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String Compactification

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Overview

- Heterotic compactification
- Perturbative IIB
- F-Theory

Part I: Heterotic compactification

Heterotic string

- Bosonic fields in ten dimensions:

$$g_{MN} \quad B_{MN} \quad \phi \quad A_M$$

- Lots of symmetry: *10d* super-Poincare, *SO(32)* or *E8 x E8* gauge symmetry

- Break the symmetry to something smaller

- Kaluza-Klein Ansatz:

$$M^{1,9} = R^{1,3} \times X_6$$

X_6 small $\longleftrightarrow$ Effective four-dimensional theory

Narain compactification

- Simplest possibility: $X_6 = T^6$
- Often referred to as Narain compactification
- Admits an elegant formulation using even, self-dual momentum lattice $\Gamma^{22,6}$

Moduli space:

$$\mathcal{M} = O(22,6;\boldsymbol{Z})\backslash O(22,6;\boldsymbol{R})/O(22;\boldsymbol{R}) \times O(6;\boldsymbol{R})$$

Describes deformations of g_{MN} B_{MN} A_M along $X_6 = T^6$

- However: N=1 supersymmetry in *10d* reduces to *N=4* supersymmetry in *4d*

$\longrightarrow$ Very restrictive

- Want to consider more general compactifications with N=1 in 4d

Supersymmetry variations

The 10d supersymmetry variations of the heterotic string are given by:

$$\delta\psi_M = \nabla_M \epsilon - \frac{1}{4} H_{MAB}\Gamma^{AB}\epsilon + (Fermi)^2$$

$$\delta\lambda = F_{MN}\Gamma^{MN}\epsilon + (Fermi)^2$$

$$\delta\chi = \nabla_M \phi \Gamma^M \epsilon + \frac{1}{24} H_{MNP}\Gamma^{MNP}\epsilon + (Fermi)^2$$

We want to find non-trivial solutions to $\delta\psi_M = \delta\lambda = \delta\chi = 0$ with 4d Poincare invariance

Assume $H = 0$ and ϕ constant

We are left with:

$$\nabla_M \epsilon = 0 \qquad\qquad F_{MN}\Gamma^{MN}\epsilon = 0$$

Assume $F = 0$ for now

Covariantly constant spinors

The spinor ϵ lives in $\mathbf{16}_R$ (Majorana-Weyl)

$$M^{1,9} = R^{1,3} \times X_6$$

Decompose under $SO(1,3) \times SO(6) \subset SO(1,9)$

$$\mathbf{16}_R = (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}})$$

$$\epsilon = \epsilon_4^+ \otimes \epsilon_6^+ + \epsilon_4^- \otimes \epsilon_6^-$$

Then $\nabla_M \epsilon = 0$ implies $\nabla_m \epsilon_6^{\pm} = 0$

i.e. X_6 admits a covariantly constant spinor

This is a very strong condition on X_6. Discuss structure implied by this requirement.

Consequences (I)

Covariantly constant $\nabla_m \epsilon_6^{\pm} = 0$

Then also $\nabla_m (\epsilon_6^* \epsilon_6) = 0$ $\longrightarrow$ Rescale and set norm to one

Consider bilinear:

$$J_m^{\ n} = i\, \epsilon_6^{+*} \Gamma_m^{\ n} \epsilon_6^{+} \qquad \Gamma_m^{\ n} = \frac{1}{2}(\Gamma_m \Gamma^n - \Gamma^n \Gamma_m)$$

Then we have $J_m^{\ n} J_n^{\ p} = -\delta_m^{\ p}$ or $J^2 = -\mathbf{1}$

This is an ``*almost complex structure*"

May not necessarily come from a complex structure

Integrability condition: vanishing of the *Nijenhuis tensor*:

$$N_{mn}^{\ \ p} = J_m^{\ q} \nabla_q J_n^{\ p} - J_m^{\ q} \nabla_n J_q^{\ p} - J_n^{\ q} \nabla_q J_m^{\ p} + J_n^{\ q} \nabla_m J_q^{\ p}$$

Consequences (I)

From $\nabla_m \epsilon_6^{\pm} = 0$ and $J_m^{\ n} = i\,\epsilon_6^{*}\Gamma_m^{\ n}\epsilon_6$ we have also $\nabla J = 0$

So the Nijenhuis tensor vanishes and the complex structure is integrable

$\longrightarrow$ X_6 is a complex manifold

Convenient to use complex coordinates $z^j, \quad j = 1,2,3$

Holomorphic tangent bundle TX spanned by $\dfrac{\partial}{\partial z^j}$

Holomorphic co-tangent bundle T^*X spanned by dz^j

Consequences (II)

From $J_m^{\ n} = i\, \epsilon_6^* \Gamma_m^{\ n} \epsilon_6$ also find that $J_m^{\ p} J_n^{\ q} g_{pq} = g_{mn}$

i.e. the metric is Hermitian.

Using complex coordinates in which $J_j^{\ k} = i\, \delta_j^{\ k}$ $\quad J_{\bar{j}}^{\ \bar{k}} = -i\, \delta_{\bar{j}}^{\ \bar{k}}$

We get $g_{ij} = g_{\bar{i}\bar{j}} = 0 \quad g_{i\bar{j}}, g_{\bar{i}j} \neq 0$

Then we can construct an associated two-form (the ``Kahler form"):

$$J = i\, g_{i\bar{j}} dz^i \wedge dz^{\bar{j}} \qquad dJ = 0$$

It is closed due to covariance constancy.

A complex manifold with closed associated two-form is called a *Kahler manifold*.

Such manifolds have very nice properties, some of which we discuss later

Consequences (III)

From $\nabla_m \epsilon_6^{\pm} = 0$ also find $[\nabla_m, \nabla_n]\epsilon_6 = \frac{1}{4} R_{mnpq}\Gamma^{pq}\epsilon = 0$

With a little manipulation, see that the Ricci curvature vanishes

→ So X_6 is a *Ricci-flat Kahler manifold*

On any Kahler manifold, construct two-form from Ricci tensor:

$$c_1(TX) = \frac{1}{2\pi}[R_{i\bar{j}} dz^i \wedge dz^{\bar{j}}]$$

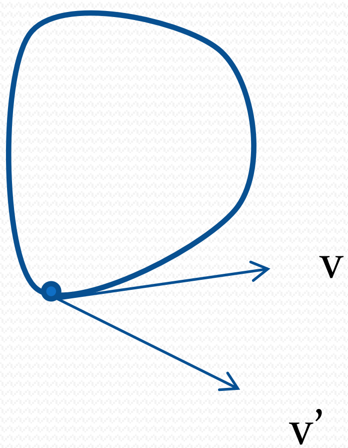
It is closed (by Bianchi) and defines a cohomology class, the *first Chern class* of X

→ Ricci flatness implies that the first Chern class vanishes

Conversely, it turns out that this is sufficient to solve $\nabla_m \epsilon_6^{\pm} = 0$, but first discuss holonomy.

Holonomy

$\gamma(t)$



Parallel transport a vector around a loop

$$\nabla_{\dot\gamma(t)} v(t) = 0 \quad v(0) = v \quad v(2\pi) = v'$$

$$\longrightarrow \qquad v' = Rv \quad R \in SO(d)$$

All loops $\longrightarrow$ subgroup of *SO(d)* called the *holonomy group*

Furthermore, close connection with curvature:

For small loops enclosing area δA^{mn} : infinitesimal rotation

$$\delta_p^{\ q} + \delta A^{mn} R_{mnp}^{\quad\ q}$$

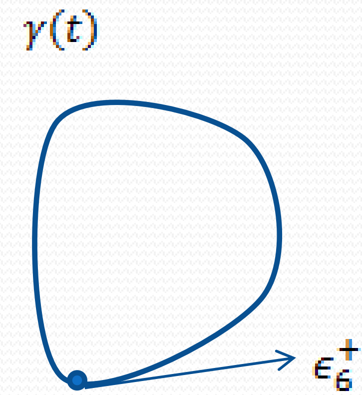
Similarly parallel transport spinors or forms around loop

Holonomy

We have equivalence $Spin(6) \cong SU(4)$

and $\epsilon_6^+ \in \mathbf{4}$ of $SU(4)$

Let's parallel transport ϵ_6^+ around a loop



Since $\nabla_m \epsilon_6^+ = 0$, just take globally defined ϵ_6^+ and restrict to $\gamma(t)$

$\longrightarrow$ ϵ_6^+ just comes back to itself

Holonomy group is subgroup of *SU(4)* that preserves ϵ_6^+

Use *SU(4)* transformation to put ϵ_6^+ in the form $(0,0,0,*)$

Subgroup that preserves ϵ_6^+ is *SU(3)*

$\longrightarrow$ Covariantly constant spinor means that X_6 has *SU(3)* holonomy

Holonomy

Kahler manifold has *U(3)* holonomy (subgroup of *SO(6)* that preserves Kahler form)

$$U(3) \cong SU(3) \times U(1)$$

Determinant comes from parallel transporting (3,0) forms

So *SU(3)* holonomy $\longrightarrow$ $\Lambda^3 T^* X$ is trivial line bundle

Another way to see this: we have a global nowhere vanishing section:

$$\Omega^{3,0} = \epsilon_6^T \Gamma_{ijk} \epsilon_6 \quad \in \Lambda^3 T^* X$$ ``holomorphic volume form"

$\longrightarrow$ Again implies $\Lambda^3 T^* X$ is trivial line bundle

Conversely, the curvature of $\Lambda^3 T^* X$ is the Ricci form $R_{i\bar{j}} dz^i \wedge dz^{\bar{j}}$

So Ricci flat & Kahler $\longrightarrow$ (3, 0) forms are parallel

$\longrightarrow$ *SU(3)* holonomy/covariantly constant spinor

Calabi-Yau manifolds

We define a Calabi-Yau manifold to be a Kahler manifold with $c_1(TX) = 0$

We have seen that Ricci-flat & Kahler implies $c_1(TX) = 0$

But converse is not obvious, since Ricci-flatness is differential geometric, and $c_1(TX) = 0$ is merely topological. Nevertheless ....

Calabi conjecture/Yau's theorem:

A compact Kahler manifold with vanishing first Chern class admits a Ricci flat metric. This metric is uniquely determined by the complex structure and the Kahler class.

This simplifies life enormously. We don't need to actually write down a Ricci flat metric or a covariantly constant spinor.

We can get away with checking $c_1(TX) = 0$

Calabi-Yau manifolds

Thus the following conditions on *X* are all equivalent:

- X admits a covariantly constant spinor ($\nabla_m \epsilon_6^+ = 0$)
- X admits a metric of *SU(3)* holonomy
- X is Kahler and admits a Ricci flat metric
- X is a Calabi-Yau manifold (Kahler and $c_1(TX) = 0$)

Examples

A canonical example of a Kahler manifold is complex projective space CP^n

$$(z_0, \dots, z_n) \cong \lambda(z_0, \dots, z_n) \qquad \lambda \in C^*$$

It is not Ricci flat.

But holomorphic submanifolds of complex projective spaces are also Kahler.

Can write explicit polynomial equations.

By choosing the degree carefully, we can make $c_1(TX) = 0$

One can show that degree *n+1* equation in $CP^n \longrightarrow c_1(TX) = 0$

- Quintic three-fold: $P_5(z) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0 \quad \subset CP^4$
- K3 surface: $P_4(z) = z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0 \quad \subset CP^3$
- Elliptic curve: $P_3(z) = z_0^3 + z_1^3 + z_2^3 = 0 \quad \subset CP^2$

Examples

Another famous example of a Calabi-Yau three-fold: the Tian-Yau manifold

Coordinates: $(z_0, z_1, z_2, z_3) \times (w_0, w_1, w_2, w_3) \in \boldsymbol{CP}^3 \times \boldsymbol{CP}^3$

Equations: $z_0^3 + z_1^3 + z_2^3 + x_3^3 = 0, \quad w_0^3 + w_1^3 + w_2^3 + w_3^3 = 0$

$$z_0 w_0 + z_1 w_1 + z_2 w_2 + z_3 w_3 = 0$$

Properties of Calabi-Yau manifolds

We are now going to discuss some general properties of Calabi-Yau manifolds.

These properties will also tell us about the structure of the effective four-dimensional theory obtained by compactifying on the Calabi-Yau.

We quickly review some properties of differentiable manifolds, and then state their complex analogues.

Hodge decomposition

Let's review some facts about manifolds.

Complex of differential forms:

$$0 \to \Omega^0(m,\boldsymbol{R}) \to \;\ldots\;\ldots. \to \Omega^d(M,\boldsymbol{R}) \to 0$$

$$d\colon \Omega^k(M,\boldsymbol{R}) \to \Omega^{k+1}(M,\boldsymbol{R}) \qquad\qquad d^2 = 0$$

De Rham cohomology:

$$H^k_{dR}(M,\boldsymbol{R}) = \frac{\{d\text{-closed k-forms}\}}{\{d\text{-exact k-forms}\}}$$

Adjoint of exterior derivative:

$$\langle d^*\alpha,\beta\rangle = \langle\alpha, d\beta\rangle \;\forall\; \alpha,\beta \qquad\qquad \langle\alpha,\beta\rangle = \int_M \alpha\wedge *\,\beta$$

Hodge Laplacian: $\Delta_d = d\,d^* + d^*d$

Hodge decomposition

The Hodge decomposition is the following orthogonal decomposition:

$$\Omega^k(m,\boldsymbol{R}) = \boldsymbol{H}^k + d\,\Omega^{k-1}(M,\boldsymbol{R}) + d^*\Omega^{k-1}(M,\boldsymbol{R})$$

"Harmonic + exact + co-exact"

In particular for closed forms: $\omega = \alpha_H + d\beta$

→ Every cohomology class has a unique harmonic representative

On a complex manifold: decomposition by (holomorphic, anti-holomorphic) indices

$$\omega_{i_1,\dots,i_p,\overline{j_1},\dots,\overline{j_q}}\, dz^{i_1}\wedge\dots\wedge dz^{i_p}\wedge dz^{\overline{j_1}}\wedge\dots\wedge dz^{\overline{j_q}} \in \Omega^{p,q}(X,\boldsymbol{C})$$

$$d = \partial + \overline{\partial}$$

$$\overline{\partial}: \Omega^{p,q}(X,\boldsymbol{C}) \to \Omega^{p,q+1}(X,\boldsymbol{C}) \qquad \partial: \Omega^{p,q}(X,\boldsymbol{C}) \to \Omega^{p+1,q}(X,\boldsymbol{C})$$

$$\overline{\partial}^2 = 0 \qquad \partial^2 = 0$$

Hodge numbers

Since $\bar\partial^2 = 0$, get a complex

$$0 \to \Omega^{p,0}(X,C) \to \;\ldots\ldots \to \Omega^{p,n}(X,C) \to 0$$

Define Dolbeault cohomology:

$$H^{p,q}_{\bar\partial}(X,C) = \frac{\{\bar\partial\text{-closed (p,q)-forms}\}}{\{\bar\partial\text{-exact (p,q)-forms}\}} \qquad h^{p,q}(X) = \dim H^{p,q}_{\bar\partial}(X,C)$$

"Hodge numbers"

Dolbeault Laplacian: $\Delta_{\bar\partial} = \overline{\partial\partial}^* + \bar\partial^*\bar\partial$

Hodge decomposition for $\bar\partial$:

$$\Omega^{p,q}(X,C) = \boldsymbol{H}^{p,q} + \bar\partial\Omega^{p,q-1}(X,C) + \bar\partial^*\Omega^{p,q+1}(X,C)$$

Every class in $H^{p,q}_{\bar\partial}(X,C)$ has a unique harmonic representative

Hodge numbers on a Kahler manifold

On a Kahler manifold it turns out that $\Delta_d = 2\,\Delta_{\bar{\partial}}$

$\longrightarrow$ Harmonic forms agree and Δ_d preserves *(p,q)* type

$$\longrightarrow \quad H^k_{dR}(X,C) = \sum_{k=p+q} H^{p,q}_{\bar{\partial}}(X,C)$$

Further symmetries:

Complex conjugation: $h^{p,q}(X) = h^{q,p}(X)$

Hodge *-isomorphism : $* : H^{p,q}_{\bar{\partial}}(X,C) \cong H^{n-p,n-q}_{\bar{\partial}}(X,C)$

On a Calabi-Yau with *SU(n)* holonomy, further have

$$h^{i,0}(X) = \begin{cases} 1 \text{ if } i = 0,n \\ 0 \text{ if } i \neq 0,n \end{cases}$$

Then we are only left with two independent Hodge numbers:

$h^{1,1}(X)$ and $h^{2,1}(X)$

Hodge diamond of a Calabi-Yau

Hodge diamond for a Calabi-Yau three-fold with *SU(3)* holonomy:

$$\begin{matrix} h^{3,0} & h^{3,1} & h^{3,2} & h^{3,3} \\ h^{2,0} & h^{2,1} & h^{2,2} & h^{2,3} \\ h^{1,0} & h^{1,1} & h^{1,2} & h^{1,3} \\ h^{0,0} & h^{0,1} & h^{0,2} & h^{0,3} \end{matrix} \longrightarrow \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & h^{2,1} & h^{1,1} & 0 \\ 0 & h^{1,1} & h^{2,1} & 0 \\ 1 & 0 & 0 & 1 \end{matrix}$$

By uniqueness of CY metric, metric deformations are complex structure and Kahler deformations.

$h^{2,1}(X)$ counts complex structure deformations

Eg. For quintic three-fold: deform to general degree five polynomial

There are 101 such deformations $\longrightarrow$ $h^{2,1}(\text{Quintic}) = 101$

$h^{1,1}(X)$ counts Kahler deformations (deformations of Kahler form *J*)

Eg for quintic: only “breathing mode” inherited from CP^4

$\longrightarrow$ $h^{1,1}(\text{Quintic}) = 1$

Moduli space of a Calabi-Yau

$$\mathcal{M}_{\text{metric}} = \mathcal{M}_{\text{complex}} \times \mathcal{M}_{\text{Kahler}}$$

$$\dim_{C} = h^{2,1}(X) \qquad \dim_{R} = h^{1,1}(X)$$

B-field deformations give further $h^{1,1}(X)$ real parameters:

$$\delta B_{mn} = b^I \omega_{I,mn} \qquad \omega_{I,mn} \in H^2(X, \boldsymbol{R})$$

We can economically think of this as deformations of a complexified Kahler form:

$$J_C = J + i\, B$$

Thus adding B-field deformations has the effect of complexifying $\mathcal{M}_{\text{Kahler}}$

Bundles

We've discussed the consequences of $\nabla_m \epsilon_6^+ = 0$ in quite some detail.

Now it is time to return to the remaining condition: $\delta\lambda = F_{mn}\Gamma^{mn}\epsilon_6^+ = 0$

This equation can be written as:

$$F^{0,2} = 0 \qquad g^{i\bar{j}}F_{i\bar{j}} = 0$$

These are called the *Hermitian Yang-Mills equations.*

We first discuss the meaning of $F^{0,2} = 0$

Split the connection into $A = A^{1,0} + A^{0,1}$. We have:

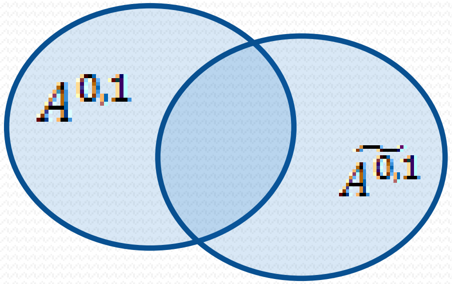
$$\left[\bar{\partial} + A^{0,1}, \bar{\partial} + A^{0,1}\right] = F^{0,2} = 0$$

So $A^{0,1}$ is pure gauge, $A^{0,1} = \Lambda^{-1} \cdot \bar{\partial}\Lambda$

We can locally set $A^{0,1} = 0$ by a (complexified) gauge transformation.

Holomorphic bundles

Now consider the gauge field on two different patches:



On the overlap we have

$$\widetilde{A^{0,1}} = \Lambda A^{0,1}\Lambda^{-1} - \bar{\partial}\Lambda \cdot \Lambda^{-1}$$

But we saw that on each patch we could set $A^{0,1} = 0$

With this choice we will have $\bar{\partial}\Lambda = 0$

i.e. the transition functions can be chosen holomorphic.

Thus the gauge field is a connection on a *holomorphic bundle.*

Stability

It turns out that curvature always decreases along holomorphic sub-bundles

If the connection satisfies $g^{i\bar{j}}F_{i\bar{j}} = 0$, then $g^{i\bar{j}}F^{U}_{i\bar{j}}$ along holomorphic sub-bundle *U* is negative (or possibly zero).

Then if *U* is a holomorphic sub-bundle of our Hermitian YM bundle *V*, it is not hard to see that the degree of *U* must be negative, where

$$\deg(U) = \frac{1}{\mathrm{vol}(X)} \int_X J \wedge J \wedge \frac{1}{2\pi} \mathrm{Tr}(F)$$

A bundle *V* is said to be *stable* if

$$U \subset V \quad \Rightarrow \quad \mathrm{slope}(U) < \mathrm{slope}(V)$$

where slope = degree/rank. A bundle is poly-stable if it is a sum of stable bundles of the same slope.

Hermitian-Yang-Mills (HYM) implies poly-stability.

Stability

Hermitian-YM implies poly-stability. Remarkably the converse is also true:

Donaldson-Uhlenbeck-Yau:

If a vector bundle V on a Kahler manifold is holomorphic and poly-stable, then there exists a unique solution to the Hermitian Yang-Mills equations.

Finding explicit solutions of the HYM equations is practically impossible.

The above says that we can get away with writing down holomorphic bundles and checking the slope of holomorphic sub-bundles.

Writing down interesting holomorphic bundles requires some more technology, which unfortunately we do not have time for. But we can give one canonical example of a Hermitian-Yang-Mills bundle.

Tangent bundle

A simple example of a HYM bundle is the holomorphic tangent bundle of a Calabi-Yau.

We have already seen that

$$[\nabla_m, \nabla_n]\epsilon_6 = \frac{1}{4} R_{mnpq}\Gamma^{pq}\epsilon = 0$$

But this is precisely $\quad F_{pq}\Gamma^{pq}\epsilon = 0$

We didn't need Donaldson-Uhlenbeck-Yau for that.

Virtually every other interesting example does require DUY.

Bundle-valued Dolbeault cohomology

Since we had $\bar{\partial}_A^2 = 0$, for holomorphic bundles we can define a generalization of the Dolbeault complex:

$$0 \to \Omega^{0,0}(X,V) \xrightarrow{\bar{\partial}_A} \ldots\ldots \xrightarrow{\bar{\partial}_A} \Omega^{0,n}(X,V) \to 0$$

$$H^p(X,V) = \frac{\{\bar{\partial}\text{-closed V-valued (0,p)-forms}\}}{\{\bar{\partial}\text{-exact V-valued (0,p)-forms}\}}$$

These cohomology groups are just what we need to understand the Kaluza-Klein reduction of the gauge sector. They have the following interpretation:

$H^0(X,V)$ $\longrightarrow$ Counts unbroken gauge generators that survive in effective 4d theory

$H^1(X,V)$ $\longrightarrow$ Counts charged matter fields in effective 4d theory

Let's illustrate this with an example.

Example: the standard embedding

We consider the *E8 x E8* heterotic string.

Take the connection for the first *E8* bundle to be the spin connection of *X_6*.

Under the maximal subgroup $SU(3) \times E_6 \subset E_8$, the adjoint representation decomposes as:

$$\mathbf{248} = (\mathbf{3}, \mathbf{27}) + (\overline{\mathbf{3}}, \overline{\mathbf{27}}) + (\mathbf{1}, \mathbf{78}) + (\mathbf{8}, \mathbf{1})$$

At the level of Dolbeault cohomology, this yields

$$H^p(X, V_{E_8}) = H^p(TX) \otimes \mathbf{27} + H^p(T^*X) \otimes \overline{\mathbf{27}} + H^p(O_X) \otimes \mathbf{78} + H^p(\mathrm{End}_0(TX)) \otimes \mathbf{1}$$

The 4d spectrum is then:

$p = 0$ Gauge field in **78** , the adjoint of E_6

$p = 1$ $h^1(TX) = h^{2,1}(X)$ chirals in **27**;

$H^1(T^*X) = h^{1,1}(X)$ chirals in $\overline{\mathbf{27}}$

$\longrightarrow$ *E_6* Grand Unified Theory with $h^{2,1}(X) - h^{1,1}(X) = -\chi(X)/2$ generations