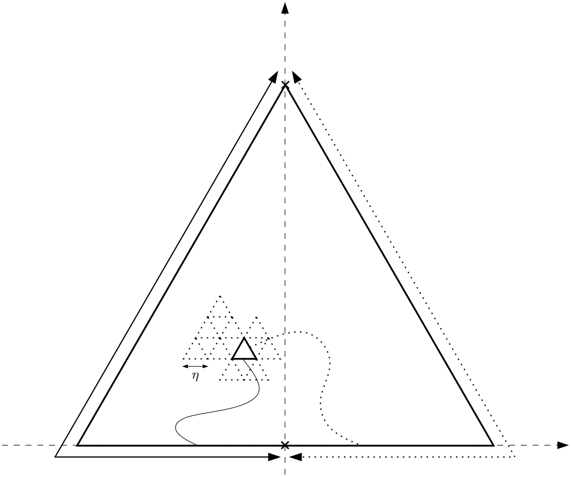
We want to detect some asymmetry in the scaling limit.

Idea: We divide T into mesoscopic triangles of size η, and count how many times the interface will turn "to its right" rather than "to its left".

There are $\approx \eta^{-7/4}$ triangles, and the probability to turn "more right than left" is $\approx 1/2 + \eta^{3/4}$. The fluctuation of order $\eta^{-7/8}$ is thus beaten by the drift η^{-1}.

In this way, we can construct an event that has a probability 0 for SLE(6), and a probability 1 for any subsequential limit γ of near-critical percolation.

Mesoscopic estimate



End

Thank you !