A theory for symmetry protected topological order

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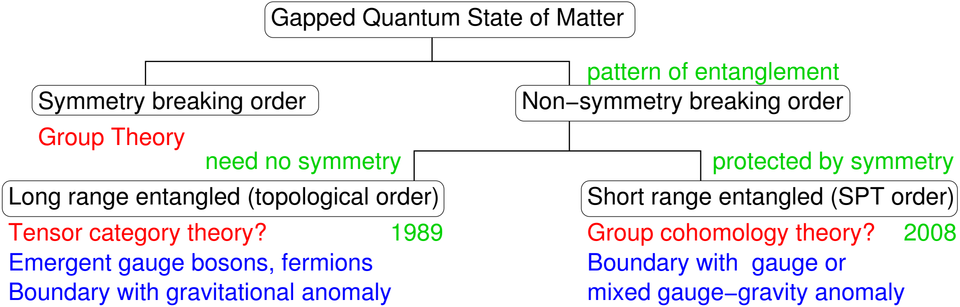
Try to classify quantum states of matter

Quantum states of matter:

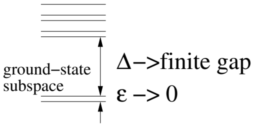
- gapless states – Very hard beyond 1+1D. I have no clue
- gapped states – A classification maybe possible:



- Group theory classifies 230 crystal orders in 3D space.
- What classifies SPT orders?
- What classifies topological orders?

What is a gapped state (or a gapped system)

It is not just the energy spectrum has a gap.



- We need to take thermal dynamical limit.

\- But how to take large-size limit without translation symmetry?

- Gapped state may have gapless boundary.

\- Avoid boundary by putting the system on manifold without boundary, but the definition will depend on "shapes/topologies" of the manifold.

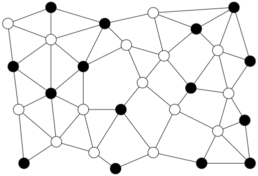
\- How to define gapped state by putting the system on a "ball"?

- Help us to understand what is the input to even define gapped phases

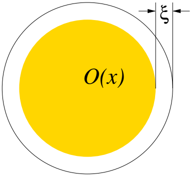
  $\to$ Mathematical foundation of gapped phases.

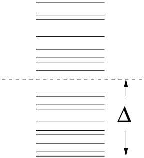
Def. of gapped liquid phase w/o translation symmetry

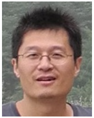
- A **local** Hamiltonian $H = \sum O_{ij} + O_{ijk} + \cdots$ on graph w/ a shape of a ball.



- A **bulk-gapped** *local* Hamiltonian:
  Excitations in bulk are all gapped.
  $|\langle\Phi_{\text{grnd}}|\hat{O}_{\text{bluk local}}|\Phi_{\text{grnd}}\rangle - \langle\Phi_{\text{exc}}|\hat{O}_{\text{bluk local}}|\Phi_{\text{exc}}\rangle| < \mathrm{e}^{-\frac{\text{distance to boundary}}{\xi}}$
  where $|\Phi_{\text{exc}}\rangle$ has an energy less than, say, $\Delta/2$.
  All low energy states are locally indistinguishable in the bulk.
  $\to$ unique bulk ground state on a "ball".





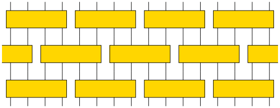


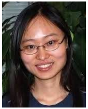
Kong-Wen 15

Def. of gapped liquid phase w/o translation symmetry

- A **gapped liquid phase** = an equivalent class of sequences of bulk-gapped local Hamiltonians H_{N_k} with size $N_k \to \infty$, where the equivalence relations are generated by

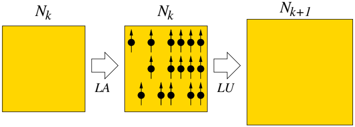
a) $H_{N_k} \sim (LU) H'_{N_k} (LU)^\dagger$

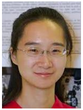




Chen-Gu-Wen, arXiv:1004.3835

b) $H_{N_{k+1}} \sim H_{N_k} \otimes H_{\text{tri}}$
$H_{\text{tri}} = -\sum S_i^z$

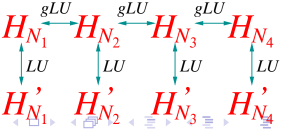




Zeng-Wen, arXiv:1406.5090

Two sequences of Hamiltonians:

$H_{N_1}, H_{N_2}, H_{N_3}, H_{N_4}, \cdots$
$H'_{N_1}, H'_{N_2}, H'_{N_3}, H'_{N_4}, \cdots$
$N_{k+1} = sN_k, \ s \sim 2$

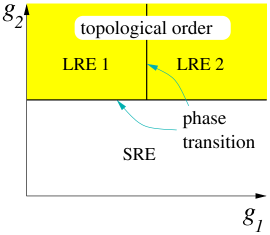


Gapped liquid phases = Topological orders

Even w/o symmetry breaking (H has no symmetry), we can have

- different gapped liquid phases
  = different **topological orders** Wen PRB 40, 7387 (1989)
  = different patterns of long range entanglement

- *Short-range entanglement (SRE):*
  = LU equivalent to product state
- *Long-range entanglement (LRE):*
  = LU inequivalent to product state



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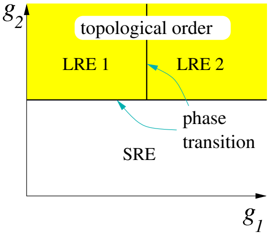


- **Counter examples** (non-liquid gapped states):

- Landau symmetry breaking state
- 3D layered quantum Hall states
- Haah's 3D cubic-code state

Jeongwan Haah, Phys. Rev. A 83, 042330 (2011) arXiv:1101.1962

Examples for gapped liquid phases (topological orders)

- **Examples**:

- Product state (trivial topological order) $H_{\text{tri}} = -\sum \sigma_i^z$
- 1+1D p-wave superconductor (fermionic) Kiteav cond-mat/0010440,
- 2+1D $p + \mathrm{i}p$ superconductor (fermionic) Read-Green cond-mat/9906453,
- IQH states (2+1D fermionic) Klitzing-Dorda-Pepper PRL 45, 494 (80),
- FQH states (2+1D fermionic) Tsui-Stormer-Gossard, PRL 48, 1559 (82),
- Chiral-spin liquids (bosonic)

Kalmeyer-Laughlin PRL 59, 2095 (87), Wen-Wilczek-Zee PRB 39, 11413 (89),

A realization by spin-1/2 on Kagome lattice

$H = J_1 \sum_{1\text{st}} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{2\text{nd}} \mathbf{S}_i \cdot \mathbf{S}_j + J_3 \sum_{3\text{rd}} \mathbf{S}_i \cdot \mathbf{S}_j$

$J_2 = J_3 > 0.1 J_1$ Gong-Zhu-Balents-Sheng arXiv:1412.1571

- Z_2-spin liquid Read-Sachdev PRL 66, 1773 (91), Wen PRB 44, 2664 (91)
- Pfaffian state $\mathcal{A}(\frac{1}{z_1-z_2}\frac{1}{z_3-z_3}..)\prod(z_i-z_j)^2$ Moore-Read NPB 360, 362 (91)
- $SU(2)_2$ state $\Psi_{\nu=1}(z_i)\Psi_{\nu=2}(z_i)\Psi_{\nu=2}(z_i)$ Wen PRL 66, 802 (91)

The last two examples are non-abelian with the same non-abelian statistics: Ising $\times$ $U(1)$ or Ising $\times$ $U^2(1)$.

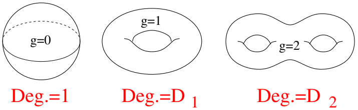
Quantitative/macroscopic characterization of topo. orders

We conjectured that topological order can be completely defined via only two topological properties (at least in 2D):

Wen IJMPB 4, 239 (90); KeskiVakkuri-Wen IJMPB 7, 4227 (93)

(1) **Topological ground state degeneracy** D_g

- degenerate only in size $\to \infty$ limit
- robust against any impurities
- depend on topology of space



Wen PRB 40, 7387 (89), Wen-Niu PRB 41, 9377 (90)

(2) **Non-Abelian goemetric phases** of the degenerate ground state from local and global deformation of space manifold.

- Local deformation detects grav. Chern-Simons term $e^{i \frac{2\pi c}{24} \int_{M^2 \times S^1} \omega_3}$
- Global deformation of torus: 90° rotation $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = S_{\alpha\beta}|\Psi_\beta\rangle$

Dehn twist: $|\Psi_\alpha\rangle \to |\Psi'_\alpha\rangle = T_{\alpha\beta}|\Psi_\beta\rangle$

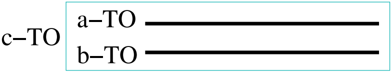
S, T generate a rep. of modular group: $S^2 = (ST)^3 = C, C^2 = 1$

Wen IJMPB 4, 239 (90); KeskiVakkuri-Wen IJMPB 7, 4227 (93)

The above properties are robust against any impurities!

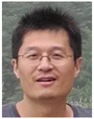
Monoid and group structures of topological orders

- Let $\mathcal{S}_{d+1} = \{a, b, c, \cdots\}$ be a set of topologically ordered phases in d spatial dimensions.
  Stacking a-TO state and b-TO state $\to$ a c-TO state:
  $a \boxtimes b = c, \quad a, b, c \in \mathcal{S}_{d+1}$



- $\boxtimes$ make $\mathcal{S}_{d+1}$ a monoid (a group without inverse). Number of the types of topological excitations $N_c = N_a N_b$.
- Some topological orders have inverse $\to$ **invertible topological orders (iTO)** which form an abelian group.
  A topological order is invertible iff it has no non-trivial topological excitations. Kong-Wen arXiv:1405.5858; Freed arXiv:1406.7278
- **Examples**:
  1. 1+1D p-wave SC (fermionic). 2. 2+1D IQH (fermionic).
  3. 2+1D $p + \mathrm{i}p$ SC (fermionic). 4. 2+1D E_8 QH (bosonic)

Classification of (potential) invertible topological orders



- Bosonic iTO:

$0+1D$	$1+1D$	$2+1D$	$3+1D$	$4+1D$	$5+1D$	$6+1D$
0	0	$\mathbb{Z}$ E_8	0	$\mathbb{Z}_2$	0	$\mathbb{Z}\oplus\mathbb{Z}$
0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0

Kapustin arXiv:1404.6659; Kong-Wen arXiv:1405.5858; Wen arXiv:1506.05768

- Fermionic iTO:

$0+1D$	$1+1D$	$2+1D$	$3+1D$	$4+1D$	$5+1D$	$6+1D$
$\mathbb{Z}_2$	$\mathbb{Z}_2$ p-wave	$\mathbb{Z}$ $p+\mathrm{i}p$	0	0	0	$\mathbb{Z}\oplus\mathbb{Z}$
$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

Kapustin-Thorngren-Turzillo-Wang arXiv:1406.7329

Kitaev cond-mat/0010440; Read-Green cond-mat/9906453

Classification of (potential) invertible topological orders

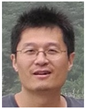
(Potential) invertible topological orders are classified by possible gravitational topological terms

- In 2+1D by possible gravitational Chern-Simons terms $\mathrm{e}^{\mathrm{i}\,\frac{2\pi c}{24}\int_{M^3}\omega_3}$, which not well defined

- Using $\mathrm{d}\omega_3 = p_1$, we define $\mathrm{e}^{\mathrm{i}\,\frac{2\pi c}{24}\int_{M^3=\partial M^4}\omega_3} = \mathrm{e}^{\mathrm{i}\,\frac{2\pi c}{24}\int_{M^4} p_1}$ which is well defined only when $\mathrm{e}^{\mathrm{i}\,\frac{2\pi c}{24}\int_{M^4} p_1} = 1$ on any closed M^4.
- Bosonic: since $\int_{M^4} p_1 = 3 \times \text{int.}$, $\to c = 8 \times \text{int.}$
- Fermionic: since $\int_{spinM^4} p_1 = 48 \times \text{int.}$ $\to c = \frac{1}{2} \times \text{int.}$

In the above we require the gravitational Chern-Simons terms to be well defined on any space-time manifolds.

- For condensed matter Hamiltonian systems, we only require the gravitational Chern-Simons terms to be well defined on space-time of form $M^2 \rtimes S^1$ (mapping torus). In this case:



- Bosonic: since $\int_{M^2 \rtimes \Sigma^2} p_1 = 12 \times \text{int.}$, $\to c = 2 \times \text{int.}$
- Fermionic: since $\int_{spin(M^2 \rtimes \Sigma^2)} p_1 = 48 \times \text{int.}$, $\to c = \frac{1}{2} \times \text{int.}$

Kong-Wen arXiv:1405.5858

2+1D bosonic topo. orders (up to invertibles) via S, T

Dim of S, T = # of topological types > 1.

- There is a basis such that T is diagonal, S unitary & symmetric

$$T_{ij} = \mathrm{e}^{2\pi\,\mathrm{i}\, s_i}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,\frac{c}{24}}\delta_{ij}, \quad S_{1i} > 0, \quad N_k^{ij} = \sum_l \frac{S_{li}S_{lj}(S_{lk})^*}{S_{1l}} = \text{integer} \geq 0.$$

$$(ST)^3 = S^2 = C, \quad C^2 = 1, \quad C_{ij} = N_1^{ij}.$$

s_i: spin of i^{th} type of particle. N_k^{ij}: fusion coeff. of the particles.

- N_k^{ij} satisfy

$$N_k^{ij} = N_k^{ji}, \quad N_j^{1i} = \delta_{ij}, \quad \sum_{k=1}^{N} N_1^{ik} N_1^{kj} = \delta_{ij},$$

$$\sum_{m=1}^{n} N_m^{ij} N_l^{mk} = \sum_{n=1}^{n} N_l^{in} N_n^{jk} \text{ or } N_k N_i = N_i N_k$$

where $i, j, \cdots = 1, 2, \cdots, n$, and the matrix N_i is given by $(N_i)_{kj} = N_k^{ij}$. In fact N_1^{ij} defines a charge conjugation $i \to \bar{i}$:

$$N_1^{ij} = \delta_{\bar{i}j}.$$

2+1D bosonic topo. orders (up to invertibles) via S, T

There exist a c (mod 8) to make s_i to satisfy the following conditions:

- N^{ij}_k and s_i satisfy

$$\sum_j \tilde M_{ij} s_j = 0 \text{ mod } 1,$$

where $\tilde M_{ij} = \delta_{ij} \frac{4}{3} \sum_k M_{ik} - M_{ij} = \text{integer}, \ M_{ij} = 2N^{\bar i \bar i}_j N^{ij}_i + N^{ii}_j N^{j\bar i}_i$

- s_i, S_{ij} satisfy

$$S_{ij} = \frac{1}{\sqrt{\sum_i d_i^2}} \sum_k N^{ij}_k \mathrm{e}^{2\pi\,\mathrm{i}\,(s_i + s_j - s_k)} d_k.$$

where d_i is the largest eigenvalue of the matrix N_i.

- Let $\nu_i = \frac{1}{D^2} \sum_{jk} N^{jk}_i d_j d_k \,\mathrm{e}^{4\pi\,\mathrm{i}\,(s_j - s_k)}$. Then, we also have $\nu_i = 0$ if $i \neq \bar i$, and $\nu_i = \pm 1$ if $i = \bar i$.

2+1D bosonic topo. orders (up to invertibles) via S, T

Wen arXiv:1506.05768 $\mathbb{Z}_{16}$: minimal bosonic topo. orders with a fermion

N_c^B	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	wave func.	N_c^B	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	wave func.
1_1^B	1	0					
2_1^B	$1, 1$	$0, \frac{1}{4}$	$\prod(z_i - z_j)^2$	2_{-1}^B	$1, 1$	$0, -\frac{1}{4}$	$\prod(z_i^* - z_j^*)^2$
$2_{14/5}^B$	$1, \zeta_3^1$	$0, \frac{2}{5}$		$2_{-14/5}^B$	$1, \zeta_3^1$	$0, -\frac{2}{5}$	
3_2^B	$1, 1, 1$	$0, \frac{1}{3}, \frac{1}{3}$		3_{-2}^B	$1, 1, 1$	$0, -\frac{1}{3}, -\frac{1}{3}$	
$3_{8/7}^B$	$1, \zeta_5^1, \zeta_5^2$	$0, -\frac{1}{7}, \frac{2}{7}$		$3_{-8/7}^B$	$1, \zeta_5^1, \zeta_5^2$	$0, \frac{1}{7}, -\frac{2}{7}$	
$3_{1/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{1}{16}$		$3_{-1/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{1}{16}$	
$3_{3/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{3}{16}$	Ψ_{Pfaffian}	$3_{-3/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{3}{16}$	
$3_{5/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{5}{16}$	$\Psi_{\nu=2}^2$ $SU(2)_2$	$3_{-5/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{5}{16}$	
$3_{7/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, \frac{7}{16}$		$3_{-7/2}^B$	$1, 1, \zeta_2^1$	$0, \frac{1}{2}, -\frac{7}{16}$	
$4_0^{B,a}$	$1, 1, 1, 1$	$0, 0, 0, \frac{1}{2}$		4_4^B	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	
4_1^B	$1, 1, 1, 1$	$0, \frac{1}{8}, \frac{1}{8}, \frac{1}{2}$		4_{-1}^B	$1, 1, 1, 1$	$0, -\frac{1}{8}, -\frac{1}{8}, \frac{1}{2}$	
4_2^B	$1, 1, 1, 1$	$0, \frac{1}{4}, \frac{1}{4}, \frac{1}{2}$		4_{-2}^B	$1, 1, 1, 1$	$0, -\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}$	
4_3^B	$1, 1, 1, 1$	$0, \frac{3}{8}, \frac{3}{8}, \frac{1}{2}$		4_{-3}^B	$1, 1, 1, 1$	$0, -\frac{3}{8}, -\frac{3}{8}, \frac{1}{2}$	
$4_0^{B,b}$	$1, 1, 1, 1$	$0, 0, \frac{1}{4}, -\frac{1}{4}$	$\prod(z_i - z_j)^4$	$4_{9/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, -\frac{1}{4}, \frac{3}{20}, \frac{2}{5}$	
$4_{-9/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{4}, -\frac{3}{20}, -\frac{2}{5}$		$4_{19/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{4}, -\frac{7}{20}, \frac{2}{5}$	
$4_{-19/5}^B$	$1, 1, \zeta_3^1, \zeta_3^1$	$0, -\frac{1}{4}, \frac{7}{20}, -\frac{2}{5}$	$\Psi_{\nu=3}^2$ $SU(2)_3$	$4_0^{B,c}$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, \frac{2}{5}, -\frac{2}{5}, 0$	
$4_{12/5}^B$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, -\frac{2}{5}, -\frac{2}{5}, \frac{1}{5}$		$4_{-12/5}^B$	$1, \zeta_3^1, \zeta_3^1, \zeta_3^1\zeta_3^1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{1}{5}$	
$4_{10/3}^B$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$	$0, \frac{1}{3}, \frac{2}{9}, -\frac{1}{3}$		$4_{-10/3}^B$	$1, \zeta_7^1, \zeta_7^2, \zeta_7^3$	$0, -\frac{1}{3}, -\frac{2}{9}, \frac{1}{3}$	
5_0^B	$1, 1, 1, 1, 1$	$0, \frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}$		5_4^B	$1, 1, 1, 1, 1$	$0, \frac{2}{5}, \frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}$	
$5_2^{B,a}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, \frac{1}{3}$		$5_2^{B,b}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, \frac{1}{3}$	
$5_{-2}^{B,b}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, \frac{1}{8}, -\frac{3}{8}, -\frac{1}{3}$		$5_{-2}^{B,a}$	$1, 1, \zeta_4^1, \zeta_4^1, 2$	$0, 0, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{3}$	
$5_{16/11}^B$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, -\frac{2}{11}, \frac{2}{11}, \frac{1}{11}, -\frac{5}{11}$		$5_{-16/11}^B$	$1, \zeta_9^1, \zeta_9^2, \zeta_9^3, \zeta_9^4$	$0, \frac{2}{11}, -\frac{2}{11}, -\frac{1}{11}, \frac{5}{11}$	
$5_{18/7}^B$	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, -\frac{1}{7}, -\frac{1}{7}, \frac{1}{7}, \frac{3}{7}$		$5_{-18/7}^B$	$1, \zeta_5^2, \zeta_5^2, \zeta_{12}^2, \zeta_{12}^4$	$0, \frac{1}{7}, \frac{1}{7}, -\frac{1}{7}, -\frac{3}{7}$	

2+1D fermionic topological orders (up to invertibles)

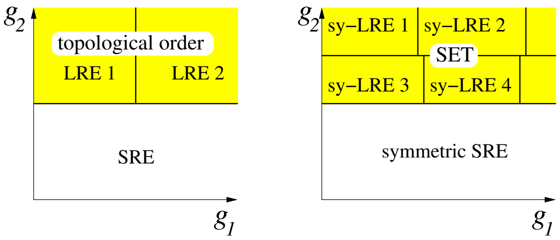
Classified by **modular BFC over sRep(Z_2^f)**.

Lan-Wen arXiv:1507.04673

N_c^F	S_{top}	D^2	$d_1, d_2, \cdots$	$s_1, s_2, \cdots$	
2_0^F	0	2	$1, 1$	$0, \frac{1}{2}$	trivial $\mathcal{F}_0$
4_0^F	0.5	4	$1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{4}, -\frac{1}{4}$	$\mathcal{F}_0 \otimes 2_1^B$
$4_{1/5}^F$	0.9276	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, \frac{1}{10}, -\frac{2}{5}$	$\mathcal{F}_0 \otimes 2_{-14/5}^B$
$4_{-1/5}^F$	0.9276	7.2360	$1, 1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{10}, \frac{2}{5}$	$\mathcal{F}_0 \otimes 2_{14/5}^B$
$4_{1/4}^F$	1.3857	13.6568	$1, 1, \zeta_6^2, \zeta_6^2$	$0, \frac{1}{2}, -\frac{1}{4}, \frac{1}{4}$	$\mathcal{F}_{(A_1,6)}$
6_0^F	0.7924	6	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, \frac{1}{6}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{3}$	$\mathcal{F}_0 \otimes 3_{-2}^B$
6_0^F	0.7924	6	$1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, -\frac{1}{6}, \frac{1}{3}, -\frac{1}{6}, \frac{1}{3}$	$\mathcal{F}_0 \otimes 3_2^B$
6_0^F	1	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{16}, -\frac{7}{16}$	$\mathcal{F}_0 \otimes 3_{1/2}^B$
6_0^F	1	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{16}, -\frac{5}{16}$	$\mathcal{F}_0 \otimes 3_{3/2}^B$
6_0^F	1	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{5}{16}, -\frac{3}{16}$	$\mathcal{F}_0 \otimes 3_{-3/2}^B$
6_0^F	1	8	$1, 1, 1, 1, \zeta_2^1, \zeta_2^1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{7}{16}, -\frac{1}{16}$	$\mathcal{F}_0 \otimes 3_{-1/2}^B$
$6_{1/7}^F$	1.6082	18.5916	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, \frac{5}{14}, -\frac{1}{7}, -\frac{3}{14}, \frac{2}{7}$	$\mathcal{F}_0 \otimes 3_{8/7}^B$
$6_{-1/7}^F$	1.6082	18.5916	$1, 1, \zeta_5^1, \zeta_5^1, \zeta_5^2, \zeta_5^2$	$0, \frac{1}{2}, -\frac{5}{14}, \frac{1}{7}, \frac{3}{14}, -\frac{2}{7}$	$\mathcal{F}_0 \otimes 3_{-8/7}^B$
6_0^F	2.2424	44.784	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, \frac{1}{3}, -\frac{1}{6}, 0, \frac{1}{2}$	primitive
6_0^F	2.2424	44.784	$1, 1, \zeta_{10}^2, \zeta_{10}^2, \zeta_{10}^4, \zeta_{10}^4$	$0, \frac{1}{2}, -\frac{1}{3}, \frac{1}{6}, 0, \frac{1}{2}$	$\mathcal{F}_{(A_1,10)}$
8_0^F	1	8	$1, 1, 1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{8}, -\frac{3}{8}, \frac{1}{8}, -\frac{3}{8}$	$\mathcal{F}_0 \otimes 4_1^B$
8_0^F	1	8	$1, 1, 1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}, \frac{1}{4}$	$\mathcal{F}_0 \otimes 4_0^B$
8_0^F	1	8	$1, 1, 1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{3}{8}, -\frac{1}{8}, \frac{3}{8}, -\frac{1}{8}$	$\mathcal{F}_0 \otimes 4_{-1}^B$
8_0^F	1	8	$1, 1, 1, 1, 1, 1, 1, 1$	$0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, 0$	$\mathcal{F}_0 \otimes 4_0^B$
$8_{1/5}^F$	1.4276	14.4720	$1, 1, 1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{10}, -\frac{2}{5}, \frac{7}{20}, -\frac{3}{20}$	$\mathcal{F}_0 \otimes 4_{-9/5}^B$
$8_{-1/5}^F$	1.4276	14.4720	$1, 1, 1, 1, \zeta_3^1, \zeta_3^1, \zeta_3^1, \zeta_3^1$	$0, \frac{1}{2}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{10}, \frac{2}{5}, \frac{3}{20}, -\frac{7}{20}$	$\mathcal{F}_0 \otimes 4_{9/5}^B$

Symmetric gapped liquid phases = enriched topo. order

- A **symmetric gapped liquid phase**
  $\equiv$ a symmetry enriched topological (**SET**) order
  = an equivalent class of gapped quantum states, with **symmetric** equivalence relations generated by
  a) $H \sim (symm.LU)H(symm.LU)^\dagger$
  b) $H \sim H \otimes H_{\text{tri}}$, with $H_{\text{tri}} = \mathbf{S}_1 \cdot \mathbf{S}_2 + \mathbf{S}_3 \cdot \mathbf{S}_4 + \cdots$

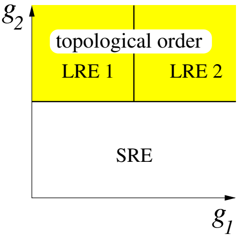


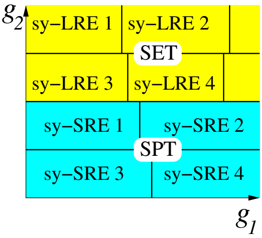
Symmetric gapped liquid phases w/o topo. order

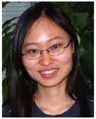
- Gapped liquid phases with symmetry and no topological order
  $\to$ **symmetry protected topological/trivial (SPT) order**

$(LU)H_{\mathrm{SPT}}(LU)^\dagger \sim H_{\mathrm{tri}}$ $\to$ trivial topological order

$(symm.LU)H_{\mathrm{SPT}}(symm.LU)^\dagger \nsim H_{\mathrm{tri}}$ $\to$ non-trivial SPT order







Gu-Wen arXiv:0903.1069; Chen-Gu-Wen arXiv:1004.3835

- **Examples**:
  1. Haldane phase of spin-1 chain (bosonic) Haldane 83
  2. Topological insulators (fermionic).

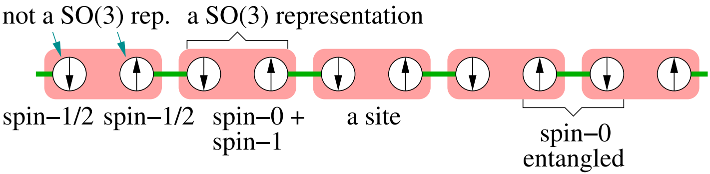
2D: Kane-Mele 05; Bernevig-Zhang 06; 3D: Moore-Balents 07; Fu-Kane-Mele 07

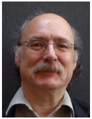
An $SO(3)$ SPT state in spin-1 chain (Haldane phase)

Affleck-Kennedy-Lieb-Tasaki PRL 59, 799 (1987)

- spin-1 chain – a spin-disordered state:

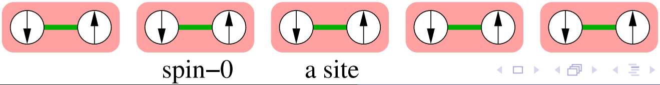
$$H_{\text{AKLT}} = \sum_i [\mathbf{S}_i \cdot \mathbf{S}_{i+1} + \frac{1}{3}(\mathbf{S}_i \cdot \mathbf{S}_{i+1})^2] \quad (+U \sum_i P(S_i = 0))$$





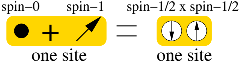
- If we do not project out the spin-0 state and consider spin-1, 0 chain → ideal $SO(3)$ SPT state (RG fixed point). Gu-Wen arXiv:0903.1069

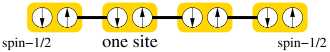
- Degenerate spin-$1/2$ doublet at each boundary, if we do not break the $SO(3)$ symmetry.
- The above is a different gapped symmetric phase than the local singlet state of spin-1, 0 chain

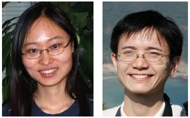


A Z_2 SPT state on square lattice   Chen-Liu-Wen arXiv:1106.4752

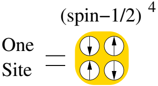
- **Haldane phase with $SO(3)$ symm.**:
  spin-1/2 is not a rep. of $SO(3)$

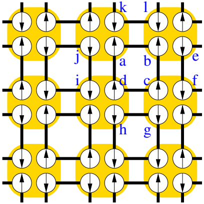




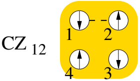


- **2D SPT phase with Z_2 symm.**:





  - Physical states on each site:
    $(\text{spin-}\frac{1}{2})^4 = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle$



  - The ground state wave function:
    $|\Psi_{CZX}\rangle = \otimes_{\text{all squares}}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$

A Z_2 SPT state on square lattice Chen-Liu-Wen arXiv:1106.4752

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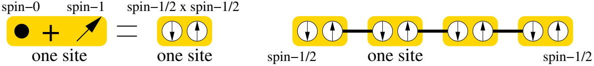

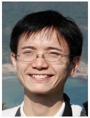

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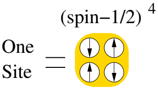
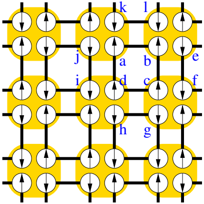
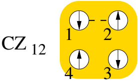




  - Physical states on each site:
    $(\text{spin-}\frac{1}{2})^4 = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle$



  - The ground state wave function:
    $|\Psi_{CZX}\rangle = \otimes_{\text{all squares}}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$
  - The on-site Z_2 symmetry: (acting on each site $|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle$):
    $U_{CZX} = U_{CZ}U_X, \quad U_X = X_1X_2X_3X_4, \quad U_{CZ} = CZ_{12}CZ_{23}CZ_{34}CZ_{41}$
    $CZ_{ij} : |\uparrow\uparrow\rangle \to |\uparrow\uparrow\rangle, \ |\uparrow\downarrow\rangle \to |\uparrow\downarrow\rangle, \ |\downarrow\uparrow\rangle \to |\downarrow\uparrow\rangle, \ |\downarrow\downarrow\rangle \to -|\downarrow\downarrow\rangle$

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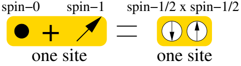
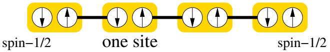
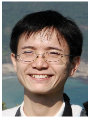
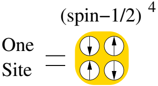
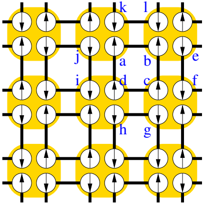






- **2D SPT phase with Z_2 symm.**:





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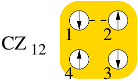

  - The ground state wave function:
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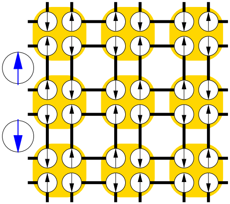
    $CZ_{ij} : |\uparrow\uparrow\rangle \to |\uparrow\uparrow\rangle, \ |\uparrow\downarrow\rangle \to |\uparrow\downarrow\rangle, \ |\downarrow\uparrow\rangle \to |\downarrow\uparrow\rangle, \ |\downarrow\downarrow\rangle \to -|\downarrow\downarrow\rangle$
  - Z_2 symm. Hamiltonian $H = \sum_\square H_p, \ H_p = -X_{abcd}P_{ef}P_{gh}P_{ij}P_{kl}$,
    $X_{abcd} = |\uparrow\uparrow\uparrow\uparrow\rangle\langle\downarrow\downarrow\downarrow\downarrow| + |\downarrow\downarrow\downarrow\downarrow\rangle\langle\uparrow\uparrow\uparrow\uparrow|, \ P = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|.$

Edge excitations for the 2D Z_2 SPT state

- Bulk Hamiltonian $H = \sum_\Box H_p$, $H_p = -X_{abcd} P_{ef} P_{gh} P_{ij}, P_{kl}$,
  $X_{abcd} = |\uparrow\uparrow\uparrow\uparrow\rangle\langle\downarrow\downarrow\downarrow\downarrow| + |\downarrow\downarrow\downarrow\downarrow\rangle\langle\uparrow\uparrow\uparrow\uparrow|$, $P = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|$.



- Edge excitations: *gapless or break the $\mathbb{Z}_2$ symmetry, robust against any perturbations that do not break the $\mathbb{Z}_2$ symmetry.*
- Edge effective spin $|\tilde{\uparrow}\rangle$ and $|\tilde{\downarrow}\rangle$.
- Edge eff. Z_2 symm. : $\tilde{U}_{Z_2} = \prod_i CZ_{i,i+1} \prod_i \tilde{X}_i$
  which cannot be written as $U_{Z_2} = \prod_i O_i$, *such as* $U_{Z_2} = \prod \tilde{X}_i$.
  Not an on-site symmetry!
- Edge effective Ham. ($c = 1$ gapless if the Z_2 is not broken)
  $$H = \sum \Big(- J\tilde{Z}_i\tilde{Z}_{i+1} + B_x[\tilde{X}_i + \tilde{Z}_{i-1}\tilde{X}_i\tilde{Z}_{i+1}] + B_y[\tilde{Y}_i - \tilde{Z}_{i-1}\tilde{Y}_i\tilde{Z}_{i+1}]\Big)$$
  $$H = \sum(\tilde{X}_i + \tilde{Z}_{i-1}\tilde{X}_i\tilde{Z}_{i+1}) \text{ dual to } H = -\sum(X_iX_{i+1} + Y_iY_{i+1})$$
- Non-on-site $\tilde{U}_{Z_2}$ $\to$ **anomalous symmetry.** Not gaugable.
  On-site U_{Z_2} $\to$ **anomaly-free symm.** – the usual global symm.

Field theory for edge of the 2+1D Z_2 SPT state

- The primary field of $U(1)$ current algebra $V_{l,m}$ has dimensions $(h_R, h_L) = (\frac{(l+2m)^2}{8}, \frac{(l-2m)^2}{8})$.
- The Z_2 symmetry action $V_{l,m} \to (-)^{l+m} V_{l,m}$
- Edge effective (chiral boson) theory: Chen-Wen arXiv:1206.3117

$$\mathcal{L}_{\text{edge}} = -\frac{1}{4\pi}(\partial_x\phi_1\partial_t\phi_2 + \partial_x\phi_2\partial_t\phi_1) - \frac{1}{4\pi}V(\partial_x\phi_1\partial_x\phi_1 + \partial_x\phi_2\partial_x\phi_2)$$

where $\phi_1 \sim \phi_1 + 2\pi$, $\phi_2 \sim \phi_2 + 2\pi$.

- Z_2 symmetry $= (\phi_1, \phi_2) \to (\phi_2, \phi_1)$
- Choose $\phi_\pm = \phi_1 \pm \phi_2$ (Under Z_2: $\phi_\pm \to \pm\phi_\pm$)

$$\mathcal{L}_{\text{edge}} = -\frac{1}{8\pi}(\partial_x\phi_+\partial_t\phi_+ - \partial_x\phi_-\partial_t\phi_-) - \frac{1}{8\pi}V[(\partial_x\phi_+)^2 + (\partial_x\phi_-)^2]$$

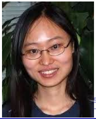
ϕ_+ right movers with no Z_2 charge. ϕ_- left movers with Z_2 charge.

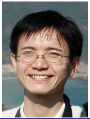
- The 1+1D theory has a Z_2 anomaly
  $\to$ It is the edge of a 2+1D Z_2 SPT state.

Non-linear σ-model (NLσM) for generic SPT states

- Consider an $d+1$D system: $S = \int d^dx dt\ \frac{1}{2\lambda}|g^{-1}(x)\partial g(x)|^2$,
  Symmetry $g(x) \to hg(x)$. h, g are unitary matrices in G.
- λ is small $\to$ the ground state is ordered $\langle g(x,t)\rangle = g_0$.
  λ is large $\to$ the ground state is disordered $\langle g(x,t)\rangle = 0$.
- The fixed point action of the disordered phases:
  Under RG, $\lambda \to \infty$ $\to$ the disordered symmetric phase is described by a fixed point theory $S_{\text{fixed}} = 0$ or $\mathrm{e}^{-S_{\text{fixed}}} = 1$.
  All correlations are short ranged. All excitations are gapped.
- We thought that the disordered ground states are trivial, no excitations at low energies.
  But the disordered ground states can be non-trivial: they can belong to different phases, even without symmetry breaking.
  $\to$ the notions of **topological orders** and **SPT orders**.

Chen-Gu-Liu-Wen arXiv:1106.4772





NLσM with topological terms

Different disordered phases can arise from NLσM with different topological terms.

NLσM with topological terms

Different disordered phases can arise from NLσM with different topological terms.

- Another G symmetric system
  $S = \int \mathrm{d}^d x \mathrm{d}t \Big(\frac{1}{2\lambda} |g^{-1}\partial g|^2 + 2\pi \mathrm{i}\, W_{\text{G-top}} \Big)$
  where $W_{\text{G-top}}[g(x^i, t)]$ is a topological term.
- The topo. term $W_{\text{G-top}}[g(x^i, t)]$ can appear if $\pi_{d+1}(G) \neq 0$.
  Example 1: $0+1$D with $G = U_1 = S^1$ ($g = \mathrm{e}^{\mathrm{i}\theta}$), $\pi_1(U_1) = \mathbb{Z}$
  $W_{\text{G-top}} = -k \frac{\mathrm{i}}{2\pi} g^{-1}\partial_t g = k \frac{\dot\theta}{2\pi}, \quad k \in \mathbb{Z}$.
  $\int \mathrm{d}t\ W_{\text{G-top}} = k \times$ winding number.
  Example 2: $2+1$D with $G = SU_2 = S^3$, $\pi_3(SU_2) = \mathbb{Z}$
  $W_{\text{G-top}} = k \frac{1}{24\pi^2} \epsilon^{\mu\nu\lambda} (\mathrm{i} g^{-1}\partial_\mu g)(\mathrm{i} g^{-1}\partial_\nu g)(\mathrm{i} g^{-1}\partial_\lambda g), \quad k \in \mathbb{Z}$.
  $\int \mathrm{d}t \mathrm{d}^2 x\ W = k \times$ winding number.
- The topological term has no dynamical effect $\mathrm{e}^{-S_{\text{fixed}}} = 1$, but can give rise to different symmetric phases classified by k

Fixed-point eff. Lagrangian for symmetric phases

- If λ is large $\to$ disordered state.
  Under RG, $\lambda \to \infty$ $\to$ disordered symmetric ground state is described by a low energy fixed-point theory $S_{\text{fixed}} = 2\pi \mathrm{i} \int W_{\text{G-top}}$ – *topological non-linear σ-model with pure topological term*.

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- Can we define topological terms and topological non-linear σ-models when space-time is a discrete lattice?

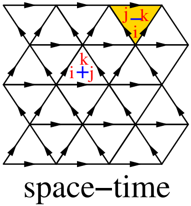
NLσM on 1+1D space-time lattice

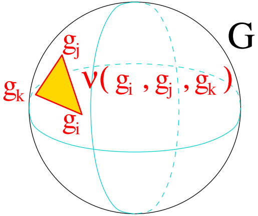
- Path integral on 1+1D space-time lattice with branching structure:

$$e^{-S} = \prod \nu_2^{s_{ijk}}(g_i, g_j, g_k),$$

where $\nu^{s_{ijk}}(g_i, g_j, g_k) = e^{-\int_\triangle L}$ and $s_{ijk} = 1, *$

- The above defines a LNσM with target space G on 1+1D space-time lattice.





- The NLσM will have a symmetry G if $g_i \in G$ and

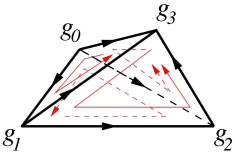
$$\nu_2(g_i, g_j, g_k) = \nu_2(hg_i, hg_j, hg_k), \quad h \in G$$

- The above is the lattice version of NLσM field theory:

$$\mathcal{L} = \frac{1}{\lambda}|g^{-1}\partial g|^2, \qquad \text{symm. } h: \ g(x) \to hg(x)$$

Topo. term and topo. NLσM on space-time lattice

- $\nu(g_i,g_j,g_k)$ give rise to a topological NLσM if $e^{-S_{\text{fixed}}} = \prod \nu^{s_{ijk}}(g_i,g_j,g_k) = 1$ on any sphere, including a tetrahedron (simplest sphere).



- $\nu(g_i,g_j,g_k) \in U_1$
- On a tetrahedron $\to$ 2-cocycle condition

$$\nu_2(g_1,g_2,g_3)\nu_2(g_0,g_1,g_3)\nu_2^{-1}(g_0,g_2,g_3)\nu_2^{-1}(g_0,g_1,g_2) = 1$$

The solutions of the above equation are called **group cocycle**.

- The 2-cocycle condition has many solutions: $\nu_2(g_0,g_1,g_2)$ and $\tilde{\nu}_2(g_0,g_1,g_2) = \nu_2(g_0,g_1,g_2)\frac{\beta_1(g_1,g_2)\beta_1(g_0,g_1)}{\beta_1(g_0,g_2)}$ are both cocycles. We say $\nu_2 \sim \tilde{\nu}_2$ (equivalent).
- The set of the equivalent classes of ν_2 is denoted as

$$\mathcal{H}^2(G,U_1) = \pi_0(\text{space of the solutions}).$$

- $\mathcal{H}^2(G,U_1)$ ($\to$ topo. terms) describes 1+1D SPT phases protected by G.

Group cohomology $\mathcal{H}^d[G, U_1]$ in any dimensions

- d-Cochain: U_1 valued function of $d+1$ variables

  $\nu_d(g_0, ..., g_d) = \nu_d(gg_0, ..., gg_d) \in U_1, \ \to$ on-site G -symmetry

- δ-map: ν_d with $d+1$ variables $\to$ $(\delta\nu_d)$ with $d+2$ variables

$$(\delta\nu_d)(g_0, ..., g_{d+1}) = \prod_i \nu_d^{(-)^i}(g_0, ..., \hat{g}_i, ..., g_{d+1})$$

- Cocycles = cochains that satisfy

$$(\delta\nu_d)(g_0, ..., g_{d+1}) = 1.$$

- Equivalence relation generated by any $d-1$-cochain:

$$\nu_d(g_0, ..., g_d) \sim \nu_d(g_0, ..., g_d)(\delta\beta_{d-1})(g_0, ..., g_d)$$

- $\mathcal{H}^{d+1}(G, U_1)$ is the equivalence class of cocycles ν_d.

  $d+1$D lattice topological NLσMs with symmetry G in are classified by $\mathcal{H}^{d+1}(G, U_1)$:

  $e^{-S} = \prod \nu_{d+1}^{s(i,j,...)}(g_i, g_j, ...), \ \nu_{d+1}(g_0, g_1, ..., g_{d+1}) \in \mathcal{H}^{d+1}(G, U_1)$

Topological invariance in topological NLσMs

As we change the space-time lattice,
the action amplitude e^{-S} does not change:

$$\nu_2(g_0,g_1,g_2)\nu_2^{-1}(g_1,g_2,g_3) = \nu_2(g_0,g_1,g_3)\nu_2^{-1}(g_0,g_2,g_3)$$

$$\nu_2(g_0,g_1,g_2)\nu_2^{-1}(g_1,g_2,g_3)\nu_2(g_0,g_2,g_3) = \nu_2(g_0,g_1,g_3)$$

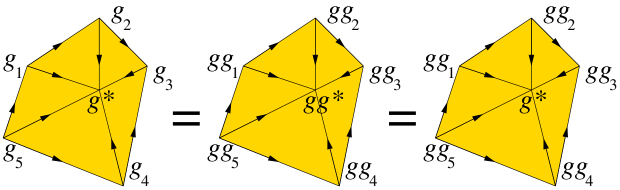
as implied by the cocycle condition:

$$\nu_2(g_1,g_2,g_3)\nu_2(g_0,g_1,g_3)\nu_2^{-1}(g_0,g_2,g_3)\nu_2^{-1}(g_0,g_1,g_2) = 1$$

The topological NLσM is a RG fixed-point.

- What is the ground state wave function of the topological NLσM?

The NLσM ground state is short-range entangled



The ground state wave function $\Psi(\{g_i\}) = \prod_i \nu_2(g_i, g_{i+1}, g^*)$

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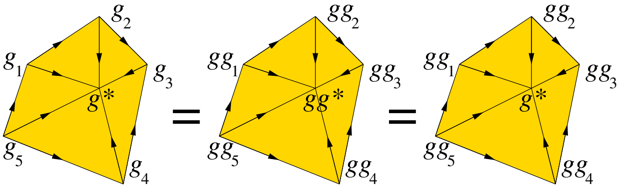

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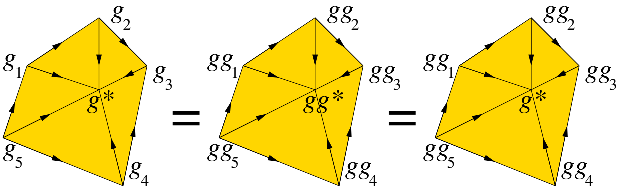
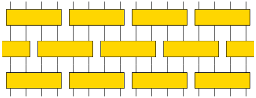


The ground state wave function $\Psi(\{g_i\}) = \prod_i \nu_2(g_i, g_{i+1}, g^*)$

- It is symmetric under the G-transformation
- It is equivalent to a product state $|\Psi_0\rangle = \otimes_i \sum_{g_i} |g_i\rangle$ under a LU transformation (note that $\Psi_0(\{g_i\}) = 1$)

$$\Psi(\{g_i\}) = \prod_{i=\text{even}} \nu_2(g_i, g_{i+1}, g^*) \prod_{i=\text{odd}} \nu_2(g_i, g_{i+1}, g^*) \Psi_0(\{g_i\})$$

$=$  $\Psi_0(\{g_i\}) \to$ Short-range entangled

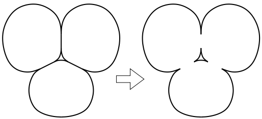
The ground state is symmetric with a trivial topo. order

The NLσM defined by ν_{d+1} has no topological order

Does the partition function $Z[M^{d+1}]$ have any no trivial dependence on the "shape" or topology of the space-time manifold M^{d+1}?

$$Z[M^{d+1}] = \sum_{\{g_i\}} \prod \nu_{d+1}^{s_{012\cdots}}(g_0, g_1, g_2, \cdots) = |G|^{N_v}$$

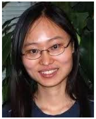
for any space-time manifold M^{d+1} obtained by gluing S^{d+1}'s.

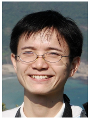


No topological order (?)

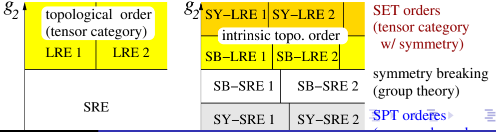
SPT phases from $\mathcal{H}^{d+1}(G, U_1)$ Chen-Gu-Liu-Wen arXiv:1106.4772

Symmetry G	$d=0$	$d=1$	$d=2$	$d=3$
$U_1 \rtimes Z_2^T$ (top. ins.)	$\mathbb{Z}$	$\mathbb{Z}_2$ (0)	$\mathbb{Z}_2$ $(\mathbb{Z}_2)$	$\mathbb{Z}_2^2$ $(\mathbb{Z}_2)$
$U_1 \rtimes Z_2^T \times$ trans	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_2^3$	$\mathbb{Z} \times \mathbb{Z}_2^8$
$U_1 \times Z_2^T$ (spin sys.)	0	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^3$
$U_1 \times Z_2^T \times$ trans	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^9$
Z_2^T (top. SC)	0	$\mathbb{Z}_2$ $(\mathbb{Z})$	0 (0)	$\mathbb{Z}_2$ (0)
$Z_2^T \times$ trans	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$
U_1	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$U_1 \times$ trans	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^4$
Z_n	$\mathbb{Z}_n$	0	$\mathbb{Z}_n$	0
$Z_n \times Z_n$	$\mathbb{Z}_n^2$	$\mathbb{Z}_n$	$\mathbb{Z}_n^3$	$\mathbb{Z}_n^2$
$Z_n \times Z_n \times Z_n$	$\mathbb{Z}_n^3$	$\mathbb{Z}_n^3$	$\mathbb{Z}_n^7$	$\mathbb{Z}_n^8$
$D_{2h} = Z_2 \times Z_2 \times Z_2^T$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$
$SU(2)$	0	0	$\mathbb{Z}$	0
$SO(3)$ (spin sys.)	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0
$SO(3) \times Z_2^T$ (spin sys.)	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$





"Z_2^T": time reversal,
"trans": translation,
$0 \to$ only trivial phase.
$(\mathbb{Z}_2) \to$ free fermion result



Universal probe for SPT orders

- How do you know the NLσM's with different cocycles produce different SPT orders? Why "seemsly-the-same" path integrals can produce different SPT phases? How do you measure SPT orders?

  Universal probe $=$ one probe to detect all possible orders.

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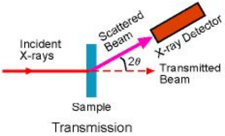
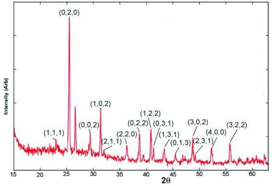
- Universal probe for crystal order = X-ray diffraction:





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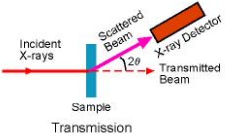
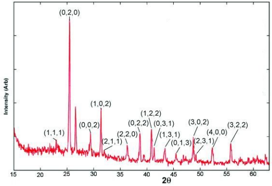




- Partitional function as an universal probe, but $Z_{\text{top}}^{SPT}(M^d) = 1 \to$ does not work.

- Twist the symmetry by "gauging" the symmetry on M^d
  $\to A - G$ gauge field.
  $\to Z_{\text{top}}^{SPT}(A, M^d) \neq 1$.

  Levin-Gu arXiv:1202.3120; Hung-Wen arXiv:1311.5539



Symmetry twist and "gauging" NLσM

- In the NLσM path integral

$$Z(M^d) = \int D[g(x)] \mathrm{e}^{-\int \mathrm{d}^d x \ \mathcal{L}(g, g^{-1} dg)},$$

we sum over all the cross-sections of a trivial bundle $G \times M^d$.

- In the "gauged" NLσM (with symmetry twist), we sum over all the cross-sections of a flat bundle $G \rtimes M^d$ with a flat connection.

"Gauging" (adding symmetry twist) in more details

- Change variable $g(x) \to h(x) g(x)$:
$\mathcal{L}(hg, (hg)^{-1} d(hg)) = \mathcal{L}(g, g^{-1} dg + A)$, $A = h^{-1} dh \to$

$$Z(M^d) = \int D[g(x)] \mathrm{e}^{-\int \mathrm{d}^d x \ \mathcal{L}(g, g^{-1}(d - \mathrm{i} A) g)}, \quad A = \mathrm{i}\, h^{-1} dh.$$

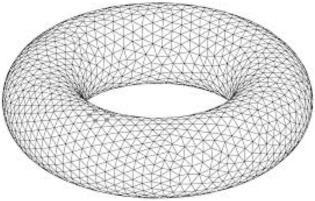
"Gauged" partition function (with symmetry twist)

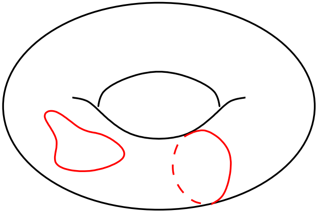
$$Z(A, M^d) = \int D[g(x)] \mathrm{e}^{-\int \mathrm{d}^d x \ \mathcal{L}(g, g^{-1}(d - \mathrm{i} A) g)}, \quad F = dA + \mathrm{i}[A, A] = 0$$

- For continuous group G, we can generalize the above to non-flat connection and non-flat bundle.

Examples of symmetry twist (gauge configuration)

- $U(1)$ symmetry twist: closed one-form A with
  $\oint_{S^1_x} A = \phi_x$,
  $\oint_{S^1_y} A = \phi_y$.





- Z_2 symmetry twist: closed quantized one-form A with
  $\oint_{S^1_x} A = 0, \pi$, $\oint_{S^1_y} A = 0, \pi$.

- We can choose the one-form A to be non-zero only on some codimension-1 closed sub-manifolds.
  Z_2 symmetry twist $\leftrightarrow$ codimension-1 sub-manifolds
  (Poincaré duality).

- Contractable loop $\to$ exact one-form $A = \mathrm{d}f$
  (pure gauge or coboundary)

Universal topo. inv.: "gauged" partition function

$$\frac{Z(A,M^d)}{Z(0,M^d)} = \frac{\int Dg\, \mathrm{e}^{-\int \mathcal{L}(g^{-1}(\mathrm{d}-\mathrm{i}\,A)g)}}{\int Dg\, \mathrm{e}^{-\int \mathcal{L}(g^{-1}\mathrm{d}g)}} = \mathrm{e}^{-\mathrm{i}2\pi \int W_{\text{A-top}}(A)}$$

- $W_{\text{A-top}}(A)$ and $W'_{\text{A-top}}(A)$ are equivalent if

$$W'_{\text{A-top}}(A) - W_{\text{A-top}}(A) = \frac{1}{\lambda_g}\mathrm{Tr}(F^2) + \cdots$$

- The equivalent class of the gauge-topological term $W_{\text{A-top}}(A)$ is the topological invariant that probe different SPT state.
- The topological invariant $W_{\text{A-top}}(A)$ are Chern-Simons terms or Chern-Simons-like terms.
- Such Chern-Simons-like terms are classified by $H^{d+1}(BG,\mathbb{Z}) = \mathcal{H}^d[G,U(1)]$ Dijkgraaf-Witten CMP 129, 393 (90)
  The topological invariant $W_{\text{A-top}}(A)$ can probe all the NLσM SPT states
- $W_{\text{A-top}}(A)$ can be viewed as a Lagrangian that defines a gauge theory:
  Dijkgraaf-Witten gauge theory

Calculate the SPT invariants for SPT phases

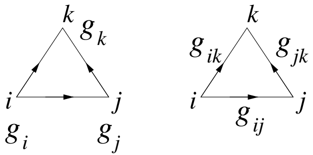
- Topological NLσM, as a fixed-point theory, contain only the pure topological term, and it is easy to calculate $W_{\text{A-top}}(A)$:

- **Lattice**: cocycle $\nu_d(\{g_i\}) \to$ DW action $W_{\text{A-top}}(A)$

- NLσM: $Z = \sum_{\{g_i\}} \prod \nu_3^{s_{ijk}}(g_i, g_j, g_k) = \int D[g] \mathrm{e}^{\mathrm{i}\, 2\pi W_{\text{G-top}}(g^{-1}\mathrm{d}g)}$

- DW-gauge theory $Z = \sum_{\{g_{ij}\}} \prod \omega_3^{s_{ijk}}(g_{ij}, g_{jk}) = \int D[A] \mathrm{e}^{\mathrm{i}\, 2\pi W_{\text{A-top}}(A)}$ where the amplitude $\mathrm{e}^{\mathrm{i}\, 2\pi W_{\text{A-top}}(A)}$ is non-zero only for flat connections: $g_{ik} = g_{ij} g_{jk}$.

- Connection: $\nu_3(g_i, g_j, g_k) = \nu_3(hg_i, hg_j, hg_k) = \omega_3(g_i^{-1} g_j, g_j^{-1} g_k)$



- **Continuum**: G-topo. term $W_{\text{G-top}}(g^{-1}\mathrm{d}g)_{g^{-1}\mathrm{d}g \to A} \to W_{\text{A-top}}(A)$

An example: SU_2 SPT state

- Topo. term for SU_2 **SPT state**:
- **In** $2+1$**D** $\pi_3(SU_2)=\mathbb{Z}$:
  $W^3_{\text{G-top}} = k\frac{\mathrm{Tr}(\mathrm{i}\,g^{-1}dg)^3}{24\pi^2}$.

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- Topo. term for SU_2 **SPT state**: SU_2 connection $A \sim \mathrm{i}g^{-1}\mathrm{d}g$
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SPT inv. → phys. measurement:
→ Spin quantum Hall conductance $\sigma^{\text{spin}}_{xy} = \frac{k}{4\pi}$
→ Gapless state if the SU_2 symm. is not broken.

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(No topo. order, need symm. protection) Liu-Wen arXiv:1205.7024

The edge of the $SU(2)$ SPT state must be gapless

Bulk fixed-point action: Liu-Wen arXiv:1205.7024

$S_{\rm bulk} = -i\frac{k}{12\pi}\int_{M^3} {\rm Tr}(g^{-1}{\rm d}g)^3, \quad k \in \mathbb{Z}, \quad g \in SU(2)$ The $SU(2)$ symmetry $g(x) \to hg(x), \quad h, g(x) \in SU(2)$

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- The edge excitations on ∂M^3 described by fixed-point WZW:

$$S_{\text{edge}} = \int_{\partial M^3} \frac{k}{8\pi}\text{Tr}(\partial g^{-1}\partial g) - i\int_{M^3} \frac{k}{12\pi}\text{Tr}(g^{-1}\text{d}g)^3,$$

- At the fixed point, we have a equation of motion

$$\partial_{\bar z}[(\partial_z g)g^{-1}] = 0, \quad \partial_z[(\partial_{\bar z} g^{-1})g] = 0, \quad z = x + \text{i}\,t.$$

Right movers $[(\partial_z g)g^{-1}](z) \to SU(2)$-charges
Left movers $[(\partial_{\bar z} g^{-1})g](\bar z) \to SU_L(2)$-charges, $g(x) \to g(x)h_L$

Level-k Kac-Moody algebra Witten NPB 223, 422 (83)

- **The $SU(2)$ symmetry is anomalous at the edge.**
In general, G **SPT state has anomalous** G**-symmetry at the boundary** – a defining property of SPT phases.

Theorem: **The boundary of any 2+1D SPT states must be gapless or symmetry breaking.** Chen-Liu-Wen arXiv:1106.4752;

$U(1)$ SPT phases and their physical properties

- Topo. terms for U_1 SPT state:
- **In** $0+1$**D**, $W^1_{\text{A-top}} = k\frac{A}{2\pi}$.

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- **In** $2+1$**D**, $W^3_{\text{A-top}} = k\frac{AF}{(2\pi)^2}$ (?)

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- **In $2+1$D**, $W^3_{\text{A-top}} = k\frac{AF}{(2\pi)^2}$ (?)

  SPT inv. → phys. measurement:

  → Hall conductance $\sigma_{xy} = 2k\frac{e^2}{h}$

  → The edge of U_1 SPT phase must be gapless with left/right movers and has anomalous $U(1)$ symm.

  → Choose space-time $S^1 \times M^2$ and put $2\pi m$ flux through M^2.

  $\mathcal{L}^{2+1D} = kA\mathrm{d}A/2\pi \to \mathcal{L}^{0+1D} = k\int_{M^2} A\mathrm{d}A/2\pi = 2kmA$.

- The 2+1D U_1 SPT state labeled by k reduces to a 0+1D U_1 SPT state labeled by $2km$ (with charge $2km$ in ground state).
- $2\pi m$ flux in space M^2 induces $2km$ unit of charge → Hall conductance $\sigma_{xy} = 2ke^2/h$. Lu-Vishwanath arXiv:1205.3156

From probe to mechanism of SPT states

- To probe:
  2π **flux inducing** $2k$ **charge probes** $U(1)$ **SPT state**.
- To create:
  Attaching $2k$ **charges to a** $U(1)$ **vortex makes** $U(1)$ **SPT state**.

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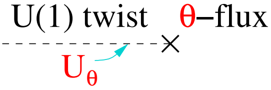
- Start with 2+1D bosonic superfluid:
  proliferate vortices $\rightarrow$ **trivial Mott insulator**.
  proliferate vortex+$2k$**-charge** $\rightarrow$ $U(1)$ **SPT state labeled by** k.

- Why? *$U(1)$ flux is the $U(1)$ symmetry twist. A vortex in $U(1)$ superfluid is a 2π $U(1)$ symmetry twist $= 2\pi$ flux.*



The vortex condensed state (or the vortex proliferated state) remembers the binding of vortex and $2k$-charge: a 2π $U(1)$ symmetry twist (2π flux) carries $2k$ charges.

Why bind even charge to vortex?

- Can we bind charge-1 to a vortex, to make a new $U(1)$ SPT state beyond group cohomology, which has $\sigma_{xy} = \frac{e^2}{h}$?

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  (1) Inserting 2π flux will always create a quasiparticle. Such a quasiparticle would carry a unit $U(1)$ charge. The bound state of charge-1+2π-flux is a fermion, which implies that the $\sigma_{xy} = \frac{e^2}{h}$ SPT state must carry a non-trivial topological order.

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  (2) The bound state of charge-1+vortex is a fermion. They cannot condense to make the superfluid into an insulator. But a (charge-1+vortex)-pair is a boson. Proliferate/condensing such (charge-1+vortex)-pairs can make the superfluid into an insulator with non-trivial Z_2 topological order described by Z_2 gauge theory.

The duality between the probe and the mechanism is a general phenomenon which also appears for other SPT orders.

Electromagnetic responce in state with no topological order

- Bosons with no topo. order: $\frac{2k}{4\pi}\int_{M^3} A\mathrm{d}A$
- Fermions with no topo. order: $\frac{k}{4\pi}\int_{M^3} A\mathrm{d}A$

An understanding via algebraic topology:

- The Chern-Simons term is better defined by going to one higher dimension:
$$\frac{\sigma_{xy}}{4\pi}\int_{M^3=\partial M^4} A\mathrm{d}A = \frac{\sigma_{xy}}{2}\,2\pi\int_{M^4}(\frac{\mathrm{d}A}{2\pi})^2$$
which well defined only when $\frac{\sigma_{xy}}{2}\int_{M^4}(\frac{\mathrm{d}A}{2\pi})^2$ is always integer for closed M^4.

- Two math relations:
1) $Sq^2(x_2) = x_2 \cup x_2$ for any 2-cocycle $x_2 \in H^2(M^4, \mathbb{Z}_2)$
2) $Sq^2(x_2) = u_2 \cup x_2 = (w_2 + w_1 \cup w_1) \cup x_2$ in 4-dimensions
Choose $x_2 = \frac{\mathrm{d}A}{2\pi}$ mod 2 $\to (\frac{\mathrm{d}A}{2\pi})^2 = (w_2 + w_1 \cup w_1) \cup \frac{\mathrm{d}A}{2\pi}$ mod 2.
- On CP^2, $\int_{M^4}(\frac{\mathrm{d}A}{2\pi})^2 = 1 \to \frac{\sigma_{xy}}{2} =$ int. for bosons.
- On spin manifold, $w_1, w_2 = 0$ and $\int_{M^4}(\frac{\mathrm{d}A}{2\pi})^2 = 0$ mod 2
$\to \sigma_{xy} =$ int. for fermions.

The boundary of the 2+1D bosonic $U(1)$ SPT state has a 1+1D bosonic $U(1)$ gauge anomaly

The boundary of the 2+1D $U(1)$ SPT state must be gapless.

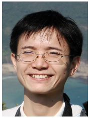
- Partition function of $U(1)$-SPT state on space-time with boundary:

$$Z(A, M^3) = \mathrm{e}^{-\epsilon Vol}\,\mathrm{e}^{\mathrm{i}\int_M \frac{2k}{4\pi} A\mathrm{d}A + \int_{\partial M} \mathrm{d}t\,\mathrm{d}x\ \mathcal{L}_{\text{edge}}}$$

- The total $Z(A, M^3)$ is gauge invariant under $A \to A + \mathrm{d}f$, but the bulk CS-term and the edge action separately are not gauge invariant if $2k \neq 0$. We need a $U(1)$ anomalous edge described by $\mathcal{L}_{\text{edge}}$ to cancell the gauge non-invariance of the CS-term.
- Such an edge must be gapless. Wen PRB 43, 11025 (89)

A mechanism for 2+1D $U_1 \rtimes Z_2^T$ SPT state

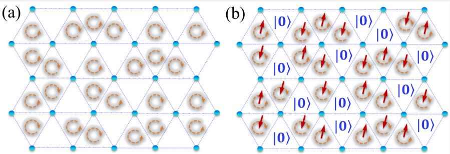
Liu-Gu-Wen arXiv:1404.2818



- 2+1D boson superfluid + gas of vortex
  $\to$ boson Mott insulator.
- 2+1D boson superfluid + gas of S^z-vortex
  $\to$ boson topological insulator ($U_1 \rtimes Z_2^T$ SPT state)

\- The boson superfluid + spin-1 system

S^z-vortex =
vortex + $(S_z = +1)$-spin
anti S^z-vortex =
anti-vortex + $(S_z = -1)$-spin



Probing 2+1D $U_1 \rtimes Z_2^T$ SPT state

Let Φ_{vortex} be the creation operator of the vortex. Then
$T^{-1}\Phi_{\text{vortex}}T = \Phi^\dagger_{\text{vortex}}$, $\Phi_{S_z\text{-vortex}} = S^+\Phi_{\text{vortex}}$, $T^{-1}\Phi_{S_z\text{-vortex}}T = -\Phi^\dagger_{S_z\text{-vortex}}$.
The π-flux in the $U_1 \rtimes Z_2^T$ SPT state is Kramer doublet:
$\Phi_{S_z\text{-vortex}}|-\pi\rangle = |\pi>$, $\Phi^\dagger_{S_z\text{-vortex}}|\pi\rangle = |-\pi>$,

Z_2 SPT phases and their physical properties

- Topological terms:

$\oint A_{Z_2} = 0, \pi$; $a \equiv \frac{A_{Z_2}}{\pi}$;

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Wen arXiv:1410.8477

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  $= e^{i k\pi \oint_{S^1} a} = e^{i k\pi} = \pm 1, \ k = 0, 1$
  SPT inv. $\to$ **phys. measurement:**
  $\to$ ground state Z_2-charge $= k = 0, 1$

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  Here we do not view a as 1-form
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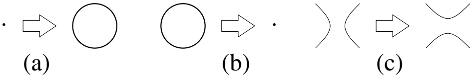
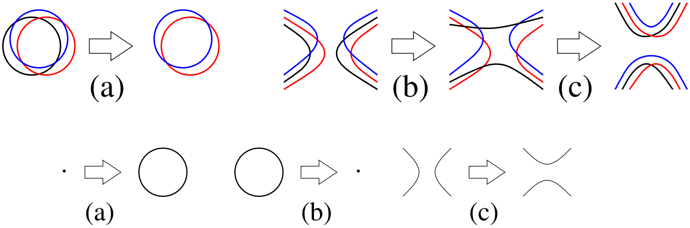
- Poincaré duality: 1-cocycle $a \leftrightarrow$ 2-cycle N^2 (2D submanifold)
  N^2 is the surface across which we do the $\mathbb{Z}_2$ symmetry twist.
  Give M^3 an time and space slices, as we evolve in time:



$\int_{M^3} a^3 = \#$ of loop creation/annihilation $+ \#$ of line reconnection

How calculate $\int_{M^3} a^3$ (which can be 0 or 1 mod 2)

- $\int_{M^3} a_1 \cup a_2 \cup a_3 = \#$ of intersections of N_1, N_2, N_3 mod 2, where $a_i \to N_i$.
- $\int_{M^3} a \cup a \cup a = \#$ of intersections of N, N', N'' mod 2, where $a \to N, N', N''$.



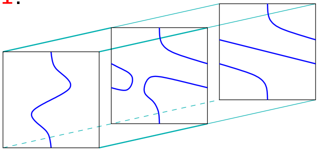
How to probe Z_2 SPT phase; How to measure $\int_{M^3} \frac{1}{2} a^3$

$$\frac{Z[a, M^3]}{Z[0, M^3]} = \mathrm{e}^{\mathrm{i}\, 2\pi \int_{M^3} \frac{1}{2} a^3}$$

How to design (a, M^3) such that $\frac{Z[a,M^3]}{Z[0,M^3]} = \mathrm{e}^{\mathrm{i}\, 2\pi \int_{M^3} \frac{1}{2} a^3} = -1$

- If we choose $M^3 = T^3$, $\mathrm{e}^{\mathrm{i}\, 2\pi \int_{M^3} \frac{1}{2} a^3} = 1$ no matter how we choose the Z_2 symmetry twists $a \in H^1(T^3, \mathbb{Z}_2)$.
- Let us choose $M^3 = T^2 \rtimes_{\mathrm{Dehn}^2} S^1$, then, we can have a Z_2 symmetry twist to make $\mathrm{e}^{\mathrm{i}\, 2\pi \int_{M^3} \frac{1}{2} a^3} = -1$.

The Z_2 symmetry twist a is represented by a 2D surface in space-time M^3, which is a curve in space.

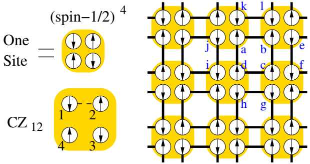


Hung-Wen arXiv:1311.5539

It is hard to probe the Z_2 SPT order (or Z_2 SPT inv.) using bulk measurement.

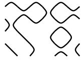
How to create 2+1D Z_2 SPT phase?

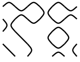
- $H = \sum_{\square} H_p$, $H_p = -X_{abcd} P_{ef} P_{gh} P_{ij} P_{kl}$,
  $X_{abcd} = |\uparrow\uparrow\uparrow\uparrow\rangle\langle\downarrow\downarrow\downarrow\downarrow| + |\downarrow\downarrow\downarrow\downarrow\rangle\langle\uparrow\uparrow\uparrow\uparrow|$,
  $P = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|$.
  Chen-Liu-Wen arXiv:1106.4752;



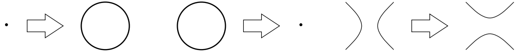
- Start with a Z_2 symmetry breaking state, the, proliferate the symmetry breaking domain walls to restore the Z_2-symmetry.

- Domain wall quantum liquid = disordered Z_2-symmetric state.

- If Domain wall quantum liquid = $\sum |$  $\rangle$, then the Z_2-symmetric state is the trivial Z_2 SPT state.

- If Domain wall quantum liquid = $\sum (-)^{\#\text{ of loops}} |$  $\rangle$, then the Z_2-symmetric state is the non-trivial Z_2 SPT state.

Levin-Gu arXiv:1202.3120



(a) (b) (c)

Why there is no non-trivial 1+1D Z_2 SPT phase?

- Because $\mathcal{H}^2(Z_2, U_1) = 0$.
  But this only implies that our NLσM construction fails to produce a non-trivial 1+1D Z_2 SPT state.
- May be non-trivial 1+1D Z_2 SPT state exists since we have a potential Z_2 SPT invariant in 1+1D $W^2_{\text{A-top}}(a) = \frac{1}{2} a \cup a$.

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- However, $\int a \cup a = 0$ mod 2 on oriented manifold. There is not even non-trivial potential Z_2 SPT invariant in 1+1D
  $\to$ there is no non-trivial 1+1D Z_2 SPT phase.

  Proof:
  $Sq^1(a) = a \cup a$ and $Sq^1(a) = u_1 \cup a = w_1 \cup a$
  $w_1 = 0$ for oriented manifold, and thus $a \cup a = 0$ mod 2.

A 1+1D $Z_{N_1} \times Z_{N_2}$ SPT state

Wang-Gu-Wen arXiv:1405.7689

$$H^3[Z_{N_1} \times Z_{N_2}, U(1)] = \mathbb{Z}_{N_{12}} = \{0, 1, \cdots, k, \cdots, N_{12} - 1\}$$

where $N_{12} = \gcd(N_1, N_2)$.

- We consider a SPT state labeled by $k \in \mathbb{Z}_{N_{12}}$ and assume $N_1 = N_2 = N$.
- **What is the SPT invariant?**

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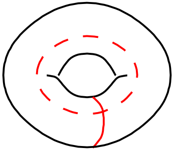
The fixed-point partition function on space-time $T^2 = S^1 \times S^1$ with symmetry twists in x, t directions:

$$\frac{Z[a_1, a_2, T^2]}{Z[0, 0, T^2]} = \mathrm{e}^{\mathrm{i}\, k \frac{2\pi}{N_{12}} \int a_1 a_2},$$

$$\oint a_1 \in \mathbb{Z}_{N_1}; \quad \oint a_2 \in \mathbb{Z}_{N_2}.$$



SPT inv. $\to$ physical measurement

A symmetry twist of Z_{N_1} carries Z_{N_2}-charge k.

Example: 1D $Z_2 \times Z_2 = D_2$ SPT state (spin-1 Haldane chain)
$Z_2 \times Z_2 = D_2 = 180^\circ$ spin rotations in S^x, S^z.
Untwisted case:

$$H_{D_2} = \sum_i J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z$$
$$+ J_x S_L^x S_1^x + J_y S_L^y S_1^y + J_z S_L^z S_1^z$$

The ground state has $e^{i\pi \sum S_i^z} = 1$.
Twisted case (by $e^{i\pi \sum S_i^x}$):

$$H_{D_2}^{\text{twist}} = \sum_i J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z$$
$$+ J_x S_L^x S_1^x - J_y S_L^y S_1^y - J_z S_L^z S_1^z$$

The ground state has $e^{i\pi \sum S_i^z} = -1$.

From a SPT invariant to a SPT mechanism

For a 1D $Z_{N_1} \times Z_{N_2}$ SPT state

- SPT invariant: a symmetry twist of Z_{N_1} carries a "charge" of Z_{N_2}

Wen arXiv:1301.7675

Since the symmetry twist of Z_{N_1} = the domain wall of Z_{N_1}

- Bind k Z_{N_2}-charge to the domain wall of Z_{N_1}
  $\to$ 1D $Z_{N_1} \times Z_{N_2}$ SPT state labeled by $k \in \mathcal{H}^2[Z_{N_1} \times Z_{N_2}, U(1)]$

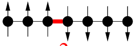
Example: A $Z_2^x \times Z_2^z$ spin-1 chain, & its symmetric phases

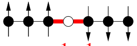
- $|x\rangle, |y\rangle, |z\rangle$ basis:

$$|\uparrow_z\rangle = \frac{|x\rangle + \mathrm{i}|y\rangle}{\sqrt{2}}, |0_z\rangle = |z\rangle, \quad |\downarrow_z\rangle = \frac{|x\rangle - \mathrm{i}|y\rangle}{\sqrt{2}}$$

$$S^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad S^y = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ -\mathrm{i} & 0 & 0 \end{pmatrix}, \quad S^z = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

- $Z_2^x \times Z_2^z$ symmetry: $U^x : (|x\rangle, |y\rangle, |z\rangle) \to (-|x\rangle, |y\rangle, |z\rangle)$
  $U^z : (|x\rangle, |y\rangle, |z\rangle) \to (|x\rangle, |y\rangle, -|z\rangle)$

- $H^0 = \sum_i -J_z S_i^z S_{i+1}^z \;\to\; Z_2^x$ breaking 



 -
  Two kinds of domain walls with the same energy, but different Z_2^z-charges and different hopping operators:

$$H_1^{\mathrm{hop}} = \sum_i -K[(S_i^+)^2 + h.c.], \qquad H_2^{\mathrm{hop}} = \sum_i -J_{xy}(S_i^+ S_{i+1}^+ + h.c.).$$

- $H^0 + H_1^{\mathrm{hop}}$ & $H^0 + H_2^{\mathrm{hop}}$ $\to$ different symm. ground states

A 2+1D $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ SPT state

$$H^3[\prod_{i=1}^{3} Z_{N_i}, U(1)] = \mathbb{Z}_{N_1} \oplus \mathbb{Z}_{N_2} \oplus \mathbb{Z}_{N_3} \oplus \mathbb{Z}_{N_{12}} \oplus \mathbb{Z}_{N_{23}} \oplus \mathbb{Z}_{N_{13}} \oplus \mathbb{Z}_{N_{123}}$$

where $N_{123} = \gcd(N_1, N_2, N_3)$.

- We consider a SPT state labeled by $k \in \mathbb{Z}_{N_{123}}$ and assume $N_1 = N_2 = N_3 = N$.
- **The SPT invariant**: Wang-Gu-Wen arXiv:1405.7689

$$\frac{Z[a_1, a_2, a_3]}{Z[0,0,0]} = \mathrm{e}^{\,\mathrm{i}\, k \frac{2\pi}{N_{123}} \int a_1 a_2 a_3}$$

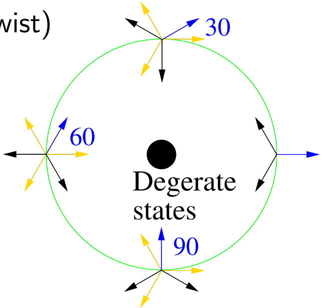
- **SPT inv. $\to$ physical measurement:** The intersection of the symmetry twists in Z_{N_1} and Z_{N_2} carries Z_{N_3}-charge k.
- **A mechanism for such a SPT state:** Bind k Z_{N_3}-charge to the intersection of the domain walls of Z_{N_1} and Z_{N_2}.

- Dimension reduction: $T^3 = T^2_{x,t} \times S^1_y$ and $\oint_{S^1_y} a_3 = 1$:

$$\frac{Z[a_1, a_2, T^2]}{Z[0, 0, T^2]} = \mathrm{e}^{\,\mathrm{i}\,\frac{kN_{12}}{N_{123}}\frac{2\pi}{N_{12}}\int a_1 a_2}$$

$\to$ A 1+1D SPT state with $\frac{kN_{12}}{N_{123}} \in H^2[Z_{N_1} \times Z_{N_2}, U(1)] = \mathbb{Z}_{N_{12}}$.
$\to$ degenerated states at the end of 1D chain that form a projective representation of $Z_{N_1} \times Z_{N_2}$.

- A Z_{N_3} "vortex" (end of Z_{N_3} symmetry twist) carries degenerated states that form a projective representation of $Z_{N_1} \times Z_{N_2}$.



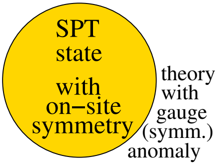
- **How to make Z_3-vortex**:
  1) Consider $U(1)$ symm. break down to Z_3 symm.
  2) A vortex of the order parameter = Z_3-vortex.

- Another mechanism for the 2+1D $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ SPT state: bind the 1+1D $Z_{N_1} \times Z_{N_2}$ SPT state to the domain wall of Z_{N_3}. Chen-Lu-Vishwanath 12

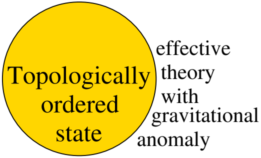
$\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$ does not produce all the SPT phases with symm. G: Topological states and anomalies

Wen arXiv:1303.1803; Kong-Wen arXiv:1405.5858

SPT order from $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$



Topological order

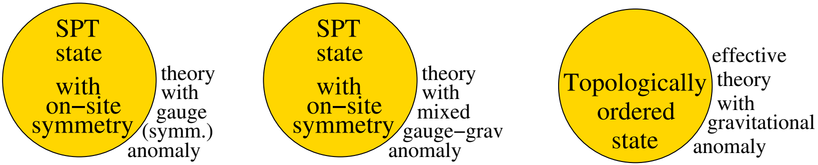


Pure SPT order within $\mathcal{H}^d(G,\mathbb{R}/\mathbb{Z})$: $W^d_{\text{A-top}} = \frac{AF}{(2\pi)^2}, \frac{1}{2}a^3$

$\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ does not produce all the SPT phases with symm. G: Topological states and anomalies

Wen arXiv:1303.1803; Kong-Wen arXiv:1405.5858

SPT order from $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ Topological order



SPT order beyond $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$

Pure SPT order within $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$: $W^d_{\text{A-top}} = \frac{AF}{(2\pi)^2}, \frac{1}{2}a^3$

mixed SPT order beyond $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$: $W^d_{\text{A-top}} = \frac{F}{2\pi}\omega_3, \frac{1}{2}ap_1$

Invertible topological order: $W^d_{\text{A-top}} = \omega_3, \frac{1}{2}\mathrm{w}_2\mathrm{w}_3$

p_1 is the first Pontryagin class, $\mathrm{d}\omega_3 = p_1$, and w_i is the Stiefel-Whitney classes.

A general theory for bosonic pure STP orders, mixed SPT orders, and invertible topological orders

Wen arXiv:1410.8477

- NLσM (group cohomology) approach to pure SPT phases:
  (1) NLσM+topo. term: $\frac{1}{2\lambda}|\partial g|^2 + 2\pi \mathrm{i}\, W(g^{-1}\partial g),\ \ g \in G$
  (2) Add symm. twist: $\frac{1}{2\lambda}|(\partial - \mathrm{i} A)g|^2 + 2\pi \mathrm{i}\, W[(\partial - \mathrm{i} A)g]$
  (3) Integrate out matter field: $Z_{\text{fixed}} = \mathrm{e}^{2\pi \mathrm{i} \int W_{\text{A-top}}(A)}$
- $G \times SO_\infty$ NLσM (group cohomology) approach:
  (1) NLσM: $\frac{1}{2\lambda}|\partial g|^2 + 2\pi \mathrm{i}\, W(g^{-1}\partial g),\ \ g \in G \times SO$
  (2) Add twist: $\frac{1}{2\lambda}|(\partial - \mathrm{i} A - \mathrm{i}\Gamma)g|^2 + 2\pi \mathrm{i}\, W[(\partial - \mathrm{i} A - \mathrm{i}\Gamma)g]$
  (3) Integrate out matter field: $Z_{\text{fixed}} = \mathrm{e}^{2\pi \mathrm{i} \int W_{\text{A-top}}(A,\Gamma)}$

Pure STP orders, mixed SPT orders, and invertible topological orders are classified by
$\mathcal{H}^d(G \times SO, \mathbb{R}/\mathbb{Z})$
$= \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \oplus_{k=1}^{d-1} \mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})$
after quotient out something $\Gamma^d(G)$.

Trying to classify bosonic pure STP orders, mixed SPT orders, and invertible topological orders

- **Pure STP orders**: $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$
- **mixed SPT orders**: $\oplus_{k=1}^{d-1}\mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})]$
- **iTO's**: $\mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})$

The above are one-to-one description of pure SPT orders, but only many-to-one description of mixed SPT orders and iTO's

- G-symmetry twists $A \to W^d_{A-top}(A)$ can fully detect/distinguish all elements of $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$.
- SO-symmetry twists $\Gamma_{SO} \to W^d_{A-top}(\Gamma_{SO})$ can fully detect/distinguish all elements of $\mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})$.
- SO-symmetry twists Γ from the tangent bundle of M^d are only special SO-symmetry twists (which are arbitrary SO-bundles on M^d) $\to W^d_{A-top}(\Gamma)$ cannot fully detect/distinguish all elements of $\mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})$. $\to$ $\text{iTO}^d = \mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})/\Gamma^d$

Trying to classify bosonic pure STP orders, mixed SPT orders, and invertible topological orders

- **Pure STP orders**: $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ (the black entries below)
- **mixed SPT order** $\oplus_{k=1}^{d-1} \mathcal{H}^k(G, \text{iTO}^{d-k}) = \frac{\oplus \mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})]}{\Gamma^d(G)}$
- **iTO's**: $\text{iTO}^d = \mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})/\Gamma^d$
- Probe mixed SPT order described by $\mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})]$: put the state on $M^d = M^k \times M^{d-k}$ and add a G-symmetry twist on M^k $\to$ Induce a state on M^{d-k} described by $\mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})$ $\to$ a iTO state in iTO^{d-k}

$G \setminus d =$	0+1	1+1	2+1	3+1	4+1	5+1	6+1
iTO^d	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	0
Z_n	$\mathbb{Z}_n$	0	$\mathbb{Z}_n$	0	$\mathbb{Z}_n \oplus \mathbb{Z}_n$	$\mathbb{Z}_{\langle n,2\rangle}$	$\mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_{\langle n,2\rangle}$
Z_2^T	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	$\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$\mathbb{Z}_2$
$U(1)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$
$U(1) \rtimes Z_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$\mathbb{Z} \oplus 2\mathbb{Z}_2 \oplus \mathbb{Z} \oplus 2\mathbb{Z}_2$
$U(1) \times Z_2^T$	0	$2\mathbb{Z}_2$	0	$3\mathbb{Z}_2 \oplus \mathbb{Z}_2$	0	$4\mathbb{Z}_2 \oplus 3\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$
$U(1) \rtimes Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$	$2\mathbb{Z}_2 \oplus 2\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus 3\mathbb{Z}_2 \oplus \mathbb{Z}_2$

Wen arXiv:1410.8477