Hubbard-Stratonovich treatment of Gross-Pitaevskii

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In my lectures, I went through a Hubbard-Stratonovich transformation of the Gross-Pitaevskii action, following Zee's work [1], but I did not explain correctly why the Goldstone boson can eventually be treated as a lagrange multiplier enforcing a divergence free condition on the four-vector whose Hodge dual winds up being the NS-NS three form field strength. Zee's treatment is correct but telegraphic. Here I try to walk through the steps more carefully.

Our starting point is the lagrangian

$$\mathcal{L}_1 = -\rho\dot{\eta} - \frac{\rho}{2m}(\nabla\eta)^2 \,, \tag{1}$$

and it's integrated over $\mathbf{R}^{3,1}$ to get part of the Gross-Pitaevskii action. η is the Goldstone boson, which in qualitative terms we want to replace with a two-form gauge field B_2. To this end, we first observe that

$$\mathcal{L}_1 = -\xi^\mu \partial_\mu \eta + \frac{m}{2\rho}\vec{\xi}^2 - \frac{m}{2\rho}\left(\vec{\xi} - \frac{\rho}{m}\nabla\eta\right)^2 \,, \tag{2}$$

where $\xi^0 = \rho$ and $\vec{\xi}$ is the Hubbard-Stratonovich field. (2) is a straightforward algebraic equality—no path integral tricks yet!

Now let's consider what's going on in the path integral when we perform Hubbard-Stratonovich. First we note that

$$\begin{aligned} e^{i\int d^4x\,\mathcal{L}_1} &= \int \mathcal{D}\vec{\xi} \, \exp\left\{ i \int d^4x \, \left[\mathcal{L}_1 + \frac{m}{2\rho}\left(\vec{\xi} - \frac{\rho}{m}\nabla\eta\right)^2 \right] \right\} \\ &= \int \mathcal{D}\vec{\xi} \, \exp\left\{ i \int d^4x \, \left[-\xi^\mu \partial_\mu \eta + \frac{m}{2\rho}\vec{\xi}^2 \right] \right\} \,. \end{aligned} \tag{3}$$

where we have defined the measure and contour so that

$$\int \mathcal{D}\vec{\xi} \, \exp\left\{ i \int d^4x \, \frac{m}{2\rho}\vec{\xi}^2 \right\} = 1 \,. \tag{4}$$

In the first equality of (3), we are assuming that the measure $\mathcal{D}\vec{\xi}$ is invariant under a shift of integration variables $\vec{\xi} \to \vec{\xi} - \frac{\rho}{m}\nabla\eta$. The second equality in (3) is a slight rearrangement of (2).

Our next step is to do the path integral over η:[1]

$$\int \mathcal{D}\eta \, e^{i\int d^4x\,\mathcal{L}_1} = \int \mathcal{D}\vec{\xi} \int \mathcal{D}\eta \, \exp\left\{ i\int d^4x \left[-\xi^\mu \partial_\mu \eta + \frac{m}{2\rho}\vec{\xi}^2 \right] \right\} , \tag{5}$$

where we have assumed that we can switch the order of integration. The first term in square brackets on the right hand side of (5) can be integrated by parts to give $\eta\partial_\mu\xi^\mu$. Then the η integral reduces to a functional delta function enforcing the constraint

$$\partial_\mu \xi^\mu = 0 \tag{6}$$

at every point on $\mathbf{R}^{3,1}$. We solve (6) by setting

$$\xi^\mu = \frac{1}{6}\epsilon^{\mu\nu\lambda\sigma} H_{\nu\lambda\sigma} \tag{7}$$

where $H_3 = dB_2$. The reason that (7) works is that (6) is implied by the equality of mixed partials acting on components of B_2. (7) means in particular that

$$\frac{m}{2\rho}\vec{\xi}^2 = \frac{m}{4\rho}\sum_{i,j} H_{0ij}^2 \, . \tag{8}$$

The H_{ijk}^2 term that is desired as part of the NS-NS two-form dynamics comes from the lagrangian I called $\mathcal{L}_2$ in the lecture. To be precise, $H_{\mu\nu\lambda} = H_{\mu\nu\lambda}^{(0)} + h_{\mu\nu\lambda}$ where the only non-zero components of $H_{\mu\nu\lambda}^{(0)}$ are $H_{123}^{(0)} = \rho_0$ and other components related by index permutations. ρ_0 is the background superfluid density, and h_{ijk} describes density fluctuations around it. We can still set $H_3 = dB_2$. To summarize, (nearly) the whole claim is

$$\begin{aligned} &\int \mathcal{D}\rho\mathcal{D}\eta \, \exp\left\{ i\int d^4x \left[-\rho\dot{\eta} - \frac{\rho}{2m}(\nabla\eta)^2 - \frac{(\nabla\rho)^2}{8m\rho} - \frac{g}{2}(\rho-\rho_0)^2 \right] \right\} \\ &\quad = \int \mathcal{D}B_2 \, \exp\left\{ i\int d^4x \left[-\frac{g}{12}\eta^{\mu\alpha}\eta^{\nu\beta}\eta^{\lambda\gamma} h_{\mu\nu\lambda}h_{\alpha\beta\gamma} - \frac{(\nabla h_{ijk})^2}{48m\rho_0} \right] \right\} , \end{aligned} \tag{9}$$

where $\eta^{\mu\nu} = \mathrm{diag}\{-\frac{1}{c^2}, 1, 1, 1\}$ and $c^2 = g\rho_0/m$.

The main way in which the claim (9) needs improvement is that it neglects the vortex strings, which sit at the locations where η has 2π monodromy. As explained in lecture,

[1] For the step (5), what we need is to assume that η is a well-defined real-valued function on $\mathbf{R}^{3,1}$. This fails in the presence of vorticity. To proceed in that case, we split $\eta = \eta_{\mathrm{vortex}} + \eta_{\mathrm{smooth}}$, where η_{vortex} is held fixed, corresponding to some fixed location of vortices in the superfluid, and η_{smooth} is the well-defined real-valued part which we regard as the fluctuating field to be integrated over in (5). Later, we can integrate over the position of the vortices to get the fully dynamical system of strings plus NS-NS two-form.

including the vortex strings leads to a term $2\pi \int_\Sigma b_2$ in the action, where b_2 is the pullback of B_2 (including the part of B_2 that gives $H^{(0)}_{ijk}$) to the string worldsheet. If we properly account for the core of the strings (which is a small region of normal phase), there is also some Nambu-like term $-\tau_{1,\mathrm{bare}} \int d^2\sigma \sqrt{-\det g_{ab}}$ that comes into the action.

There are additional ways in which the claim (9) is slightly tricky and/or incomplete:

- The superfluid density ρ_0 appearing on the right hand side of (9) should really have been ρ, and the proper thing to do is to expand it around ρ_0 to get some interactions of the H_3 field with itself. These interactions were uniformly neglected in my lectures, but in principle they are there, and a diagrammatic treatment of them could be developed.

- There's a non-trivial claim that the Jacobian between $\mathcal{D}\rho\mathcal{D}\vec{\xi}$ and the natural measure $\mathcal{D}B_2$ for the two-form gauge field can be neglected. It seems to me that there is not much possibility of a problem in the long-distance physics arising from this technical issue. Of course, to properly describe the path integral over B_2, one must develop some sort of gauge-fixing technology.

- The higher derivative term in square brackets on the right hand side of (9) gets dropped once we pass to the infrared limit.

Now let me answer a few questions during or after the lecture which I didn't fully address:

1. **Q:** It doesn't look like η enters simply enough into (2) to be treated as a Lagrange multiplier. When we integrate η out of (2), doesn't it result in some extra terms for ξ?

   A: Good point! This is why it's important to make the slight rearrangement of (2) that came up in (3). In the last expression in (3), η appears just once, and after an integration by parts, you can see that it appears as $\eta\partial_\mu\xi^\mu$.

2. **Q:** If ρ/m is treated as small, the extra dependence of (2) on η goes away. Are we relying on small ρ/m in order to treat η as a Lagrange multiplier?

   A: The slight rearrangement discussed in previous answer shows that η truly becomes a Lagrange multiplier, and we don't rely on the smallness of ρ/m or any other combination of parameters.

3. **Q:** Would it be more proper to dualize $\xi_\mu = \frac{1}{6}\epsilon^{\mu\nu\lambda\sigma}H_{\nu\lambda\sigma}$ without demanding $H_3 = dB_2$?

   A: It should be fine to make the substitution $\xi_\mu = \frac{1}{6}\epsilon^{\mu\nu\lambda\sigma}H_{\nu\lambda\sigma}$ early on, as a convenient algebraic shuffling of variables. At some point we must integrate over η, and as soon as we do so, it should enforce the condition $dH_3 = 0$, which we can immediately solve as $H_3 = dB_2$ since $\mathbf{R}^{3,1}$ is topologically trivial. In short, an early substitution of this sort should result in an equivalent treatment to the one given here.

References

[1] A. Zee, "Vortex strings and the antisymmetric gauge potential," *Nucl.Phys.* **B421** (1994) 111–124.