The Gopakumar-Vafa Formula

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In topological string theory, as you have heard in the lectures by Hirosi Ooguri, one counts the holomorphic maps from a Riemann surface Σ of genus g (with unspecified complex structure) to a target space Y. Such a map $\Phi : \Sigma \to Y$ determines a class $\vec{d} \in H_2(Y, \mathbb{Z})$ which measures the "wrapping" of Σ around various cycles in Y. For each class $\vec{d}$, we let $a_{g,\vec{d}}$ be the "number" of holomorphic maps with that homology class where Σ has genus g. I put the words "number" in quotes because there are many subtleties in defining this number correctly.

(This talk is based on a paper to appear with M. Dedushenko.)

We introduce a set of coupling constants $\vec{t}$ that is dual to the homology class $\vec{d}$, and define the genus g topological string partition free energy

$$\mathcal{F}_g(\vec{t}) = \sum_{\vec{d} \in H_2(Y,\mathbb{Z})} a_{g,\vec{d}} \exp(2\pi i \vec{d} \cdot \vec{t}). \tag{1}$$

The full topological string partition function free energy is

$$\mathcal{F}(g_{\mathrm{st}}, \vec{t}) = \sum_{g=0}^{\infty} g_{\mathrm{st}}^{2g-2} \mathcal{F}_g(\vec{t}). \tag{2}$$

(The partition function including disconnected contributions is $\mathcal{Z} = e^{\mathcal{F}}$.)

What does topological string theory have to do with physical string theory? A remarkable answer was proposed by Bershadsky, Cecotti, Ooguri and Vafa (BCOV; 1993) and confirmed in a hard calculation by Antoniadis, Gava, Narain, and Taylor (AGNT; 1993). One focuses on Type IIA superstring theory compactified on $\mathbb{R}^4 \times Y$ where Y is a Calabi-Yau three-manifold. (This case is important both in physical string theory – it is used to make semi-realistic models of particle physics – and topological string theory, which turns out to be particularly interesting in this case.)

In this theory, there are $b_2 + 1$ $U(1)$ gauge fields that come from the Ramond-Ramond (RR) sector of superstring theory. One of them is simply the RR gauge field that is already present in ten dimensions and the other b_2 arise in Kaluza-Klein reduction of the RR three-form

$$C = \sum_{I=1}^{b_2} A^I(x) \cdot \omega_I(y)$$

where $\omega_I(y)$ are harmonic two-forms on Y, normalized to give a basis of $H^2(Y, \mathbb{Z})$ mod torsion, and $A^I(x)$ are $U(1)$ gauge fields on $\mathbb{R}^4$.

Type IIA on a Calabi-Yau manifold Y has eight unbroken supersymmetries and the effective action can be described in a superspace with four bosonic coordinates x^μ, four fermionic coordinates θ^{Ai} of negative chirality, and four more $\bar\theta^{\dot A i}$ of positive chirality. (Here all indices $A, \dot A, i$ take the values 1,2. A and $\dot A$ are spinor indices of respectively negative and positive chirality in four dimensions, and the index i is there because Type IIA on a Calabi-Yau has $\mathcal{N} = 2$ supersymmetry in four dimensions.) A chiral superfield is a function $\Psi(x, \theta)$ that does not depend on the $\bar\theta$'s, and conversely an antichiral superfield $\widetilde\Psi(x, \bar\theta)$ does not depend on the θ's. A generic superfield is a function of all eight fermionic coordinates.

Of the $b_2 + 1$ $U(1)$ gauge fields, b_2 linear combinations are in vector multiplets, which for $\mathcal{N} = 2$ supersymmetry in four dimensions are described by chiral superfields

$$\mathcal{X}^\Lambda(x, \theta) = X^\Lambda + \theta\Psi^\Lambda + \theta^2 F^\Lambda + \dots$$

where F^Λ is a $U(1)$ gauge field strength. The remaining $U(1)$ gauge field is the "graviphoton," which is part of the supergravity multiplet. It also is part of a chiral superfield, but this is not a chiral superfield of spin zero. Rather, the anti-selfdual part of the graviphoton field strength, which I will write as a bispinor $W_{AB}(x)$ $(= W_{BA}(x) = \sigma^{\mu\nu}_{AB} W_{\mu\nu}(x))$, partly because this is natural in string perturbation theory, is the bottom component of a chiral superfield that itself is a bispinor $\mathcal{W}_{AB}(x, \theta) = W_{AB}(x) + \dots$.

In general, in a supersymmetric theory, a term in the effective action might be a "D-term," meaning that it can be written as an integral over all of superspace, in our case $\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\bar{\theta}(\dots)$, or an "F-term," meaning that it cannot be so written. For example, an F-term might be $\int \mathrm{d}^4x\mathrm{d}^4\theta\mathcal{O}$, where the operator $\mathcal{O}$ has to be chiral (independent of $\bar{\theta}$) so that one can get a supersymmetric result just by integration over the four θ's. It usually is only the F-terms about which one can make nonperturbative statements and my lecture today is concerned with F-terms.

In the case at hand, for every $g \geq 0$, there is a possible F-term

$$\mathcal{I}_g = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{F}_g(\mathcal{X}^0, \ldots, \mathcal{X}^{b_2})(\mathcal{W}_{AB}\mathcal{W}^{AB})^g.$$

These particular interactions have the remarkable property that, from the standpoint of Type IIA superstring perturbation theory, $\mathcal{I}_g$ is generated only in genus g. The proof of this just follows from the low energy supergravity. One transforms the interaction $\mathcal{I}_g$ to the string frame and finds (to simplify slightly) that it is proportional to g_{st}^{2g-2}.

The insight of BCOV is that the function $\mathcal{F}_g(\mathcal{X}^\Lambda)$ is the topological string partition function $\mathcal{F}_g(\vec{t})$ if one sets $t^I = \mathcal{X}^I/\mathcal{X}^0$, $I = 1, \ldots, b_2$. (Technically, here $\mathcal{X}^0$ is the multiplet that corresponds to the RR 1-form in ten dimensions and $\mathcal{X}^I$, $I = 1, \ldots, b_2$ correspond to the vector fields that arise from the C-field). This was already known for $g = 0$ but the generalization to $g > 0$ is quite remarkable. Unfortunately, the only complete proof that is presently known is rather technical (AGNT).

What did Gopakumar and Vafa (1998) add to this story? They suggested that one should view Type IIA on $\mathbb{R}^4 \times Y$ in terms of M-theory on $\mathbb{R}^4 \times S^1 \times Y$, where S^1 is sometimes called the M-theory circle. In this correspondence, the string coupling constant is determined by the radius of the M-theory circle. Since $\mathcal{I}_g$ for given g has a known dependence on $g_{\rm st}$, its dependence on the radius of the S^1 is essentially also known and we can calculate in the region where the S^1 is large and the M-theory description is useful.

In string perturbation theory, $\mathcal{F}_g(\mathcal{X}^\Lambda)$ is computed by counting superstring worldsheets wrapped on a complex submanifold $\Sigma \subset Y$. This corresponds in M-theory to an M2-brane wrapped on $S^1 \times \Sigma$. Such an M2-brane can be studied in a Hamiltonian formalism in terms of states propagating in the S^1 direction. So it should be possible to compute the F-terms $\mathcal{I}_g$ by summing over contributions of wrapped M2-brane states.

It possibly is intuitively obvious that the only M2-brane states that can contribute to an F-term are BPS states and later I will explain why this is true. For now I give this naive explanation: BPS states are states that are invariant under some supersymmetries, and they can generate terms in the effective action that are integrals of operators that are likewise invariant under some supersymmetries, in other words F-terms. A state not invariant under any supersymmetries, when it propagates around the circle, generates only an operator not invariant under any supersymmetries, in other words a D-term.

Moreover, Gopakumar and Vafa had a remarkable idea on how to compute the $\mathcal{F}_g$. Their proposal was that instead of computing one particular interaction

$$\mathcal{I}_g = \int \mathrm{d}^4 x \, \mathrm{d}^4 \theta \, \mathcal{F}_g(\mathcal{X}^0, \ldots, \mathcal{X}^{b_2})(\mathcal{W}_{AB}\mathcal{W}^{AB})^g,$$

we should study the whole sum

$$\mathcal{I} = \sum_{g=0}^{\infty} \mathcal{I}_g = \int \mathrm{d}^4 x \, \mathrm{d}^4 \theta \, \sum_{g=0}^{\infty} \mathcal{F}_g(\mathcal{X}^0, \ldots, \mathcal{X}^{b_2})(\mathcal{W}_{AB}\mathcal{W}^{AB})^g.$$

Moreover, they suggested that this sum should be viewed as the superspace effective action evaluated in a background in which a constant anti-selfdual graviphoton field is turned on. This superspace effective action is supposed to be computed by a one-loop calculation, summing over one-loop diagrams with BPS states running around the loop.

This calculation is supposed to be a supersymmetric variant of Schwinger's celebrated computation of a one-loop effective action due to a charged particle in a constant magnetic field. The upshot then would be to express the interaction $\mathcal{I}$ as a sum of contributions of BPS states. Moreover, Gopakumar and Vafa proposed what the resulting formula would look like, and the resulting GV formula has had many applications and enormous success.

There is also an Ooguri-Vafa (1999) analog of the GV formula. In this case, one considers Type IIA on $\mathbb{R}^4 \times Y$ with D4-branes on $\mathbb{R}^2 \times L$ where $\mathbb{R}^2 \subset \mathbb{R}^4$ and $L \subset Y$ is a Lagrangian submanifold. The M-theory lift is to $\mathbb{R}^4 \times S^1 \times Y$ with M5-branes on $\mathbb{R}^2 \times S^1 \times L$. Certain chiral interactions in Type IIA on $\mathbb{R}^4 \times Y$ can be computed in the M-theory description by counting BPS states that live on the M5-branes. I will not really have time to explain this.

Before explaining the computation that leads to the GV formula, I want to first address some general questions about it. Perhaps the first question is whether contributions to the interactions $\mathcal{I}_g$ can arise by dimensional reduction from a supersymmetric action in five dimensions. To the extent that the $\mathcal{I}_g$ can arise that way, we can only determine them by knowing the 5d effective action for M-theory on $\mathbb{R}^5 \times Y$; an interaction that is already present in five dimensions cannot be determined by a Schwinger computation that involves further compactification from $\mathbb{R}^5$ to $\mathbb{R}^4 \times S^1$.

However, the $\mathcal{I}_g$ do not arise this way; only certain very special and known terms in $\mathcal{I}_0$ and $\mathcal{I}_1$ can arise by dimensional reduction from five dimensions. A not quite complete explanation of this statement is that the superfields $\mathcal{X}^\Lambda$ contain axion-like modes that decouple at zero momentum in any interaction that arises by dimensional reduction from five dimensions. But (except for very special terms in $\mathcal{I}_0$ and $\mathcal{I}_1$) the interactions $\mathcal{I}_g$ spoil this decoupling, so they cannot arise by classical dimensional reduction. This means that in principle we can do the calculation that we want to do.

The next basic question is whether a supersymmetric background with a graviphoton field turned on does exist. A "no" answer would mean we could not get anywhere. In field theory, we could compute an effective action in an arbitrary background that is not necessarily supersymmetric or a solution of any classical equations of motion. We don't really know how to do that in string/M-theory. But more important, even if we could do a calculation, we would not learn anything from it. The reason is that in a non-supersymmetric background, in which say there is a superfield S with $\int \mathrm{d}^4\overline{\theta}\, S \neq 0$, it is hard to distinguish an F-term $\int \mathrm{d}^4x\mathrm{d}^4\theta\, \Phi(x,\theta)$ (for some chiral superfield Φ) from a D-term $\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\overline{\theta}\, S\Phi$.

So we can only learn about F-terms by expanding about a supersymmetric background. In field theory, one can possibly use a background that is supersymmetric but not a classical solution, but it is doubtful that we would understand how to do that in string/M-theory. But at a minimum we need a supersymmetric background.

If one turns on an arbitrary linear combination of anti-selfdual $U(1)$ gauge fields F^{Λ}_{AB} on $\mathbb{R}^4$, one will get no gravitational backreaction – since the energy momentum tensor of a selfdual or anti-selfdual Maxwell field vanishes. But generically one will get scalar backreaction. It turns out that scalar backreaction is avoided and moreover one gets a supersymmetric background precisely if one turns on only the graviphoton and not any of the gauge fields that are in vector multiplets.

So a supersymmetric classical solution with the necessary properties actually exists. It is most simply described as a solution in a 5d spacetime with Lorentz signature. The solution was called the supersymmetric Gödel solution by Gauntlett, Gutowski, Hull, Pakis and Reall (GGHPR, 2003):

$$\mathrm{d}s^2 = -(\mathrm{d}t + V_\mu \mathrm{d}x^\mu)^2 + \sum_{\mu=1}^{4} (\mathrm{d}x^\mu)^2, \quad V_\mu = \frac{1}{8} T^-_{\mu\nu} x^\nu,$$

where $T^-_{\mu\nu}$ is the constant anti-selfdual graviphoton. To use this in M-theory compactification to Type IIA, we need to compactify the t direction but also we want the metric on the circle to by Euclidean. So we rotate $t \to iy$ where y is a periodic variable.

It is not possible to make both the graviphoton $T^-_{\mu\nu}$ and the metric $\mathrm{d}s^2$ real. The lesser evil is to take the graviphoton imaginary so that the metric is real. The graviphoton being imaginary in the background is really not a problem since Schwinger's original calculation worked fine in either an electric or a magnetic field; an imaginary magnetic field in Euclidean space is somewhat like a real electric field in Lorentz signature.

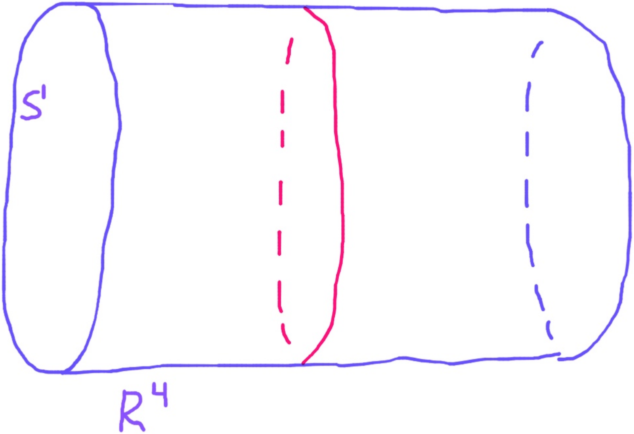
This background has a remarkable property that was discovered by GGHPR and also from a different perspective by Berkovits and Seiberg (2003). Naively, if one turns on an anti-selfdual gauge field, one preserves at most one-half of the supersymmetry, which in our problem would mean that one preserves only four of the eight supercharges. However, it turns out (GGHPR, BeS) that the supersymmetric Gödel solution preserves all eight supercharges that one would have before turning on the graviphoton field. But the supersymmetry algebra is deformed from what it is before turning on the graviphoton.

It turns out that the reason that this is important is that to compute the GV formula, one needs to know the magnetic moments of the BPS states. The unexpected extra four supersymmetries, and the deformed supersymmetry algebra, uniquely determine those magnetic moments.

There are a few more questions we should ask and here is another one: Quantum mechanics has wave-particle duality but the range of validity of a calculation based on waves can be quite different from that of a calculation based on particles. Is the Schwinger calculation that leads to the GV formula supposed to be a calculation based on particles or on fields?

To answer this question, observe that – to the extent that we do understand M-theory – we understand it when the length scales involved are much greater than the Planck length. This means that we should think of the Calabi-Yau manifold Y as being much bigger than the Planck length, and therefore a massive BPS state coming from a wrapped M2-brane has a mass much greater than $M_{\rm Pl}$. We would be wary of using field theory for particles much heavier than the Planck mass. Moreover, wrapped M2-branes can have arbitrarily large spin, and for particles of spin > 2 coupled to the background supergravity multiplet, it is not clear that we have a sensible field theory to use. So we will do a particle calculation for BPS states that are massive in $d = 5$, and only use field theory for the (few) BPS states that are massless in $d = 5$.

To do the particle calculation, one starts by thinking of the M2-brane wrapped on $p \times S^1 \times \Sigma \subset \mathbb{R}^4 \times S^1 \times Y$ (where p is a point in $\mathbb{R}^4$ and $\Sigma \subset Y$) as a sort of instanton:



In drawing the picture, I actually ignored Σ and just depicted the M2-brane worldvolume $p \times S^1 \times \Sigma$ as a particle worldline $p \times S^1 \subset \mathbb{R}^4 \times S^1$, with a point $p \in \mathbb{R}^4$. This is actually the right thing to do in the following sense: If the S^1 is sufficiently large, which is the limit in which our approximations are going to be good, then we can ignore the internal structure of the BPS state and the existence of the compact manifold Y and just think of the BPS state as a point particle propagating in $\mathbb{R}^4 \times S^1$. We will proceed in that way. We view the particle winding on $p \times S^1$ as a sort of instanton in 5d supergravity on $\mathbb{R}^4 \times S^1$.

To get the effective action, we have to integrate over the moduli of the instanton. The simplest case is that the only moduli are the ones that have to exist because of translation invariance and supersymmetry. The instanton inevitably has at least four bosonic moduli associated to translation symmetry. These moduli are the coordinates x^μ of the point $p \in \mathbb{R}^4$. If the particle is not BPS, then its worldline is not invariant under any of the eight supersymmetries and therefore there will be eight fermionic moduli generated by the broken supersymmetries. I will call them θ^{Ai} and $\theta^{\dot{A}j}$, $A, \dot{A}, i, j = 1, 2$, where A and $\dot{A}$ are indices of negative or positive chirality. Thus for a non-BPS particle the effective action will be $\int \mathrm{d}^4x\mathrm{d}^4\theta\mathrm{d}^4\overline{\theta}\,\mathcal{O}$ where $\mathcal{O}$ is the instanton amplitude without integrating over the moduli that are generated by spacetime symmetries (but after integrating over other moduli, if there are any). In other words, a non-BPS particle will generate a D-term.

Let us see instead what happens for a half-BPS particle that is invariant under the four supersymmetries $Q^{\dot{A}i}$. Such a BPS particle comes from an M2-brane wrapped on a complex Riemann surface $\Sigma \subset Y$. For such a particle we only have four fermionic moduli θ^{Ai} generated by the Q^{Ai} and therefore the effective action is going to be $\int \mathrm{d}^4x \mathrm{d}^4\theta \, \mathcal{O}$, where $\mathcal{O}$ is whatever we get before integrating over the moduli generated by symmetries. In other words, a half BPS particle will generate an F-term. These are the particles that contribute to the GV formula.

The BPS particles that have no moduli except the ones forced by the symmetries are massive hypermultiplets; they come from an M2-brane wrapped on an isolated genus 0 curve $\Sigma \subset Y$. Let us analyze the contribution of such a particle. First we need to understand its mass. The Kahler form of Y is

$$\omega = \sum_I h^I \omega_I$$

where ω_I, $I = 1, \ldots, b_2$ are a basis of $H^2(Y, \mathbb{Z})$ and h^I are Kahler moduli. (I will take some minor shortcuts to avoid too many details.) The charges of the BPS state are $q_I = \int_\Sigma \omega_I$. We abbreviate $q_1, \ldots, q_{b_2}$ as $\vec{q}$. The area of Σ is the central charge

$$\zeta(\vec{q}) = \sum_I q_I h^I$$

and (in units in which the M2-brane tension is 1) the mass of the BPS state is

$$M(\vec{q}) = \zeta(\vec{q}).$$

(We don't take the absolute value since a holomorphically wrapped M2-brane has $\zeta > 0$.)

The 5d theory has gauge fields A^I and corresponding magnetic fields $F^I = \mathrm{d}A^I$. In the graviphoton background, these are

$$F^I = h^I T^-$$

where T^- is the graviphoton. (I did not write this formula before because I wrote the supersymmetric Gödel solution only for pure 5d supergravity without vector multiplets. To embed this solution in a model with vector multiplets, one needs this formula.) The effective magnetic field seen by a particle of charges $\vec{q}$ is $F(\vec{q}) = \sum_I q_I F^I$, which in the graviphoton background for a BPS particle is

$$F(\vec{q}) = \sum_I q_I h^I T^- = \zeta(\vec{q}) T^- = M T^-.$$

The fact that the effective magnetic field seen by the BPS state depends only on its mass is very important in getting the GV formula.

Now let us discuss the action for the BPS particle. We can make a nonrelativistic approximation since the worldline of the BPS state will be fluctuating only slightly around the static orbit $p \times S^1$. We include a constant $-Mc^2$ for the rest mass. We also have the nonrelativistic kinetic energy $\frac{1}{2}M\dot{x}^2$ for bosons, and $\frac{iM}{2}\psi_{Ai}\frac{\mathrm{d}}{\mathrm{d}t}\psi^{Ai}$ for the fermions. So the minimal action that we can write is

$$I_0 = \int \mathrm{d}t \left(-M + \frac{M}{2}\sum_{\mu}(\dot{x}^{\mu})^2 + \frac{iM}{2}\psi_{Ai}\frac{\mathrm{d}}{\mathrm{d}t}\psi^{Ai} \right).$$

It turns out that this is enough since any higher order terms are "irrelevant" when the radius R of the M-theory circle is large and don't contribute to the F-terms.

One can verify the supersymmetry of the action. The conserved momenta P^μ that generate translations of x^μ are the canonical momenta p^μ:

$$P^\mu = p^\mu = M\dot{x}^\mu$$

They commute

$$[P^\mu, P^\nu] = 0.$$

The supercharges are

$$Q^{Ai} = M\psi^{Ai}, \quad Q^{\dot{A}i} = \gamma_\mu^{A\dot{A}} P_\mu \psi_A^i$$

and are obviously conserved. The Hamiltonian is

$$H = M + \frac{P^2}{2M}.$$

It is straightforward to verify (the appropriate nonrelativistic limit of) the supersymmetry algebra.

Now let us turn on the graviphoton field. There is one obvious term that we need to add to the action: a coupling $\int \mathrm{d}t\, A_\mu(\vec{q})\dot{x}^\mu$ to the background gauge field $A(\vec{q}) = \sum_I q_I A^I$. As we have seen, this field has the constant field strength $MT^-_{\mu\nu}$ so the term we have to add is

$$I_1 = \frac{M}{2}\int \mathrm{d}t T^-_{\mu\nu} x^\mu \dot{x}^\nu.$$

The minimal possible action is thus

$$I = I_0 + I_1 = M \int \mathrm{d}t \left(-1 + \frac{1}{2}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\right)^2 + \frac{i}{2}\psi_{Ai}\dot{\psi}^{Ai} + \frac{1}{2}T^-_{\mu\nu}x^\mu \dot{x}^\nu \right).$$

It turns out that this is the complete supersymmetric action, modulo irrelevant terms of higher dimension.

Once one turns on the graviphoton, one has to modify the translation generators P^μ, which no longer coincide with the canonical momenta p^μ

$$P^\mu = p^\mu - \frac{M}{2} T^{-\mu\nu} x_\nu$$

and no longer commute

$$[P_\mu, P_\nu] = iMT^-_{\mu\nu}.$$

This is not surprising for a charged particle in a constant magnetic field. We can still use the old formulas for the supercharges

$$Q^{Ai} = M\psi^{Ai}, \quad Q^{\dot{A}i} = \gamma_\mu^{A\dot{A}} P_\mu \psi^i_A,$$

and they are obviously still conserved. But since the definition of P_μ has changed, the algebra they generate is deformed – in precisely the way seen in supergravity.

Now let us evaluate the path integral for fluctuations around the particle trajectory $p \times S^1$. The answer is going to be schematically

$$\int \mathrm{d}^4 x \mathrm{d}^4 \theta \exp(-I_{\mathrm{cl}}) \cdot \frac{\sqrt{\det \mathcal{D}_F}}{\sqrt{\mathcal{D}_B}},$$

where I_{cl} is the classical action and $\det \mathcal{D}'_F$, $\det \mathcal{D}'_B$ are fermionic and bosonic one-loop determinants with zero-modes removed. As is usual in instanton physics, we remove the zero-modes from the determinants and replace them with an integral over the corresponding moduli or collective coordinates x^μ and θ^{Ai}.

One term in the classical action is $2\pi RM$, where $2\pi R$ is the circumference of the M-theory circle. However, once we compactify the time direction to a circle, we should allow for the possibility that the gauge fields A^I can have monodromies $\exp(-2\pi i\alpha^I)$ around the circle. The parameters α^I are important moduli in the Type IIA description. They are the real parts (at $\theta = 0$) of the chiral superfields $\mathcal{X}^I$ that we discussed before that are in vector multiplets, or more exactly they are the real parts of $\mathcal{Z}^I = \mathcal{X}^I/\mathcal{X}^0$. In the presence of these monodromies, a particle of charges q_I propagating around the circle gets a phase $\exp(-2\pi i q_I\alpha^I)$. It turns out that the product $\exp(-2\pi i q_I\alpha^I)\exp(-2\pi RM)$ is none other than

$$\exp(-2\pi i q_I \mathcal{Z}^I).$$

So our answer is going to be

$$\int \mathrm{d}^4x\mathrm{d}^4\theta\ \exp(-2\pi i q_I \mathcal{Z}^I)\,\frac{\sqrt{\det\,\mathcal{D}_F}}{\sqrt{\mathcal{D}_B}}. \qquad (3)$$

We still have to evaluate the determinants, but they are more or less the simplest functional determinants one will see. The fermionic determinant is completely trivial, since the fermions are free ($\mathcal{D}_F = i\mathrm{d}/\mathrm{d}t$), and the bosonic determinant is a classical example since $\mathcal{D}_B = \mathcal{D}_1\mathcal{D}_2$ where $\mathcal{D}_1 = -\mathrm{d}/\mathrm{d}t$ and $\mathcal{D}_2 = \mathrm{d}/\mathrm{d}t + T^-$, where T^- is a constant matrix. After a little work one gets a result that is part of the GV formula:

$$\int \mathrm{d}^4x\mathrm{d}^4\theta \; \exp(-2\pi i q_I \mathcal{Z}^I)\frac{T^2}{\sinh^2(\pi R T)}$$

where $T = \sqrt{T^-_{\mu\nu}T^{-\mu\nu}}$.

One has to add further contributions from the case that the BPS particle wraps k times around the circle for $k > 1$. In the preceeding calculation, this multiplies the classical action by k. It also multiplies R by k, since the particle is effectively going once around a circle of circumference $2\pi kR$. Finally, one has to divide the result by k to account for the k-fold cyclic symmetry among the different sheets of the particle orbit. So the result is that the contribution to the GV formula of a massive BPS hypermultiplet of charge $\vec{q}$ is

$$\int \mathrm{d}^4x\mathrm{d}^4\theta \sum_{k=1}^{\infty} \frac{1}{k} \exp(-2\pi i k q_I \mathcal{Z}^I) \frac{T^2}{\sinh^2(\pi k R T)}.$$

(It takes some discussion to justify ignoring the self-interactions of the BPS particles for $k > 1$, since after all the interactions between M2-branes are strong.)

We still have to go on and analyze the contribution to the GV formula of other BPS states. For massive BPS states that are not in hypermultiplets, the idea is the same, though some details are different. One uses supersymmetry – notably the fact that the graviphoton background preserves eight rather than four supercharges – to determine the appropriate Hamiltonian for these more general massive BPS states. Then one computes in much the way as I have described, with a few minor twists.

For those (few) BPS states that are *massless* in five dimensions, the “instanton” point of view does not work well, but there is a different simplification. These modes just come from 11d supergravity, and unlike the massive BPS states that come from wrapped M2-branes, they have spins ≤ 2 and we do have a good field theory that describes them, namely 11-dimensional supergravity. Actually, since these modes are zero-modes along Y, one can do the calculation in 5d supergravity.

One just needs to compute one-loop determinants for massless fields in the 5d graviphoton background. It is convenient to do this computation by making a Kaluza-Klein reduction along the M-theory circle, to reduce to a sum of one-loop determinants of 4d (massive) BPS states with different values of the Kaluza-Klein momentum. Here the calculation one has to perform really is (a sum over spins and masses of) Schwinger's classic calculation for a 4d charged particle of mass m in a constant magnetic field F.