BLOWING UP THE PCMI 2008 T-SHIRT

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1. Introduction

This paper was written during the Park City Mathematics Institute 2008 Summer Session. On the back of the conference t-shirt was depicted the surface "Seepferdchen," which is the German word for "Seahorse,"given by the equation $p(x_0, x_1, x_2) = (x_0^2 - x_1^3)^2 - (x_0 + x_1^2)x_2^3 = 0$. This surface has noticeable singularities, and so I decided to resolve them using standard blow-up methods as will be shown. Surprisingly, the resolution required four blow-ups in total. The computing power of *Mathematica* and *CoCoA* significantly aided in the process. I would like to thank Herb Clemens for initiating the project and formalizing the first blow-up and David Perkinson, who helped with the calculations and provided pictures.

2. Resolving The Singularity

2.1. First Blow-Up.

Call the surface of interest W and define it by

$$W : p(x_0, x_1, x_2) = (x_0^2 - x_1^3)^2 - (x_0 + x_1^2)x_2^3 = 0$$

To find the singularities, we must find the locus of points for which the curve and all its partial derivatives simultaneously vanish.

$$\begin{aligned} \frac{\partial p}{\partial x_0} &= 4x_0(x_0^2 - x_1^3) - x_2^3 = 0 \\ \frac{\partial p}{\partial x_1} &= -6x_1^2(x_0^2 - x_1^3) - 2x_1x_2^3 = 0 \\ \frac{\partial p}{\partial x_2} &= -3(x_0 + x_1^2)x_2^2 = 0 \end{aligned}$$

These equations imply that $x_2 = 0$ and $x_0^2 - x_1^3 = 0$, so let

$$\begin{aligned} p_0(x_0, x_1, x_2) &= x_0^2 - x_1^3 \\ p_1(x_0, x_1, x_2) &= x_2 \end{aligned}$$

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be the defining equations for the singularity $Z \subset W$. The blow-up B_Z is then subject to the constraint

$$\left\{ \begin{vmatrix} y_0 & y_1 \\ p_0 & p_1 \end{vmatrix} = 0 \right\}$$

We find that our equation for B_Z is

$$y_0 p_1 - y_1 p_0 = y_0(x_2) - y_1(x_0^2 - x_1^3) = 0$$

Now we may rewrite W in terms of p_0 and p_1 using the relation $p_1 = p_0 y_1 / y_0$ for $y_0 \neq 0$.

$$\begin{aligned} p_0^2 - (x_0 + x_1^2)p_1^3 &= y_0^3 p_0^2 - (x_0 + x_1^2)p_0^3 y_1^3 \\ &= p_0^2(y_0^3 - (x_0 + x_1^2)p_0 y_1^3) = 0 \end{aligned}$$

Let $\tilde{W}$ denote the pre-image of W upstairs. It is given in B_Z by

$$y_0^3 - (x_0 + x_1^2)p_0 y_1^3 = 0$$

To begin the blow-up we must consider the different coordinate charts. First let us consider the chart $\mathbb{C}^3 \times \mathbb{P}^1$ with coordinates $\{(x_0, x_1, x_2), [y_0 : y_1]\}$ defined by letting $y_0 = 1$. Then our equation in $\tilde{W}$ becomes

$$1 - (x_0 + x_1^2)p_0 y_1^3 = 0$$

This equation is free of singularities, therefore we consider the chart $\mathbb{C}^3 \times \mathbb{P}^1$ defined by letting $y_1 = 1$. Then the relation becomes $p_0 = p_1 y_0$ together with the equation $p_0^2 - (x_0 + x_1^2)p_1^3 = 0$ characterizing W. We now substitute as follows

$$\begin{aligned} p_0^2 - (x_0 + x_1^2)p_1^3 &= 0 \\ (p_1 y_0)^2 - (x_0 + x_1^2)p_1^3 &= 0 \\ p_1^2(y_0^2 - (x_0 + x_1^2)p_1) &= 0 \end{aligned}$$

Finally we can explicitly state defining equations for $\tilde{W}$ on our chart $\mathbb{C}^3 \times \mathbb{P}^1$ where $y_1 = 1$.

$$\begin{aligned} q_0(x_0, x_1, x_2, y_0) &= p_0 - p_1 y_0 \\ &= (x_0^2 - x_1^3) - x_2 y_0 = 0 \\ q_1(x_0, x_1, x_2, y_0) &= y_0^2 - (x_0 + x_1^2)x_2 = 0 \end{aligned}$$

These equations define the blow up, $\tilde{W}$, of the original surface in the coordinate chart given by $y_1 = 1$. A point $p := (x_0, x_1, x_2, y_0) \in \tilde{W}$ is singular if

$$\operatorname{rank}\left(\begin{bmatrix} \frac{\partial q_0}{\partial x_0}|_p & \frac{\partial q_0}{\partial x_1}|_p & \frac{\partial q_0}{\partial x_2}|_p & \frac{\partial q_0}{\partial y_0}|_p \\ \frac{\partial q_1}{\partial x_0}|_p & \frac{\partial q_1}{\partial x_1}|_p & \frac{\partial q_1}{\partial x_2}|_p & \frac{\partial q_1}{\partial y_0}|_p \end{bmatrix} \right) < 2$$

Using *CoCoA*, we find two additional singular points, namely $(0,0,0,0)$ and $(-1,1,0,0)$ both in $\tilde{W}$. Therefore, we must blow-up again at these two points. It would be optimal to work with only one defining equation instead of both q_0 and q_1. Notice that if we solve q_0 and q_1 for x_2 and then set them equal, we are able to get the new defining equation

$$(x_0^2 - x_1^3)(x_0 + x_1^2) = y_0^3$$

To eliminate the writing of subscripts we change notation to $x_0 = x, \; x_1 = y, \; x_2 = z$, and $y_0 = v$. Our new equation will be defined in these terms in the following section.

2.2. **Second Blow-Up.** We work with coordinates $\{(x,y,v),[s:t:u]\} \in \mathbb{C}^3 \times \mathbb{P}^2$, then our blow-up conditions are

$$\begin{vmatrix} x & y \\ s & t \end{vmatrix} = \begin{vmatrix} x & v \\ s & u \end{vmatrix} = \begin{vmatrix} y & v \\ t & u \end{vmatrix} = 0$$

together with the equation

$$(x^2 - y^3)(x + y^2) = v^3$$

First, consider the chart where $s = 1$ so that $y = xt, \; v = xu$, and $yu = vt$. Making these substitutions yields an equation in the variables (x,t,u) as follows

$$(1 - xt^3)(1 + xt^2) = u^3$$

Our second blow-up on this chart has successfully eliminated the singular point $(0,0,0)$, however, *CoCoA* indicates that the singular point $(-1,-1,0)$ still remains. So yet another blow-up is required, but let us consider the other charts first.

Next, consider the chart where $t = 1$ so that $x = ys, \; xu = vs$, and $v = yu$. Making these substitutions yields an equation in the variables (y,s,u) as follows

$$(s^2 - y)(s + y) = u^3$$

Calculations with *CoCoA* reveal that our blow-up on this coordinate chart has failed to eliminate either singularity! This is surprising indeed, and thus will require at least two more blow-ups! We now check our final chart.

Let $u = 1$ so that $xt = ys, \; x = vs$, and $y = vt$. Making these substitutions yields an equation in the variables (v,s,t) as follows

$$(s^2 - vt^3)(s + vt^2) = 1$$

This equation has no singularities, as desired. Therefore we may continue with our calculations on the chart where $t = 1$ which still has two singularities.

2.3. **Third Blow-Up.** We work with coordinates $\{(y, s, u), [a : b : c]\} \in \mathbb{C}^3 \times \mathbb{P}^2$, then our blow-up conditions are

$$\begin{vmatrix} y & s \\ a & b \end{vmatrix} = \begin{vmatrix} y & u \\ a & c \end{vmatrix} = \begin{vmatrix} s & u \\ b & c \end{vmatrix} = 0$$

together with the equation

$$(s^2 - y)(s + y) = u^3$$

which has been carried over from the previous calculation on the coordinate chart defined by letting $t = 1$.

First, consider the chart where $a = 1$ so that $s = yb,\ u = yc,$ and $sc = bu$. Making these substitutions yields an equation in the variables (y, b, c) as follows

$$(yb^2 - 1)(b + 1) = yc^3$$

Next, consider the chart where $b = 1$ so that $y = as,\ yc = au,$ and $u = sc$. Making these substitutions yields an equation in the variables (s, a, c) as follows

$$(s - a)(1 + a) = sc^3$$

Finally, consider the chart where $c = 1$ so that $as = yb,\ y = au,$ and $s = bu$. Making these substitutions yields an equation in the variables (u, a, b) as follows

$$(b^2u - a)(b + a) = u$$

The first two charts yield equations with a common singularity at the point $(1, -1, 0)$, so that $(0, 0, 0)$ is no longer singular. The last equation is actually singularity free. Progress is being made, therefore we may choose either of the first two charts for the final blow-up. We choose the chart defined by letting $a = 1$ without loss of generality.

2.4. **Fourth Blow-Up.** Finally, we must blow up once more, this time around the point $(1, -1, 0)$. We work with coordinates $\{(y, b, c), [i : j : k]\} \in \mathbb{C}^3 \times \mathbb{P}^2$, then our blow-up conditions are

$$\begin{vmatrix} y & b \\ i & j \end{vmatrix} = \begin{vmatrix} y & c \\ i & k \end{vmatrix} = \begin{vmatrix} b & c \\ j & k \end{vmatrix} = 0$$

together with the equation

$$[(y + 1)(b - 1)^2 - 1][(b - 1) + 1] = (y + 1)c^3$$

where we have changed coordinates for computational convenience. (Replacing y by $y + 1$ and b by $b - 1$ and leaving c alone in the original equation corresponds to the translation taking $(1, -1, 0)$ to $(0, 0, 0)$.)

First, consider the chart where $i = 1$ so that $b = yj,\ c = yk,$ and $bk = jc$. Making these substitutions yields an equation in the variables (y, j, k) as follows

$$[(yj-1)^2 + j(yj-2)]j = (y+1)yk^3$$

Next, consider the chart where $j = 1$ so that $y = bi,\ ic = yk,$ and $c = bk$. Making these substitutions yields an equation in the variables (b, i, k) as follows

$$[bi(b-2) + i + b - 2] = (bi+1)bk^3$$

Finally, consider the chart where $k = 1$ so that $yj = bi,\ y = ic,$ and $b = cj$. Making these substitutions yields an equation in the variables (c, i, j) as follows

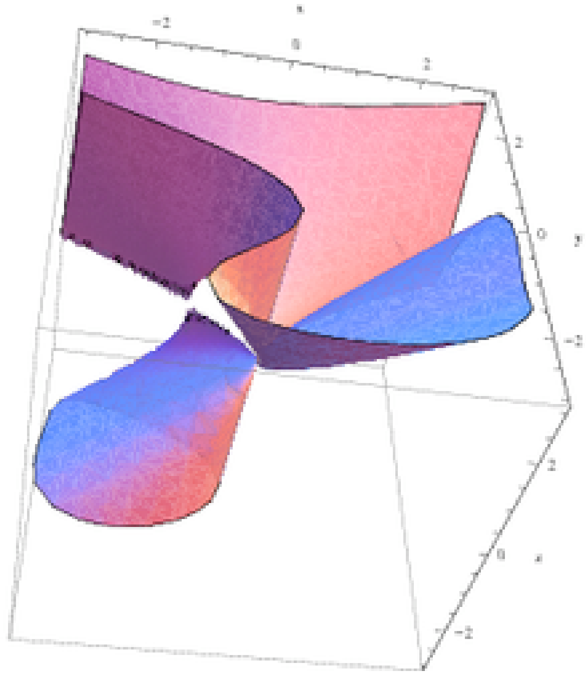
$$[i(cj-1)^2 + j(cj-2)]j = (ic+1)c$$

Using *CoCoA*, we verify that none of these equations contain any further singular points. And so the point $(1, -1, 0)$ has been completely smoothed. Thus our blow-up of the PCMI 2008 conference t-shirt curve $p(x_0, x_1, x_2) = (x_0^2 - x_1^3)^2 - (x_0 + x_1^2)x_2^3 = 0$ is finally resolved after four steps.

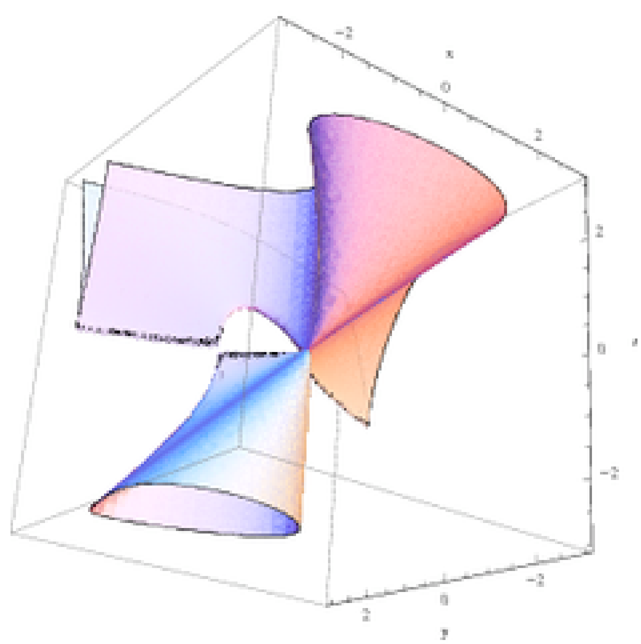
We started with a singular curve on the original surface. After the first blow-up the singularity was reduced to just two points. A second blow-up failed to resolve either singularity which was disheartening. However, the third and fourth blow-ups eliminated the singularities at $(0, 0, 0)$ and $(1, -1, 0)$, respectively. Interesting pictures drawn using *Mathematica* as well as sample code from the *CoCoA* calculations can be found in the appendix following this section.

Appendix A: Pictures of the surfaces.

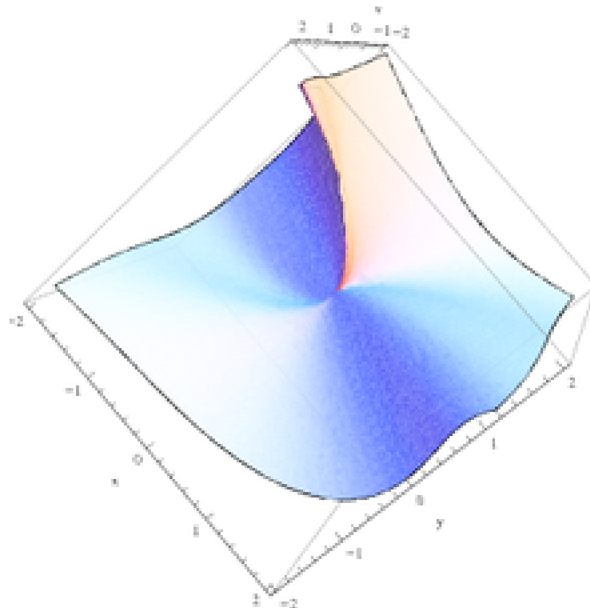
The seahorse: $(x_0^2 - x_1^3)^2 - (x_0 + x_1^2)x_2^3 = 0$.



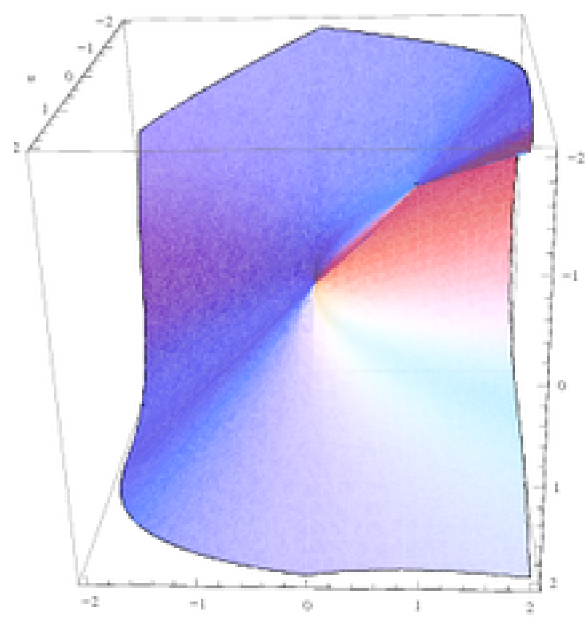
Another view of the seahorse.



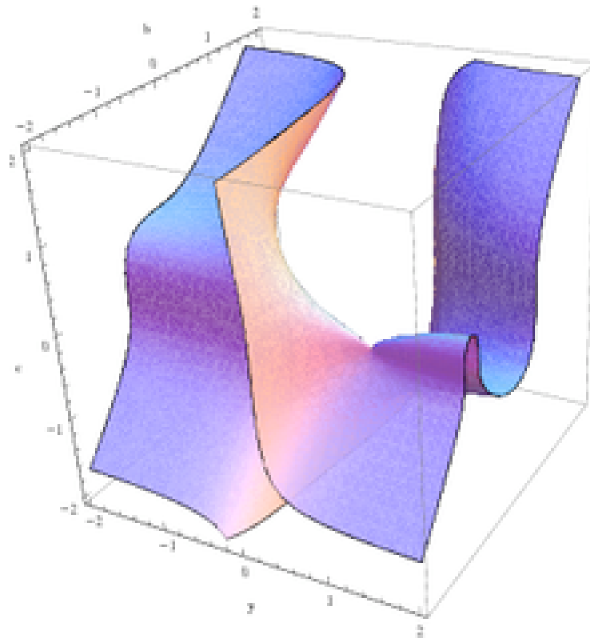
The first blow-up: $(x^2 - y^3)(x + y^2) = v^3$.



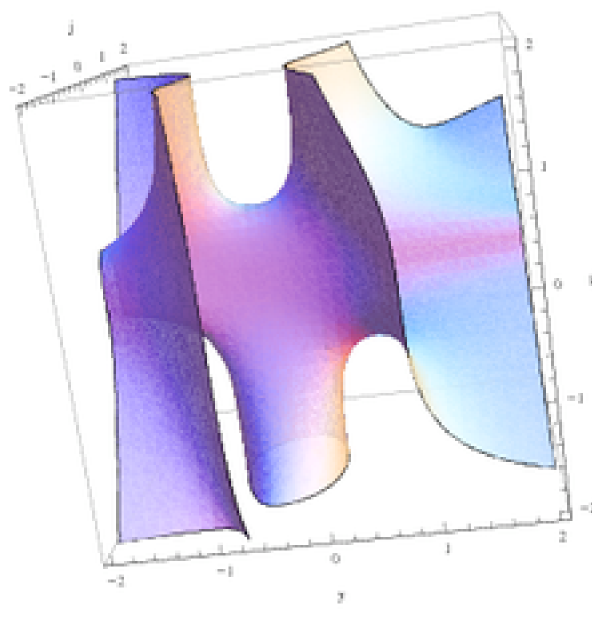
The second blow-up: $(s^2 - y)(s + y) = u^3$.



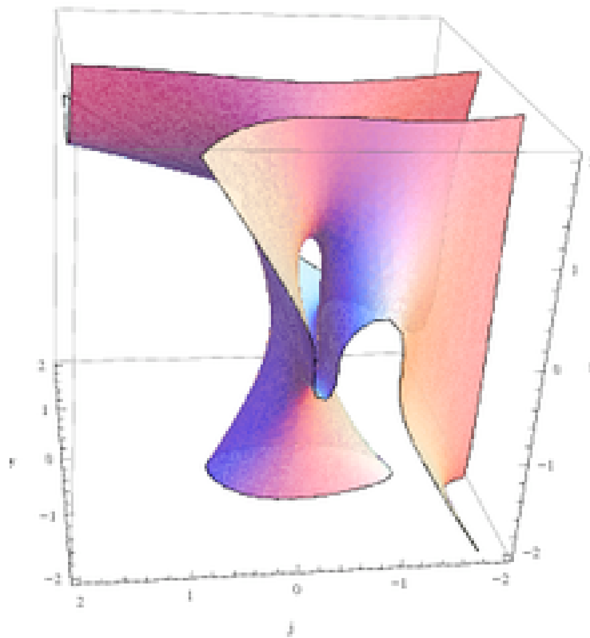
The third blow-up: $((y+1)(b-1)^2-1)b=(y+1)c^3$.



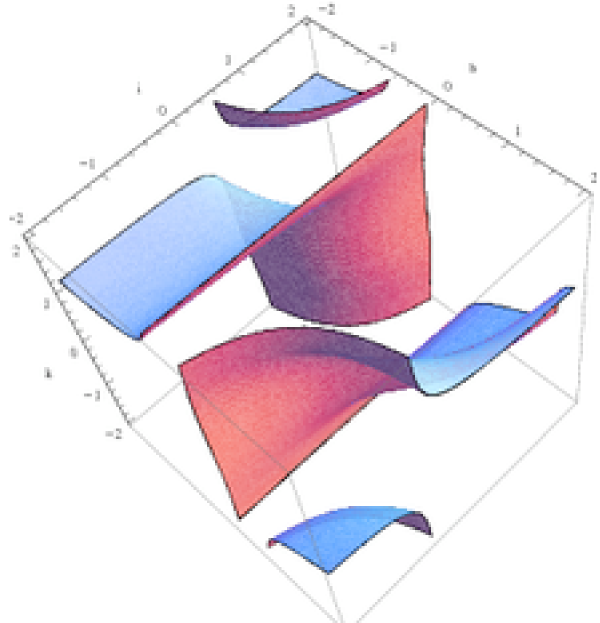
Desingularized surface, chart 1:



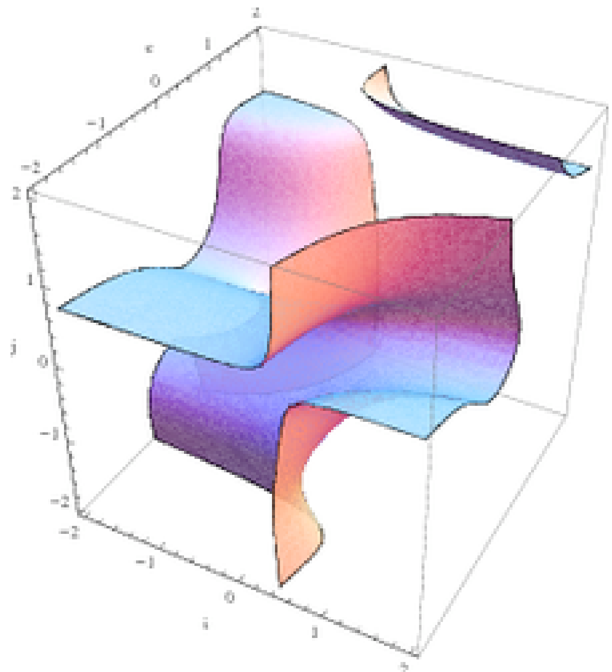
Another view:



Desingularized surface, chart 2:



Desingularized surface, chart 3:



k
j
i

Appendix B: *CoCoA* calculations.

Here are samples of the code used to find the singularities.

I. The original surface is defined by the equation $(x^2 - y^3)^2 - (x + y^2)z^3 = 0$.

```
Use R::=Q[x,y,z],Lex;  -- Lex is an elimination term ordering,
                       -- good for solving equations
F:=(x^2-y^3)^2-(x+y^2)z^3;
J:=Jacobian([F]);
J;
Mat([
  [4x^3 - 4xy^3 - z^3, -6x^2y^2 + 6y^5 - 2yz^3, -3xz^2 - 3y^2z^2]
])
-------------------------------
J:=Flatten(List(J));
J;
[4x^3 - 4xy^3 - z^3, -6x^2y^2 + 6y^5 - 2yz^3, -3xz^2 - 3y^2z^2]
-------------------------------
I:=Ideal(F)+Ideal(J);
G:=ReducedGBasis(I);
G;
[z^5, xz^2 + y^2z^2, x^3 - xy^3 - 1/4z^3, x^2y^2 - y^5 + 1/3yz^3,
       y^2z^3, y^6z^2 - y^5z^2]
-------------------------------
-- We see that z=0.
Subst(G,[[z,0]]);
[0, 0, x^3 - xy^3, x^2y^2 - y^5, 0, 0]
-------------------------------
-- Now it's easy to see that the solution set is defined by z=0 and
-- x^2-y^3=0
```

II. The first blow-up is defined (on one of its charts) by the system of equations

$$\begin{aligned} x^2 - y^3 - zv &= 0 \\ v^2 - (x + y^2)z &= 0. \end{aligned}$$

```
Use R::=Q[x,y,z,v],Lex;
Q0:=x^2-y^3-zv;
Q1:=v^2-(x+y^2)z;
J:=Jacobian([Q0,Q1]);
N:=Minors(2,J);
N:=Flatten(List(N));
I:=Ideal(N)+Ideal(Q0,Q1);
G:=ReducedGBasis(I);
G;
[v^2, y^4 - y^3 - 2/3yzv - 3/2zv, z^2v, z^3, y^2z, xz, xv - 1/4z^2,
      x^2 - y^3 - zv, y^2v + 1/3yz^2, xy^2 + y^3 + 3/2zv]
-------------------------------
-- So v=0.
G:=Subst(G,[[v,0]]);
G;
[0, y^4 - y^3, 0, z^3, y^2z, xz, -1/4z^2, x^2 - y^3, 1/3yz^2, xy^2 + y^3]
-------------------------------
-- And z=0.
G:=Subst(G,[[z,0]]);
G;
[0, y^4 - y^3, 0, 0, 0, 0, 0, x^2 - y^3, 0, xy^2 + y^3]
-------------------------------
-- Thus, x=y=z=v=0 or x=-1,y=1,z=v=0.
```

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