SOLUTION SET FOR ADVANCED LECTURES, WEEK 2

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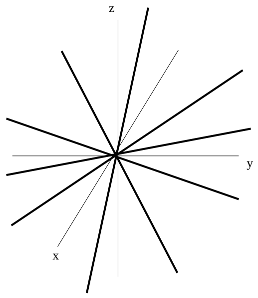
This is a solution set to the exercises given by Prof. Herb Clemens for the Advanced Undergraduate Course in week 2 of the 2008 PCMI summer math program. I neither claim that the following solutions are perfect nor that they are correct. Many solutions are quite terse, so one should expect to have to do some minor computations, most of which can be done without pen and paper. Any corrections and/or better solutions can be sent to either Brian Mann (brmann "at" umich "dot" edu) or Prof. Herb Clemens.

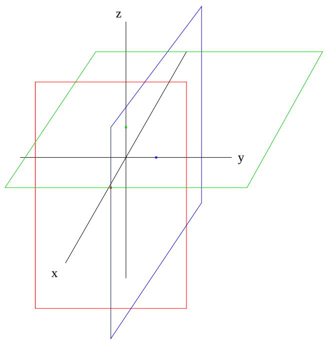
1. Using Projective Space 6.14.08

Exercise 1. Draw a picture of $\mathbf{RP}^2$.

The set of one-dimensional subspaces of $\mathbf{R}^3$, i.e. $\mathbf{RP}^2$:

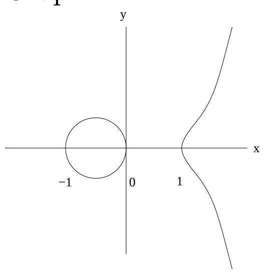


Exercise 2. Draw the affine charts for $\mathbf{RP}^2$.



Exercise 3. Graph the solution set of $y^2 - x_1^3 + x = 0$ in the $\mathbf{R}^2$ plane. Find the limits (in $\mathbf{RP}^2$) when we go off to infinity along the graph.

Graph:



To find the points at ∞ homogenize and look at another coordinate chart. See solution to exercise 6 for details.

Exercise 4. Exactly when do two points in $\mathbf{C}^{n+1}$ define the same point in $\mathbf{CP}^n$?

Proposition 1. *Two distinct (non-zero) points p_1 and p_2 in $\mathbf{C}^{n+1}$ determine the same point in $\mathbf{P}^n$ exactly when $p_1 = \lambda p_2$ for $\lambda \in \mathbf{C}^*$.*

Proof. Two points are scalar multiples if and only if they are on the same line. This gives the proposition. □

Exercise 5. Show that a polynomial is homogeneous iff each of its monomial terms have the same total degree.

Suppose the polynomial $p(x_0, \ldots, x_n)$ is homogeneous. Since $p = \sum_{i=1}^{k} m_i$ of monomials m_i, with the $\deg m_i = d_i$, we have for any non-zero scalar $c \in \mathbf{C}$ that $\sum_{i=1}^{k} c^{d_i} m_i$ and $\sum_{i=1}^{k} m_i$ vanish identically, so in particular, all the d_i must be equal.

Now suppose, with notation as in the first part, that all the m_i have the same total degree. Then we have that $p(cx_0, \ldots, cx_n) = c^d p(x_0, \ldots, x_n)$ for all $c \in \mathbf{C}^*$, where d is the degree of each monomial. So the polynomials vanish on the same set.

Exercise 6. Find the points at infinity in the equation in Ex. 3 by homogenizing.

Homogenizing the equation in exercise gives: $y^2 z - x^3 + xz^2 = 0$, letting the corresponding algebraic set intersect the $z = 0$ plane will give us our "point at ∞". We get $0 = x^3$, so the point at infinity is precisely the homogeneous 3-tuple $[0 : 0 : 1]$.

Exercise 7. Show that there is an invertible 3×3 matrix M such that MAM^t is diangonal with entries only 0 or 1.

As A is the matrix of a symmetric bilinear form defined by p, by the Principle Axis Theorem, we can conjuagate by a change of basis matrix, M, to diagonalize A. Since $\mathbf{C} = \bar{\mathbf{C}}$, we can construct this M to scale all non-zero entries to 1.

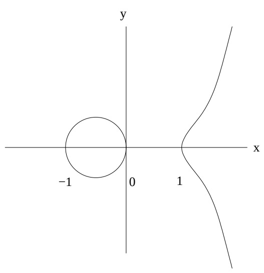
There is definitely a more elementary way of doing this.

Exercise 8. Show that every smooth conic has an algebraic, injective, and surjective map to $\mathbf{P}^1$.

Will we show instead that there is an bijective map in the other direction.

To show this consider the de-homogenizing C in the open set U_0. Then we can parametrize C exactly like the circle by considering the "slice" at $x_0 = 1$. We get the map $\mathbf{P}^1 \to C$ defined by $(t, 1) \mapsto (1, \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}) = (1+t^2, 1-t^2, 2t)$ (notice that homogenizing gives us a map defined on all of $\mathbf{P}^1$), and sends the point at infinity, $(1, 0) \mapsto [1 : 0 : 1]$. This map is clearly injective, so it remains to show that it is a surjection. However, the parametrization we used is surjective on the circle minus a point, so when we homogenize, it remains surjective on $C - [1 : 0 : 1]$.

Exercise 9. Graph the solution set of $y^2 = x(x + 1)(x - 1)$ in $\mathbf{R}^2$.



Exercise 10. Show that there are no everywhere holomorphic differentials on $\mathbf{P}^1$.

Suppose the $g(z)dz$ is an everywhere holomorphic differential on $\mathbf{P}^1$ on the open set defined by $x_0 \neq 0$, where $z = x_1/x_0$. On the overlap with the open defined by $x_1 \neq 0$ we can change coordinates so $w = x_0/x_1$, so we have $z = \frac{1}{w}$. Then we have $g(z)dz = g(1/w)d(1/w) = g(1/w)(-1/w^2)dw$. Since g is holomorphic, $g(z) = \sum a_i z^i = \sum a_i(1/w^i)$, which is not defined for $w = 0$. But our assumption was that g was everywhere holomorphic. Thus there exist no everywhere holomorphic differentials on $\mathbf{P}^1$.

2. Blowing Up Subvarieties of Projective Space 6.15.08

Exercise 11. Show that the map $pr_{\mathbf{C}} : B \to (\mathbf{C}^{n+1} - \mathbf{0})$ is injective and surjective.

To show that the map is a surjection, note that for any point $(a_0, \ldots, a_n) \in \mathbf{C}^{n+1} - \mathbf{0}$, a preimage is the point $(a_0, \ldots, a_n, a_0, \ldots, a_n) \in \mathbf{C}^{n+1} \times \mathbf{P}^n$.

For injectivity, note that for a point $(a_0, \ldots, a_n) \in \mathbf{C}^{n+1} - \mathbf{0}$ since the defining equations for the B are $x_i y_j = x_j y_i$, each y_j is uniquely determined. Indeed, for $a_i \neq 0$, we have $y_j = \frac{a_j}{a_i} y_i$, so setting $y_i = a_i$ gives $(y_0, \ldots, y_n) = (a_0, \ldots, a_n)$. Thus the map is injective.

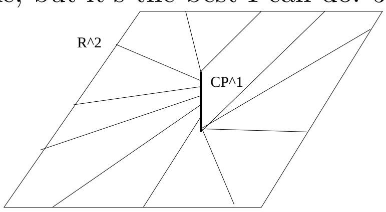
In fact, this map is an isomorphism. It is a morphism, since it is just projection onto the first $n+1$ coordinates. The inverse morphism is just given by $(a_0, \ldots, a_n) \mapsto (a_0, \ldots, a_n, a_0, \ldots, a_n)$.

Exercise 12. Show that $pr_{\mathbf{C}}^{-1}(\mathbf{0}) = \mathbf{P}^n$.

The defining equations for B are all 0 as polynomials for the point $\mathbf{0}$. So they provide no vanishing condition. Thus the fiber over $\mathbf{0}$ is all of $\mathbf{P}^1$.

Exercise 13. Draw the real blow-up of (0,0) in $\mathbf{R}^2$.

This is actually nothing like what the blow up of $\mathbf{R}^2$ at the origin looks like, but it's the best I can do. Just a warning...

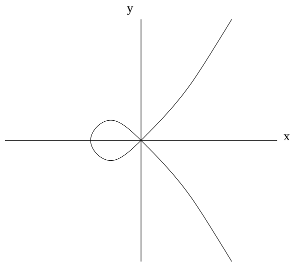


Exercise 14. Show that the blow-up of an ideal does not depend on the choice of generators for that ideal.

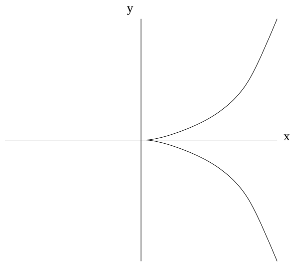
Suppose $I = (f_1, \ldots, f_r)$. Let $q_1, \ldots, q_s$ be another generating set for I. Since each q_i is in I, it can be written as a linear combination of the f_i. Applying the previous discussion in the handout s times, we have that the blow-up of the zeroes of $(f_1, \ldots, f_r)$ is the same as the blow-up of the zeroes of $(f_1, \ldots, f_r, q_1, \ldots, q_s)$. Since the q_i also generate the ideal, we can repeat the same process, using the q_i in place of the f_i, giving that the blow-up of the zeroes of $(f_i) =$ blow-up of the zeroes of (g_j).

3. What Can You Do With a Blow-up? 6.16.08 amd 6.17.08

Exercise 15. Graph the real points of the curve $y^2 = x^2 + x^3$ in $\mathbf{R}^2$.



Exercise 16. Graph the real points of the curve $y^2 = x^3$ in $\mathbf{R}^2$.



Exercise 17. Check the above assertion about the equations for Z.

The conditions $x_0^2 - x_1^3 = 0$ and $x_2 = 0$ are clearly sufficient for the above equations to hold. To show these are necessary conditions, we must solve the above equations.

Exercise 18. Check the above algebra.

More or less we did this in class while Prof. Clemens was presenting the material. If you weren't there, it's just high school algebra.

Exercise 19. Show that $(0,0,0,0)$ is a singular point of $\tilde{W}$.

We have $q_0(x,y,z,u) = (x^2 - y^3) - zu = 0$ and $q_1(x,y,z,u) = u(u^2 - (x + y^2)z) = 0$. So the matrix of partials at a point (x_0, y_0, z_0, u_0) is

$$\begin{bmatrix} 2x & -3y & -u & -z \\ -zu & -2yzu & -xu - y^2u & 3u^2 - xz - y^2z \end{bmatrix}$$

evaluated at (x_0, y_0, z_0, u_0). When $(x_0, y_0, z_0, u_0) = (0,0,0,0)$, each of the terms evaluates to 0, so the rank of the above matrix is 0. Thus $(0,0,0,0)$ is a singular point of $\tilde{W}$.

Exercise 20. See whether there are any other singular points on $\tilde{W}$.

See solution from Stefan Sabo and David Perkinson on the website.

4. Cubic Plane Curves 6.17.08 and 6.18.08

Exercise 21. Show that

$$d \cdot F = x_0 \frac{\partial F}{\partial x_0} + x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2}$$

Proof. For $F(x_0, x_1, x_2) = \sum_i a_i x_0^{d_{0,i}} x_1^{d_{1,i}} x_2^{d_{2,i}}$ we have by some simple computation with derivatives:

$$x_0 \tfrac{\partial F}{\partial x_0} + x_1 \tfrac{\partial F}{\partial x_1} + x_2 \tfrac{\partial F}{\partial x_2} = \sum_i x_0 x_1 x_2 \left(\sum_{j \in \{0,1,2\}} d_{j,i} \right) a_i x_0^{d_{0,i}-1} x_1^{d_{1,i}-1} x_2^{d_{2,i}-1}$$

and since F is homogeneous,

$$\sum_{j \in \{0,1,2\}} d_{j,i} = d$$

So $x_0 \frac{\partial F}{\partial x_0} + x_1 \frac{\partial F}{\partial x_1} + x_2 \frac{\partial F}{\partial x_2} = d \cdot F$. □

Exercise 22. Show that a point $(x, y, z) \in \mathbf{P}^2$ is a singular point of the curve C iff

$$0 = \frac{\partial F}{\partial x_0}|_x = \frac{\partial F}{\partial x_1}|_y = \frac{\partial F}{\partial x_2}|_z$$

Proof. A point $[x_0, x_1, x_2]$ of a curve $F = 0$ in $\mathbf{P}^2$ is singular iff the points (x_0, x_1, x_2) of the ray above that point are singular for the surface in $\mathbf{C}^3$ given by $F = 0$. Since the Jacobian matrix is:

$$J = \begin{bmatrix} \frac{\partial F}{\partial x_0} & \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} \end{bmatrix}$$

the rank of J at (x, y, z) is < 1 iff $0 = \frac{\partial F}{\partial x_0}|_x = \frac{\partial F}{\partial x_1}|_y = \frac{\partial F}{\partial x_2}|_z$. So the proposition is proved. □

Exercise 23. Show that the line $\frac{\partial F}{\partial x_0}(p)x_0 + \frac{\partial F}{\partial x_1}(p)x_1 + \frac{\partial F}{\partial x_2}(p)x_2 = 0$ is tangent to C at p.

Suppose that $p(t) = (x_0(t), x_1(t), x_2(t))$ is a holomorphic path on C. Then we have $F(p(t)) = 0$. By the chain rule we have

$$\frac{\partial F}{\partial x_0}\frac{\partial x_0}{\partial t} + \frac{\partial F}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial F}{\partial x_2}\frac{\partial x_2}{\partial t} = 0 \tag{1}$$

and so the tangent plane to C in $\mathbf{C}^3$ is

$$\frac{\partial F}{\partial x_0}(p)(x_0 - x_0(p)) + \frac{\partial F}{\partial x_1}(p)(x_1 - x_1(p)) + \frac{\partial F}{\partial x_2}(p)(x_2 - x_2(p)) = 0 \tag{2}$$

which, by Ex. 21 reduces to

$$\frac{\partial F}{\partial x_0}(p)x_0 + \frac{\partial F}{\partial x_1}(p)x_1 + \frac{\partial F}{\partial x_2}(p)x_2 = 0 \tag{3}$$

Thus, the above line in equation (3) is the tangent line in $\mathbf{P}^2$ to C.

Exercise 24. Show that p is an inflection point of C if and only if the intersection of the line (9) and C is $3p$ plus $d-3$ points on (9).

First we show that p is an inflection point if and only if the tangent line intersects C at p with multiplicity at least 3. We homogenize on the chart $y_2 \neq 0$, and make a linear change of coordinates moving the tangent line to the x-axis and p to $(0,0,1)$. Since F is homogeneous and $F(0,0,1) = 0$, it must be of the form $F(x_0, x_1, 1) = x_0(\dots) + x_1^k(\dots)$. A little computation yields that the determinant of the Hessian matrix of second partials vanishes at p if and only if $k \geq 3$. Thus p is an inflection point if and only if the tangent line intersects C at p with multiplicity 3. Since by Bezout's Theorem we have that the tangent line and C intersect in exactly d points (counted with multiplicity), the points of intersection must be $3p$ and $d-3$ other points on C.

Exercise 25. Show that any transformation $\begin{bmatrix} c_{00} & c_{01} & c_{02} \\ 0 & c_{11} & c_{12} \\ 0 & 0 & c_{22} \end{bmatrix}$ doesn't move the point $(0,0,1)$ or the line $y_0 = 0$ and so is an affine transformation

$$\begin{aligned} y_1 &= \frac{c_{11}z_1 + c_{01}}{c_{00}} \\ y_2 &= \frac{c_{22}z_2 + c_{12}z_1 + c_{02}}{c_{00}} \end{aligned}$$

from the $(1, z_1, z_2)$ to the $(1, y_1, y_2)$ plane. Show that there is always such an invertible matrix as above that does what we want.

That $\begin{bmatrix} c_{00} & c_{01} & c_{02} \\ 0 & c_{11} & c_{12} \\ 0 & 0 & c_{22} \end{bmatrix}$ doesn't move (0,0,1) or the line $y_0 = 0$ is a simple check by multiplying against the matrix. So we must show that there is always such an *invertible* matrix.

We have from the notes that $y_1 = \frac{c_{11}z_1+c_{01}}{c_{00}}$ and $y_2 = \frac{c_{22}z_2+c_{12}z_1+c_{02}}{c_{00}}$. So for $(z_0, z_1, z_2)(q') = (1, 0, 0)$, $y_1 = c_{01}/c_{00}$ and $y_2 = c_{02}/c_{00}$, and for $(z_0, z_1, z_2)(q'') = (1, 1, 0)$ we have $y_1 = (c_{11}+c_{01})/c_{00}$ and $y_2 = (c_{12}+c_{02})/c_{00}$.

Homogenizing wrt to the first coordinate, we can assume WLOG that $c_{00} = 1$. Then $y_1(q') = c_{01}$, $y_2(q') = c_{02}$, $y_1(q'') = c_{11} + c_{01}$, and $y_2(q'') = c_{12} + c_{02}$. Clearly we can pick c_{22} to be any non-zero scalar, so it remains to show that c_{11} can be chosen to be non-zero, since then the determinant of the above matrix will be non-zero as well. But by the above equations, we have $y_1(q') = c_{01}$, which yields for us the equation $y_1(q'') = c_{11} + y_1(q')$. Since q' and q'' are distinct lines through $[0, 0, 1]$, c_{11} must be non-zero. So the above matrix is invertible.

Exercise 26. Change variables in $y^2 = x(x-1)(x-\lambda)$ by replacing x by $x - c$ and leaving y as it is. Pick the c such that the new equation becomes $y^2 = x^3 + a(\lambda)x + b(\lambda)$.

Replacing x by $x + \frac{\lambda+1}{3}$ works. Check it.

Exercise 27. Let $f(x)$ be a polynomial of one complex variable and no multiple roots. Homogenize the affine curve $y^2 = f(x)$. Does the resulting curve have any singularities? If so resolve them by blowing up.

Unless I get more time or feel particularly zealous, the rest of this exercise is left to the reader.