PCMI2008 Undergraduate Algebraic Geometry Course

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1 Using projective space

1.1 Projective space

We will work over the complex numbers $\mathbb{C}$. We work in the ring

$$\mathbb{C}\left[x_1, \ldots, x_n\right]$$

of polynomials in the variables $x_1, \ldots, x_n$.

Complex affine variety: Solution set Z of a finite system of polynomial equations: The set of $(x_1, \ldots, x_n) \in \mathbb{C}^n$ such that

$$\begin{aligned} p_0\left(x_1, \ldots, x_n\right) &= 0 \qquad (1) \\ &\ldots \\ p_r\left(x_1, \ldots, x_n\right) &= 0. \end{aligned}$$

You have seen that, even though Z is given by a finite set of equations, the set of polynomials that vanish on Z, called I_Z is quite large. In particular, it is an *ideal*, even a *radical ideal*, that is,

$$\begin{aligned} p, q \in I_Z &\Longrightarrow p + q \in I_Z \\ f \in \mathbb{C}\left[x_1, \ldots, x_n\right], p \in I_Z &\Longrightarrow f \cdot p \in I_Z \\ p^n \in I_Z &\Longrightarrow p \in I_Z. \end{aligned}$$

The problem with affine varieties is that they have 'uncontrollable ends.' The space in which we will work is complex projective space

$$\mathbb{CP}^n$$

which is the set of one-dimensional subspaces of the $\mathbb{C}$-vector space

$$V = \mathbb{C}^{n+1} = \left\{(x_0, \ldots, x_n) : x_j \in \mathbb{C}\right\}.$$

To get a picture of how this looks, make things easier by replacing $\mathbb{C}$ with the smaller field $\mathbb{R}$ and think of the set $\mathbb{RP}^2$ of one-dimensional subspaces of $\mathbb{R}^3$, that is, the set of lines through the origin in $\mathbb{R}^3$.

Exercise 1 *Draw a picture.*

Even though $\mathbb{P}^n$ doesn't fit in any affine variety, we set up coordinate charts on $\mathbb{CP}^n$ which are affine varieties.

Exercise 2 *Draw the affine charts for* $\mathbb{RP}^2$

Exercise 3 *Staying with* $\mathbb{RP}^2$ *show how it takes care of the loose ends of*

$$x_2^2 - x_1\left(x_1^2 - 1\right) = 0. \tag{2}$$

That is, graph the solution set to (2) *in the* $\mathbb{R}^2$*-plane. Find the limit(s) in* $\mathbb{RP}^2$ *when you go off to infinity along the graph. Hint: Find the point(s) at infinity on one of the other coordinate charts of* $\mathbb{RP}^2$*.*

1.2 Homogeneous coordinates

Each non-zero point $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$ determines a unique point $[x_0 : \ldots : x_n]$ in $\mathbb{CP}^n$. But two different points in $\mathbb{C}^{n+1}$ may determine the same point in $\mathbb{CP}^n$.

Exercise 4 *Exactly when is this the case? Prove your assertion.*

We call a polynomial $p(x_0, \ldots, x_n)$ *homogeneous* if its vanishing set depends only on $[x_0 : \ldots : x_n]$, that is, if the vanishing of the polynomial at $(x_0, \ldots, x_n)$ implies its vanishing at all points $(y_0, \ldots, y_n)$ such that

$$[y_0 : \ldots : y_n] = [x_0 : \ldots : x_n].$$

Exercise 5 *Show that a polynomial* $p(x_0, \ldots, x_n)$ *is homogeneous if and only if each of its monomial terms have the same total degree.*

We can take any affine variety (1) and homogenize it or 'close it up at infinity' by writing

$$p(x_1, \ldots, x_n) = p\left(\frac{y_1}{y_0}, \ldots, \frac{y_n}{y_0}\right)$$

and then multiplying $p\left(\frac{y_1}{y_0}, \ldots, \frac{y_n}{y_0}\right)$ by the smallest power of y_0 which clears denominators.

Exercise 6 *Find the 'points at infinity' by homogenizing the equation in* (2)*.*

1.3 Curves in $\mathbb{CP}^2$

Let $p(x_0, x_1, x_2)$ be a homogeneous polynomial of degree 2. We call the solution set $p = 0$ in $\mathbb{CP}^2$ a *conic*. Can write

$$p(x_0, x_1, x_2) = (x_0, x_1, x_2) \cdot A \cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

where A is a symmetric 3×3 matrix. To compare solution sets $p = 0$ we use some linear algebra.

Exercise 7 *Show that there is an invertible* 3×3 *matrix* M *such that*

$$MAM^t$$

is diagonal with entries only 0 *or* 1*.*

The group of such M is just the group of changes of coordinates or automorphisms of $\mathbb{CP}^2$, that is,

$$[y_0, \ldots, y_n] = [(x_0, \ldots, x_n) M].$$

(Similarly for $\mathbb{CP}^n$.)

We conclude that, up to change of coordinates for $\mathbb{CP}^2$, there are only three conics:

$$\begin{aligned} x_0^2 - x_1^2 - x_2^2 &= 0 \\ x_0^2 - x_1^2 &= 0 \\ x_0^2 &= 0. \end{aligned}$$

The first one is smooth, the second has one singular point (where is it?), and the third is singlar at all its points.

Exercise 8 *Show that every smooth conic has an algebraic,* $1-1$ *and onto map to* $\mathbb{CP}^1$*. Hint: Stereographically project the smooth conic*

$$x_0^2 - x_1^2 - x_2^2 = 0$$

from the point $(1, 0, 1)$ *onto the line* $x_2 = 0$*.*

However if $p(x_0, x_1, x_2)$ is homogeneous of degree d with $d \geq 3$ and $C \subseteq \mathbb{CP}^2$ is its solution set, there is, in general no way to define a non-constant differentiable map

$$f : \mathbb{CP}^1 \to C \tag{3}$$

To see why, we'll need some calculus.

1.4 Differentials

Suppose that I have a curve C' in $\mathbb{C}^2$ given by the equation

$$p(x, y) = 0.$$

Then, if I have a smooth path $(x(t), y(t))$ in $\mathbb{C}^2$ that lies entirely inside C', the Chain Rule tells me that, for the function $p(x(t), y(t))$,

$$0 = \frac{dp}{dt} = \frac{dp}{dx}\frac{dx}{dt} + \frac{dp}{dy}\frac{dy}{dt}.$$

Since this is true no matter what the parameter t is, we simply surpress it in the notation and write

$$0 = \frac{dp}{dx}dx + \frac{dp}{dy}dy.$$

We call this *implicit differentiation*.

1.5 Non-rationality of cubics

Let's show that there is no non-constant map f in (3) when the curve C given by

$$x_0 x_2^2 = x_1 (x_1 + x_0)(x_1 + \lambda x_0). \tag{4}$$

To get an idea why this is true, write this curve on the set $x_0 \neq 0$ using coordinate changes

$$\begin{aligned} x &= \frac{x_1}{x_0} \\ y &= \frac{x_2}{x_0}. \end{aligned}$$

We get

$$y^2 = x (x + 1)(x + \lambda). \tag{5}$$

Exercise 9 *Suppose $\lambda = -1 \in \mathbb{R}$ and graph the (real) solution set of this curve in $\mathbb{R}^2$.*

Next implicitly differentiate the equation (5).

$$2y dy = ((x + 1)(x + \lambda) + x (x + \lambda) + x (x + 1)) dx.$$

We see that the expression

$$\frac{dx}{y} = \frac{2dy}{(x + 1)(x + \lambda) + x (x + \lambda) + x (x + 1)}$$

is everywhere bounded in the solution set C' (5). Next we checked above that the rest of the curve C is given by

$$\begin{aligned} z &= w (w + z)(w + \lambda z) \\ \frac{z}{w} &= (w + z)(w + \lambda z) \end{aligned}$$

on the set $x_2 \neq 0$. In fact the only point of C not on C' is the point $(z, w) = (0, 0)$. But, when z and w are both small, so is $\frac{z}{w}$, and so

$$\frac{z}{w^2} = \left(1 + \frac{z}{w}\right)\left(1 + \lambda \frac{z}{w}\right)$$

is bounded away from zero and finite. So

$$z = w^2 u$$

where $u (0, 0) \neq 0$. With coordinate changes

$$\begin{aligned} z &= \frac{x_0}{x_2} \\ w &= \frac{x_1}{x_2} \end{aligned}$$

and

$$\begin{aligned}
x &= \frac{x_1}{x_0} = \frac{w}{z} \\
y &= \frac{x_2}{x_0} = \frac{1}{z}
\end{aligned}$$

and

$$\begin{aligned}
\frac{dx}{y} &= zd\frac{w}{z} \\
&= dw - \frac{wdz}{z} \\
&= dw - \frac{wd\left(w^2u\right)}{w^2u} \\
&= dw - \left(\frac{2uwdw + w^2du}{wu}\right) \\
&= -dw - w\frac{du}{u}.
\end{aligned}$$

So C has an everywhere holomorphic differential. Via the mapping f in (3) this differential would give an everywhere holomorphic differential on the Riemann sphere.

Exercise 10 *Show that there are no everywhere holomorphic differentials on $\mathbb{CP}^1$. Hint: On $x_0 \neq 0$, write the differential as*

$$g(z)\,dz.$$

Then change coordinates to the coordinate $w = \frac{1}{z}$ on the open set $x_1 \neq 0$. So

$$g\left(\frac{1}{w}\right)d\frac{1}{w}$$

has to be nice at $w = 0$. Why is this impossible?

However, if $\lambda = 0$ in (4) so that

$$x_0x_2^2 = x_1^2\left(x_1 + x_0\right).$$

we can put

$$\begin{aligned}
p_0 &= y_0^3 \\
p_1 &= y_0\left(y_1^2 - y_0^2\right) \\
p_2 &= y_1\left(y_1^2 - y_0^2\right).
\end{aligned}$$

2 Blowing up subvarieties of projective space

2.1 Blowing up zero: Making a hole in affine space at zero and sticking in a projective space (of one lower dimension)

Consider the set

$$\mathbb{C}^{n+1} \times \mathbb{CP}^n$$

with coordinates

$$((x_0, \ldots, x_n), \ [y_0 : \ldots : y_n]).$$

Consider the subset

$$B \subseteq \mathbb{C}^{n+1} \times \mathbb{CP}^n$$

given by the set of equations

$$\left\{ \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} = 0 \right\}_{0 \leq j < k \leq n}.$$

We have two projection maps

$$\begin{gathered} pr_{\mathbb{C}} : B \to \mathbb{C}^{n+1} \\ ((x_0, \ldots, x_n), \ [y_0 : \ldots : y_n]) \mapsto (x_0, \ldots, x_n) \end{gathered}$$

and

$$\begin{gathered} pr_{\mathbb{P}} : B \to \mathbb{P}^n \\ ((x_0, \ldots, x_n), \ [y_0 : \ldots : y_n]) \mapsto [y_0 : \ldots : y_n]. \end{gathered}$$

Exercise 11 *Show that the map*

$$pr_{\mathbb{C}} : pr_{\mathbb{C}}^{-1} \left(\mathbb{C}^{n+1} - \{(0, \ldots, 0)\} \right) \to \left(\mathbb{C}^{n+1} - \{(0, \ldots, 0)\} \right)$$

is $1 - 1$ *and onto.*

Exercise 12 *Show that*

$$pr_{\mathbb{C}}^{-1} (\{(0, \ldots, 0)\}) = \mathbb{CP}^n.$$

Next we will show that B is a nice, 'smooth' algebraic set. We take any of the coordinate charts for $\mathbb{CP}^n$, for example

$$\begin{aligned} U_0 &= \{[y_0 : \ldots : y_n] \in \mathbb{CP}^n : y_0 \neq 0\} \\ &= \{[1 : y_1 : \ldots : y_n] \in \mathbb{CP}^n : y_0 \neq 0\} \\ &= \{(y_1, \ldots, y_n) \in \mathbb{C}^n\}. \end{aligned}$$

Then

$$B \cap \left(\mathbb{C}^{n+1} \times U_0 \right)$$

is given by the equations

$$\left\{ \begin{vmatrix} x_0 & x_k \\ 1 & y_k \end{vmatrix} = 0 \right\}_{1 \le k \le n}$$

and

$$\left\{ \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} = 0 \right\}_{1 \le j < k \le n}.$$

But the first set of equations make each $x_k = x_0 y_k$ for $k \geq 1$ and substituting in the second set we get

$$\left\{ \begin{vmatrix} x_0 \cdot y_j & x_0 \cdot y_k \\ y_j & y_k \end{vmatrix} = 0 \right\}_{1 \le j < k \le n}$$

which is identically satisfied. Thus $B \cap \left(\mathbb{C}^{n+1} \times U_0\right)$ is simply the graph of the smooth function

$$F\left(x_0, y_1, \ldots, y_n\right) = \left(x_1, \ldots, x_n\right)$$

given by the rule $x_k = x_0 y_k$ for $k \geq 1$. The same argument works for the part of B in any of the other coordinate charts $\mathbb{C}^{n+1} \times U_k$.

Exercise 13 *Draw the (real) blow-up of* $(0,0)$ *in* $\mathbb{R}^2$.

2.2 Blowing up an ideal

What we have just done is to blow up the point $(0, \ldots, 0)$ in $\mathbb{C}^{n+1}$. Said another way, we have blown up the (zeros of) the ideal

$$I_0 = \left\{x_0, \ldots, x_n\right\}.$$

Here we consider the coordinate functions x_j as the polynomials which determines the set we are blowing up. But we can blow up any ideal I, for example the ideal generated by the polynomials in (1). Consider the set

$$\mathbb{C}^{n+1} \times \mathbb{CP}^r$$

with coordinates

$$\left(\left(x_0, \ldots, x_n\right),\ \left[y_0 : \ldots : y_r\right]\right).$$

Consider the set of equations

$$\left\{ \begin{vmatrix} p_j & p_k \\ y_j & y_k \end{vmatrix} = 0 \right\}_{0 \le j < k \le r}.$$

Clearly the subset $\left(Z \times \mathbb{CP}^r\right) \subseteq \left(\mathbb{C}^{n+1} \times \mathbb{CP}^r\right)$ is in the solution set. But there is another piece of the solution set, namely the closure B of the graph of the function

$$\left(\mathbb{C}^{n+1} - Z\right) \to \mathbb{CP}^r$$
$$\left(x_0, \ldots, x_n\right) \mapsto \left[p_0\left(x_0, \ldots, x_n\right), \ldots, p_r\left(x_0, \ldots, x_n\right)\right]$$

Again we have

$$pr_{\mathbb{C}} : B \to \mathbb{C}^{n+1}$$

which is $1-1$ and onto over $\left(\mathbb{C}^{n+1} - Z\right)$.

2.3 B independent of choice of polynomials defining Z

Suppose we tack on a useless extra equation, for example

$$q = h_0 p_0 + \ldots + h_r p_r.$$

The map

$$\mathbb{C}^{n+1} \times \mathbb{CP}^r \to \mathbb{C}^{n+1} \times \mathbb{CP}^{r+1}$$
$$\left(\left(x_0, \ldots, x_n\right), \ [y_0 : \ldots : y_{r+1}]\right) \mapsto \left(\left(x_0, \ldots, x_n\right), \ \left[y_0 : \ldots : y_r : \sum\nolimits_{i=1}^{r} h_j y_j\right]\right)$$

takes $\mathbb{C}^{n+1} \times \mathbb{CP}^r$ isomorphically onto a smooth closed subvariety of $\mathbb{C}^{n+1} \times \mathbb{CP}^{r+1}$ and takes B exactly onto the set we would get by blowing up Z using the equations

$$p_0 = \ldots = p_r = q = 0.$$

So if we have a second set of equations

$$\begin{aligned} q_0\left(x_1, \ldots, x_n\right) &= 0 \\ &\ldots \\ q_s\left(x_1, \ldots, x_n\right) &= 0. \end{aligned} \tag{6}$$

defining Z, we use this reasoning $s+1$ times to conclude that B is the same as the blow-up we get using the set of polynomials $\{p_0, \ldots, p_r, q_0, \ldots, q_s\}$.

Exercise 14 *Show that the blow-up B_I of an ideal I does not depend on the choice of generators for the ideal of Z.*

2.4 The inverse image of Z is given (locally)

A property that characterizes the blow-up B_Z of an ideal I_Z in $\mathbb{C}^{n+1}$ is that, for the mapping

$$pr_{\mathbb{C}} : B_Z \to \mathbb{C}^{n+1},$$

the set

$$pr_{\mathbb{C}}^{-1}(Z)$$

is everywhere locally defined by a single equation. (Remember that it took $n+1$ equations to define the set $\{(0, \ldots, 0)\}$ in $\mathbb{C}^{n+1}$.) To see this, restrict B_Z, for example, to the open set

$$\left(\mathbb{C}^{n+1} \times U_0\right) \cap B_Z.$$

The equations for the inverse image of Z are

$$\begin{aligned} p_0 = p_0 \circ pr_{\mathbb{C}} &= 0 \\ &\ldots \\ p_r = p_r \circ pr_{\mathbb{C}} &= 0. \end{aligned}$$

But upstairs on B_Z we also have the equations

$$\left\{ \begin{vmatrix} y_j & y_k \\ p_j & p_k \end{vmatrix} = 0 \right\}_{0 \leq j < k \leq n}.$$

So, on the open set

$$\left(\mathbb{C}^{n+1} \times U_0 \right) \cap B_Z,$$

where we have coordinates

$$(x_0, \ldots, x_n), \left(\frac{y_1}{y_0}, \ldots, \frac{y_r}{y_0} \right)$$

since $y_0 \neq 0$, we get

$$p_k = \frac{y_k}{y_0} p_0.$$

So the ideal of

$$pr_{\mathbb{C}}^{-1}(Z)$$

is given by the single equation

$$p_0 \circ pr_{\mathbb{C}} = 0.$$

We call such an ideal a *principal* ideal. There is a little problem with this. We may start with Z given by a radical ideal, I_Z, but the generator p_0 could happen to be of the form

$$p_0 = p^2 q.$$

Then on B_Z,

$$(pq)^2$$

is in the ideal of $p_0 \circ pr_{\mathbb{C}}$ but

$$pq$$

isn't. This, among other things, makes working with only radical ideals a bad idea. So algebraic geometers instead replace the notion of a variety with the notion of a *scheme*.

The ideal $\left\{ p^2 q \right\}$ gives a scheme in B_Z. This scheme is 'bigger' than

$$pr_{\mathbb{C}}^{-1}(Z) = \{zeros\,(p)\} \cup \{zeros\,(q)\}.$$

Intuitively, it has the zeros of p in it twice, much in the same way that you count the root 1 twice when accounting for the roots of the equation

$$x^3 - 2x^2 + x = 0.$$

3 What can you do with a blow-up?

3.1 Smoothing

Now suppose I have a subvariety

$$W \subseteq \mathbb{C}^{n+1}$$

that has some 'bad' or singular subset Z. Then we blow up Z in $\mathbb{C}^{n+1}$ and look at what happens to W, that is, we lift

$$W - Z$$

into B_Z and close it up. The resulting variety $\tilde{W}$ sits over W,

$$pr : \tilde{W} \to W,$$

in fact is identical to W except over the bad subset Z:

$$\tilde{W} - pr^{-1}Z = W - Z.$$

But often $\tilde{W}$ straightens out the badness of Z. Let's see some examples

3.2 Smoothing a node

Node: The curve

$$W : x_0^2 = x_1^2 + x_1^3$$

in $\mathbb{C}^2$.

Exercise 15 *Graph the real points of this curve in* $\mathbb{R}^2$.

Now blow up $(0,0)$ in $\mathbb{C}^2$ and restrict our attention to

$$\mathbb{C}^2 \times U_0$$

with coordinates (x_0, y_1) where $x_1 = x_0 y_1$. Substituting we get

$$\begin{gathered} x_0^2 = x_0^2 y_1^2 + x_0^3 y_1^3 \\ x_0^2 - \left(x_0^2 y_1^2 + x_0^3 y_1^3\right) = 0 \\ x_0^2 \left(1 - y_1^2 \left(1 + x_0 y_1\right)\right) = 0. \end{gathered}$$

If $x_0 \neq 0$, that is, if we are over $\mathbb{C}^2 - \{(0,0)\}$, we have the curve given by

$$1 - y_1^2 \left(1 + x_0 y_1\right) = 0.$$

This curve meets the set $x_0 = 0$ in the points $(x_0, y_1) = (0, 1)$ and $(x_0, y_1) = (0, -1)$. and is smooth!

3.3 Smoothing a cusp

Cusp: The curve

$$W : x_0^2 = x_1^3$$

in $\mathbb{C}^2$.

Exercise 16 *Graph the real points of this curve in* $\mathbb{R}^2$*.*

Now blow up $(0,0)$ in $\mathbb{C}^2$ and restrict our attention to

$$\mathbb{C}^2 \times U_0$$

with coordinates (x_0, y_1) where $x_1 = x_0 y_1$. As above get

$$1 - y_1^2 (x_0 y_1) = 0$$

with no points on $x_0 = 0$. So let's try

$$\mathbb{C}^2 \times U_1$$

with coordiates

$$(x_1, y_0)$$

where $x_0 = x_1 y_0$. We get

$$\begin{aligned} (x_1 y_0)^2 &= x_1^3 \\ x_1^2 \left(y_0^2 - x_1\right) &= 0 \end{aligned}$$

So now we get

$$y_0^2 - x_1$$

with only the point $(x_1, y_0) = (0,0)$ when $x_1 = 0$. (If we were in the Graduate Summer School we would have to blow this point up, then blow up one more time to get a *simple normal-crossing divisor*.)

3.4 Taming the PCMI tee shirt

PCMI tee shirt: The surface W given by

$$W : p(x_0, x_1, x_2) = \left(x_0^2 - x_1^3\right)^2 - \left(x_0 + x_1^2\right) x_2^3 = 0$$

in $\mathbb{C}^3$. To find the singular set Z of W, find the solution set Z of the system of equations

$$\begin{aligned} \frac{\partial p}{\partial x_0} &= 4x_0 \left(x_0^2 - x_1^3\right) - x_2^3 \\ \frac{\partial p}{\partial x_1} &= 6x_1^2 \left(x_0^2 - x_1^3\right) - 2x_1 x_2^3 \\ \frac{\partial p}{\partial x_2} &= -3 \left(x_0 + x_1^2\right) x_2^2 \end{aligned}$$

that lie on W. Some algebra gives the equations

$$\begin{aligned} x_0^2 - x_1^3 &= 0 \\ x_2 &= 0 \end{aligned}$$

for Z.

Exercise 17 *Check this assertion about the equations for Z.*

So Z is our friend, the plane curve cusp singularity. Now blow up $\mathbb{C}^3$ at 0. What happens to the PCMI tee shirt? The equations for Z are

$$\begin{aligned} p_0 (x_0, x_1, x_2) &= x_0^2 - x_1^3 \\ p_1 (x_0, x_1, x_2) &= x_2 \end{aligned}$$

so the equation for B_Z is

$$\left\{ \begin{vmatrix} y_0 & y_1 \\ p_0 & p_1 \end{vmatrix} = 0 \right\}. \tag{7}$$

That is, the equation of B_Z is

$$y_0 x_2 - y_1 \left(x_0^2 - x_1^3\right) = 0.$$

Now remember that $Z \subseteq W$. So $pr_{\mathbb{C}}^{-1}(W)$ is given by writing

$$\left(x_0^2 - x_1^3\right)^2 - \left(x_0 + x_1^2\right) x_2^3 = 0$$

as

$$p_0^2 - \left(x_0 + x_1^2\right) p_1^3 = 0. \tag{8}$$

But on $y_0 \neq 0$ we have

$$p_1 = \frac{p_0 y_1}{y_0}$$

so we can rewrite this as

$$\begin{aligned} y_0^3 p_0^2 - \left(x_0 + x_1^2\right) p_0^3 y_1^3 &= 0 \\ p_0^2 \left(y_0^3 - \left(x_0 + x_1^2\right) p_0 y_1^3\right) &= 0. \end{aligned}$$

So putting $y_0 = 1$, $\tilde{W}$ is given inside B_Z by

$$1 - \left(x_0 + x_1^2\right) p_0 y_1^3 = 0. \tag{9}$$

How is $\tilde{W}$ different from W? We need only over the set Z that we blew up. Namely look at the intersection

$$\tilde{W} \cap \left(Z \times \mathbb{CP}^1\right).$$

That is

$$p_0 = p_1 = 0.$$

So, substituting in (9) , we get

$$\begin{aligned} p_0 &= p_1 = 0 \\ y_0^3 &= 0. \end{aligned}$$

This set has no points in $\mathbb{C}^3 \times U_0$ because $1 \neq 0$, but it has a lot of points in $\mathbb{C}^3 \times U_1$. But, from (8) and (9), the equations for $\tilde{W} \cap \left(\mathbb{C}^3 \times U_1\right)$ are calculated using

$$\begin{gathered} p_0 = p_1 y_0 \\ (p_1 y_0)^2 - \left(x_0 + x_1^2\right) p_1^3, \end{gathered}$$

that is,

$$\begin{gathered} p_0 = p_1 y_0 \\ p_1^2 \left(y_0^2 - \left(x_0 + x_1^2\right) p_1\right) \end{gathered}$$

so that $\tilde{W} \cap \left(\mathbb{C}^3 \times U_1\right)$ becomes

$$\begin{gathered} q_0\left(x_0, x_1, x_2, y_0\right) = \left(x_0^2 - x_1^3\right) - x_2 y_0 = 0 \\ q_1\left(x_0, x_1, x_2, y_0\right) = y_0^2 - \left(x_0 + x_1^2\right) p_1 = 0. \end{gathered}$$

Exercise 18 *Check my algebra.*

Have we smoothed the tee shirt? Unfortunately not! A point $z := (x_0, x_1, x_2, y_1) \in \tilde{W}$ is a smooth point if

$$Rank \begin{pmatrix} \left.\frac{\partial q_0}{\partial x_0}\right|_z & \left.\frac{\partial q_0}{\partial x_1}\right|_z & \left.\frac{\partial q_0}{\partial x_2}\right|_z & \left.\frac{\partial q_0}{\partial y_1}\right|_z \\ \left.\frac{\partial q_1}{\partial x_0}\right|_z & \left.\frac{\partial q_1}{\partial x_1}\right|_z & \left.\frac{\partial q_1}{\partial x_2}\right|_z & \left.\frac{\partial q_1}{\partial y_1}\right|_z \end{pmatrix} = 2.$$

Exercise 19 *Show that* $(0, 0, 0, 0)$ *is a singular point of* $\tilde{W}$*.*

Exercise 20 *(Maybe quite hard to do by hand.) See whether there are any other singular points on* $\tilde{W}$*. (I don't know the answer.)*

4 Cubic plane curves

4.1 Criterion for smooth points

Let

$$C \subseteq \mathbb{CP}^2$$

be a projective plane curve, that is, the solution set of one homogeneous form $F(x_0, x_1, x_2)$ of degree d.

Exercise 21 *Show that*

$$d \cdot F = x_0 \frac{\partial F}{\partial x_0} + x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2}.$$

Exercise 22 *Show that a point* $(x_0, x_1, x_2) \in \mathbb{CP}^2$ *is a singular point of the curve* C *if and only if*

$$\frac{\partial F}{\partial x_0} = \frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial x_2} = 0.$$

Now suppose that C is a smooth curve. Fix a point $p \in C$. Consider the line

$$\frac{\partial F}{\partial x_0}(p) \cdot x_0 + \frac{\partial F}{\partial x_1}(p) \cdot x_1 + \frac{\partial F}{\partial x_2}(p) \cdot x_2. \tag{10}$$

This line certainly passes through p.

Exercise 23 *Show that the above line is the tangent line to* C *at* p.

By this last exercise, the intersection of the line (10) and C is $2p$ plus $d-2$ points on (10).

4.2 Inflection points

Now consider the map

$$\begin{array}{rcl} C & \to & \left\{ \textit{lines in } \mathbb{CP}^2 \right\} \cong \mathbb{CP}^2 \\ p & \mapsto & \left(\frac{\partial F}{\partial x_0}(p), \frac{\partial F}{\partial x_1}(p), \frac{\partial F}{\partial x_2}(p) \right). \end{array}$$

p is called an *inflection point* of C if

$$\begin{vmatrix} \frac{\partial^2 F}{\partial x_0^2}(p) & \frac{\partial^2 F}{\partial x_0 \partial x_1}(p) & \frac{\partial^2 F}{\partial x_0 \partial x_2}(p) \\ \frac{\partial^2 F}{\partial x_1 \partial x_0}(p) & \frac{\partial^2 F}{\partial x_1^2}(p) & \frac{\partial^2 F}{\partial x_1 \partial x_2}(p) \\ \frac{\partial^2 F}{\partial x_2 \partial x_0}(p) & \frac{\partial^2 F}{\partial x_2 \partial x_1}(p) & \frac{\partial^2 F}{\partial x_2^2}(p) \end{vmatrix} = 0. \tag{11}$$

Exercise 24 *Show that* p *is an inflection point of* C *if and only if the intersection of the line* (10) *and* C *is* $3p$ *plus* $d-3$ *points on* (10).

For the rest of this section we will restrict our attention to plane cubics, that is, to the case $d=3$. Then the matrix (11) of second partials is a matrix of linear forms, so its determinant $G(x_0,x_1,x_2)$ is a homogeneous form of degree 3. Since F and G both have degree 3, the system of equations

$$\begin{array}{rcl} F(x_0,x_1,x_2) & = & 0 \\ G(x_0,x_1,x_2) & = & 0 \end{array}$$

has $3\cdot 3-9$ solutions. Said otherwise, there are 9 inflection points on C. Said still another way, there are 9 points p on C such that, if $L(x_0,x_1,x_2)$ is the tangent line to C at p, then the solution set of the system of equations

$$\begin{array}{rcl} F(x_0,x_1,x_2) & = & 0 \\ L(x_0,x_1,x_2) & = & 0 \end{array}$$

is just the point p counted 3 times.

4.3 Weierstrass normal form for a cubic

Pick one of these inflection points, call it p_∞ and call its tangent line L_∞. Using an invertible 3×3 matrix M, change coordinates

$$(x_0,x_1,x_2)=((y_0,y_1,y_2)\,M)$$

so that

$$(x_0(p_\infty),x_1(p_\infty),x_2(p_\infty))=((0,0,1)\,M)$$

and

$$\begin{array}{rcl} L_\infty(x_0,x_1,x_2) & = & L_\infty((y_0,y_1,y_2)\,M) \\ & = & y_0. \end{array}$$

Then

$$\begin{array}{rcl} F(x_0,x_1,x_2) & = & F((y_0,y_1,y_2)\,M) \\ & = & \hat{F}(y_0,y_1,y_2) \end{array}$$

gives a curve $\hat{C}$ in the $\mathbb{CP}^2$ with coordinates $[y_0,y_1,y_2]$. $\hat{C}$ is isomorphic to C so we might as well study $\hat{C}$.

But this means that in the $(y_0,y_1,1)$-plane, $\hat{C}$ is given by a curve which has the form

$$dy_1^3+y_0(\ldots)$$

and so can be written in homogeneous form as

$$ay_0y_2^2+y_2b(y_0,y_1)+c(y_0,y_1)=0. \tag{12}$$

This equation in y_2 has multiple roots when

$$b(y_0,y_1)^2-4ay_0c(y_0,y_1)=0 \tag{13}$$

Let

$$y_0 = d, y_1 = -c$$

be one of those multiply roots. This means that the line

$$cy_0 + dy_1 = 0$$

which passes through $(0,0,1)$ hits $\hat{C}$ at only one other point q and is tangent to $\hat{C}$ there. One place where this happens is when $d = 0$ and $c \neq 0$. Therefore

$$b\,(y_0, y_1)$$

has no y_1^2-term, that is

$$b\,(y_0, y_1) = y_0\,(b' y_0 + b'' y_1)\,,$$

so we rewrite (13) as

$$y_0^2\,(b' y_0 + b'' y_1)^2 - 4a y_0 c\,(y_0, y_1) = 0$$

The four solutions to (13) must be distinct, since otherwise $\hat{C}$ would be singular. So the other three solutions are given by

$$y_0\,(b' y_0 + b'' y_1)^2 - 4ac\,(y_0, y_1) = 0 \tag{14}$$

Take two other solutions and call q' and q'' the points at which they are tangent to $\hat{C}$.

Now write a new change of coordinates

$$(y_0, y_1, y_2) = (z_0, z_1, z_2) \left(\begin{array}{ccc} c_{00} & c_{01} & c_{02} \\ 0 & c_{11} & c_{12} \\ 0 & 0 & c_{22} \end{array} \right)$$

such that $(z_0\,(q'), z_1\,(q'), z_2\,(q')) = (1,0,0)$ and $(z_0\,(q''), z_1\,(q''), z_2\,(q'')) = (1,1,0)$.

Exercise 25 *a) Show that any transformation*

$$\left(\begin{array}{ccc} c_{00} & c_{01} & c_{02} \\ 0 & c_{11} & c_{12} \\ 0 & 0 & c_{22} \end{array} \right) \tag{15}$$

doesn't move the point $(0,0,1)$ *or the line* $y_0 = 0$ *and so is an affine transformation*

$$\begin{array}{rcl} y_1 & = & \dfrac{c_{11} z_1 + c_{01}}{c_{00}} \\ y_2 & = & \dfrac{c_{22} z_2 + c_{12} z_1 + c_{02}}{c_{00}} \end{array}$$

from the $(1, z_1, z_2)$ *to the* $(1, y_1, y_2)$*-plane. Show that there always is such an invertible matrix* (15) *that does what we want.*

Now we rewrite(12)as

$$a z_0 z_2^2 + z_2 b\left(z_0, z_1\right) + c\left(z_0, z_1\right) = 0 \tag{16}$$

(with different a, b, c) and (14) as

$$z_0\left(b' z_0 + b'' z_1\right)^2 - 4ac\left(z_0, z_1\right) = 0 \tag{17}$$

(with different b', b'') and we have that two of the three factors of this cubic are

$$\begin{aligned} z_1 &= 0 \\ z_1 - z_0 &= 0. \end{aligned}$$

That is, for some $\lambda \neq 0, 1$,

$$z_0\left(b' z_0 + b'' z_1\right)^2 - 4ac\left(z_0, z_1\right) = e z_1\left(z_1 - z_0\right)\left(z_1 - \lambda z_0\right). \tag{18}$$

So z_0 does not divide $c\left(z_0, z_1\right)$. On the other hand, if we set $z_0 = 1$, $z_2 = 0$ in (17),

$$\begin{aligned} \frac{z_1}{z_0} &= 0 \\ \frac{z_1}{z_0} &= 1 \end{aligned}$$

are still solutions. So

$$c\left(z_0, z_1\right) = z_1\left(\left(z_1 - z_0\right)\left(r z_0 + s z_1\right)\right).$$

Substituting this in (18) we get that $z_1\left(z_1 - z_0\right)$ divides $\left(b' z_0 + b'' z_1\right)^2$ which is impossible unless $\left(b' z_0 + b'' z_1\right) = 0$. Thus (16) becomes

$$a z_0 z_2^2 + c\left(z_0, z_1\right) = 0.$$

where $c\left(z_0, z_1\right)$ is a cubic not divisible by z_0. Then by (18) we write (16) as

$$\frac{4a^2}{e} z_0 z_2^2 = z_1\left(z_1 - z_0\right)\left(z_1 - \lambda z_0\right)$$

or

$$z_0\left(\frac{2a}{\sqrt[2]{e}} z_2\right)^2 = z_1\left(z_1 - z_0\right)\left(z_1 - \lambda z_0\right).$$

An easy change of coordinates lets us rewrite this in the plane $z_0 = 1$ $z_1 = x$, $z_2 = y$ as

$$y^2 = x\left(x - 1\right)\left(x - \lambda\right). \tag{19}$$

This shows us that there are only a one-parameter family of cubics which are in any way different from each other. In fact, the “moduli space of plane cubic curves" has dimension exactly one.

Exercise 26 *Change variables in* (19) *by replacing* x *by* $x - c$ *and leaving* y *as it is. Pick* c *such that the new equation for our curve becomes*

$$y^2 = x^3 + a(\lambda)\,x + b(\lambda)\,.$$

Exercise 27 *Let* $f(x)$ *be a polynomial of one complex variable with no multiple roots. Homogenize the affine curve*

$$y^2 = f(x)$$

Does the resulting curve have singularities? If so resolve them by blowing up. (The curves you get this way are called hyperelliptic curves.)