A theory for symmetry protected topological order

Xiao-Gang Wen, MIT
PiTP, IAS, July., 2014
Try to classify quantum states of matter

Quantum states of matter:
- gapless states – Very hard beyond 1+1D. I have no clue
- gapped states – A classification maybe possible:
  
  Gapped Quantum State of Matter

  Symmetry breaking order

  Group Theory

  need no symmetry

  Long range entangled (topological order)

  Tensor category theory? 1989

  Emergent gauge bosons, fermions

  Boundary with gravitational anomaly

  pattern of entanglement

  Non–symmetry breaking order

  Group cohomology theory? 2008

  Boundary with gauge or

  mixed gauge–gravity anomaly

  Short range entangled (SPT order)

  • **Group theory** classifies 230 crystal orders in 3D space.
  • **What** classifies SPT orders?
  • **What** classifies topological orders?
What is a gapped state (or a gapped system)

It is not just the energy spectrum has a gap.

- We need to take thermal dynamical limit.
- But how to take large-size limit without translation symmetry?

- Gapped state may have gapless boundary.
- Avoid boundary by putting the system on manifold without boundary, but the definition will depend on “shapes/topologies” of the manifold.
- How to define gapped state by putting the system on a “ball”?

- Help us to understand what is the input to even define gapped phases
  → Mathematical foundation of gapped phases.
Def. of gapped liquid phase w/o translation symmetry

- A **local** Hamiltonian $H = \sum O_{ij} + O_{ijk} + \cdots$ on graph w/ a shape of a ball.

- A **bulk-gapped local** Hamiltonian:
  Excitations in bulk are all gapped.
  
  \[
  |\langle \Phi_{\text{grnd}} | \hat{O}_{\text{bluk local}} | \Phi_{\text{grnd}} \rangle - \langle \Phi_{\text{exc}} | \hat{O}_{\text{bluk local}} | \Phi_{\text{exc}} \rangle| < e^{-\frac{\text{distance to boundary}}{\xi}}
  \]

  where $|\Phi_{\text{exc}}\rangle$ has an energy less than, say, $\Delta/2$.

  *All low energy states are locally indistinguishable in the bulk.*

  $\rightarrow$ unique bulk ground state on a “ball”.

Kong-Wen 15
Def. of gapped liquid phase w/o translation symmetry

- A **gapped liquid phase** = an equivalent class of sequences of bulk-gapped local Hamiltonians $H_{N_k}$ with size $N_k \to \infty$, where the equivalence relations are generated by
  
  a) $H_{N_k} \sim (LU)H'_{N_k}(LU)\dagger$

  
  b) $H_{N_{k+1}} \sim H_{N_k} \otimes H_{\text{tri}}$

  
  $H_{\text{tri}} = -\sum S_i^z$

Two sequences of Hamiltonians:

$H_{N_1}, H_{N_2}, H_{N_3}, H_{N_4}, \ldots$

$H'_{N_1}, H'_{N_2}, H'_{N_3}, H'_{N_4}, \ldots$

$N_{k+1} = sN_k, s \sim 2$
Gapped liquid phases $\equiv$ Topological orders

Even w/o symmetry breaking ($H$ has no symmetry), we can have
- different gapped liquid phases
  $\Rightarrow$ different topological orders
  $\Rightarrow$ different patterns of long range entanglement

- Short-range entanglement (SRE):
  $\Rightarrow$ LU equivalent to product state

- Long-range entanglement (LRE):
  $\Rightarrow$ LU inequivalent to product state

Wen PRB 40, 7387 (1989)

Counter examples (non-liquid gapped states):
- Landau symmetry breaking state
- 3D layered quantum Hall states
- Haah's 3D cubic-code state


A theory for symmetry protected topological order
Gapped liquid phases = Topological orders

Even w/o symmetry breaking ($H$ has no symmetry), we can have
- different gapped liquid phases
- different topological orders
- different patterns of long range entanglement

- Short-range entanglement (SRE):
  $= LU$ equivalent to product state
- Long-range entanglement (LRE):
  $= LU$ inequivalent to product state

- **Counter examples** (non-liquid gapped states):
  - Landau symmetry breaking state
  - 3D layered quantum Hall states
  - Haah’s 3D cubic-code state

Wen PRB 40, 7387 (1989)
Examples for gapped liquid phases (topological orders)

- **Examples:**
  - Product state (trivial topological order) $H_{\text{tri}} = - \sum \sigma_i^z$
  - 1+1D $p$-wave superconductor (fermionic) Kiteav cond-mat/0010440,
  - 2+1D $p + ip$ superconductor (fermionic) Read-Green cond-mat/9906453,
  - IQH states (2+1D fermionic) Klitzing-Dorda-Pepper PRL 45, 494 (80),
  - FQH states (2+1D fermionic) Tsui-Stormer-Gossard, PRL 48, 1559 (82),
  - Chiral-spin liquids (bosonic) Kalmeyer-Laughlin PRL 59, 2095 (87), Wen-Wilczek-Zee PRB 39, 11413 (89),
    A realization by spin-1/2 on Kagome lattice
    \[
    H = J_1 \sum_{1\text{st}} \mathbf{S}_i \cdot \mathbf{S}_j + J_2 \sum_{2\text{nd}} \mathbf{S}_i \cdot \mathbf{S}_j + J_3 \sum_{3\text{rd}} \mathbf{S}_i \cdot \mathbf{S}_j \\
    J_2 = J_3 > 0.1J_1 
    \]
    Gong-Zhu-Balents-Sheng arXiv:1412.1571
  - $Z_2$-spin liquid Read-Sachdev PRL 66, 1773 (91), Wen PRB 44, 2664 (91)
  - Pfaffian state $\mathcal{A}(\frac{1}{z_1-z_2} \frac{1}{z_3-z_3} \ldots) \prod (z_i - z_j)^2$ Moore-Read NPB 360, 362 (91)
  - $SU(2)_2$ state $\Psi_{\nu=1}(z_i)\Psi_{\nu=2}(z_i)\Psi_{\nu=2}(z_i)$ Wen PRL 66, 802 (91)

The last two examples are non-abelian with the same non-abelian statistics: $Ising \times U(1)$ or $Ising \times U^2(1)$. 
We conjectured that topological order can be completely defined via only two topological properties (at least in 2D):

Wen IJMPB 4, 239 (90); KeskiVakkuri-Wen IJMPB 7, 4227 (93)

(1) **Topological ground state degeneracy** $D_g$
- degenerate only in size $\rightarrow \infty$ limit
- robust against any impurities
- depend on topology of space

Wen PRB 40, 7387 (89), Wen-Niu PRB 41, 9377 (90)

(2) **Non-Abelian geometric phases** of the degenerate ground state from local and global deformation of space manifold.
- Local deformation detects grav. Chern-Simons term $e^{\frac{i}{24} \int_{M^2 \times S^1} \omega_3}$
- Global deformation of torus: $90^\circ$ rotation $|\Psi_\alpha\rangle \rightarrow |\Psi'_\alpha\rangle = S_{\alpha\beta} |\Psi_\beta\rangle$
  Dehn twist: $|\Psi_\alpha\rangle \rightarrow |\Psi'_\alpha\rangle = T_{\alpha\beta} |\Psi_\beta\rangle$

$S, T$ generate a rep. of modular group: $S^2 = (ST)^3 = C, C^2 = 1$

The above properties are robust against any impurities!
Monoid and group structures of topological orders

- Let $S_{d+1} = \{a, b, c, \cdots\}$ be a set of topologically ordered phases in $d$ spatial dimensions.

Stacking $a$-TO state and $b$-TO state $\rightarrow$ a $c$-TO state:

$\begin{align*}
  a \boxtimes b &= c, \\
  a, b, c &\in S_{d+1}
\end{align*}$

- $\boxtimes$ make $S_{d+1}$ a monoid (a group without inverse). Number of the types of topological excitations $N_c = N_a N_b$.

- Some topological orders have inverse $\rightarrow$ invertible topological orders (iTO) which form an abelian group.

A topological order is invertible iff it has no non-trivial topological excitations.

- Examples:
  1. 1+1D $p$-wave SC (fermionic).
  2. 2+1D IQH (fermionic).
  3. 2+1D $p + i p$ SC (fermionic).
  4. 2+1D $E_8$ QH (bosonic)
Classification of (potential) invertible topological orders

- **Bosonic iTO:**

  \[
  \begin{array}{cccccccc}
  0 + 1D & 1 + 1D & 2 + 1D & 3 + 1D & 4 + 1D & 5 + 1D & 6 + 1D \\
  0 & 0 & \mathbb{Z} & E_8 & 0 & \mathbb{Z}_2 & 0 & \mathbb{Z} \oplus \mathbb{Z} \\
  0 & 0 & 0 & \mathbb{Z} & \mathbb{Z}_2 & 0 & 0 & 0
  \end{array}
  \]


- **Fermionic iTO:**

  \[
  \begin{array}{cccccccc}
  0 + 1D & 1 + 1D & 2 + 1D & 3 + 1D & 4 + 1D & 5 + 1D & 6 + 1D \\
  \mathbb{Z}_2 & \mathbb{Z}_2 & \text{p-wave} & \mathbb{Z} & p+i \rho & 0 & 0 & \mathbb{Z} \oplus \mathbb{Z} \\
  \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathbb{Z} & 0 & 0 & 0 & 0
  \end{array}
  \]

  Kitaev cond-mat/0010440; Read-Green cond-mat/9906453
(Potential) invertible topological orders are classified by possible gravitational topological terms

- In 2+1D by possible gravitational Chern-Simons terms $e^{i \frac{2\pi c}{24} \int_{M^3} \omega_3}$, which are not well defined.
  - Using $d\omega_3 = p_1$, we define $e^{i \frac{2\pi c}{24} \int_{M^3=\partial M^4} \omega_3} = e^{i \frac{2\pi c}{24} \int_{M^4} p_1}$ which is well defined only when $e^{i \frac{2\pi c}{24} \int_{M^4} p_1} = 1$ on any closed $M^4$.
  - Bosonic: since $\int_{M^4} p_1 = 3 \times \text{int.}$, $\rightarrow c = 8 \times \text{int.}$
  - Fermionic: since $\int_{\text{spin} M^4} p_1 = 48 \times \text{int.}$, $\rightarrow c = \frac{1}{2} \times \text{int.}$

*In the above we require the gravitational Chern-Simons terms to be well defined on any space-time manifolds.*

- For condensed matter Hamiltonian systems, we only require the gravitational Chern-Simons terms to be well defined on space-time of form $M^2 \times S^1$ (mapping torus). In this case:
  - Bosonic: since $\int_{M^2 \times \Sigma^2} p_1 = 12 \times \text{int.}$, $\rightarrow c = 2 \times \text{int.}$
  - Fermionic: since $\int_{\text{spin}(M^2 \times \Sigma^2)} p_1 = 48 \times \text{int.}$, $\rightarrow c = \frac{1}{2} \times \text{int.}$. 

Kong-Wen arXiv:1405.5858

Xiao-Gang Wen, MIT PiTP, IAS, July., 2014

A theory for symmetry protected topological order
2+1D bosonic topo. orders (up to invertibles) via $S, T$

Dim of $S, T = \#$ of topological types $> 1$.

- There is a basis such that $T$ is diagonal, $S$ unitary & symmetric

$$T_{ij} = e^{2\pi i s_i} e^{-2\pi i \frac{c}{24} \delta_{ij}}, \quad S_{1i} > 0, \quad N_{ij}^k = \sum_l \frac{S_{li} S_{lj} (S_{lk})^*}{S_{1l}} = \text{integer} \geq 0.$$ 

$$(ST)^3 = S^2 = C, \quad C^2 = 1, \quad C_{ij} = N_{1ij}.$$ 

$s_i$: spin of $i^{th}$ type of particle. $N_{ij}^k$: fusion coeff. of the particles.

- $N_{ij}^k$ satisfy

$$N_{k}^{ij} = N_{k}^{ji}, \quad N_{j}^{1i} = \delta_{ij}, \quad \sum_{k=1}^{N} N_{ik}^k N_{1k}^j = \delta_{ij},$$

$$\sum_{m=1}^{n} N_{i}^{mj} N_{jm}^{mk} = \sum_{n=1}^{n} N_{ij}^{in} N_{nj}^{jk} \text{ or } N_{k} N_{i} = N_{i} N_{k}$$

where $i, j, \cdots = 1, 2, \cdots, n$, and the matrix $N_i$ is given by $(N_i)_{kj} = N_{ij}^k$. In fact $N_{1ij}$ defines a charge conjugation $i \rightarrow \bar{i}$:

$$N_{1ij} = \delta_{i\bar{j}}.$$
There exist a $c \pmod{8}$ to make $s_i$ to satisfy the following conditions:

- $\tilde{M}_{ij}$ and $s_i$ satisfy
  \[ \sum_j \tilde{M}_{ij} s_j = 0 \pmod{1}, \]
  where $\tilde{M}_{ij} = \delta_{ij} \frac{4}{3} \sum_k M_{ik} - M_{ij} = \text{integer}$, $M_{ij} = 2N_{j\bar{i}}N_{i\bar{j}} + N_{ij}N_{i\bar{j}}$

- $s_i, S_{ij}$ satisfy
  \[ S_{ij} = \frac{1}{\sqrt{\sum_i d_i^2}} \sum_k N_{ij}^k e^{2\pi i (s_i + s_j - s_k)} d_k. \]
  where $d_i$ is the largest eigenvalue of the matrix $N_i$.

- Let $\nu_i = \frac{1}{D^2} \sum_{jk} N_{i}^{jk} d_j d_k e^{4\pi i (s_j - s_k)}$. Then, we also have $\nu_i = 0$ if $i \neq \bar{i}$, and $\nu_i = \pm 1$ if $i = \bar{i}$.
<table>
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<tr>
<th>$N^B_c$</th>
<th>$d_1, d_2, \cdots$</th>
<th>$s_1, s_2, \cdots$</th>
<th>wave func.</th>
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<th>$s_1, s_2, \cdots$</th>
<th>wave func.</th>
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<td>1</td>
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<td>$\Pi(z_i - z_j)^2$</td>
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<td>$\Pi(z_i^* - z_j^*)^2$</td>
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<td>$3^B_{0}$</td>
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<td>0, 0, 0</td>
<td>$\Pi(z_i - z_j)^4$</td>
<td>$4^B_{1/2}$</td>
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<td>$4^B_{0}$</td>
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<td>$5^B_{0}$</td>
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</table>

**Note:** The $\Psi_{\nu}^2$ are the Pfaffian and determinant of the wave functions, representing the symmetry protected topological order.
2+1D fermionic topological orders (up to invertibles)

Classified by modular BFC over $s\text{Rep}(Z_2^f)$.

Lan-Wen arXiv:1507.04673

<table>
<thead>
<tr>
<th>$N_f$</th>
<th>$S_{\text{top}}$</th>
<th>$D^2$</th>
<th>$d_1, d_2, \ldots$</th>
<th>$s_1, s_2, \ldots$</th>
<th>Domain</th>
<th>Residue</th>
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<td>14.4720</td>
<td>1, 1, 1, 1, $\zeta_3^1$, $\zeta_3^{-1}$, $\zeta_3^1$, $\zeta_3^{-1}$</td>
<td>$0, \frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}, 10, -\frac{7}{20}, -\frac{3}{20}$</td>
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<td></td>
</tr>
</tbody>
</table>
• A **symmetric gapped liquid phase**
  \equiv a symmetry enriched topological (SET) order
  \equiv an equivalent class of gapped quantum states, with **symmetric** equivalence relations generated by
  a) \( H \sim (\text{symm.} LU)H(\text{symm.} LU)^\dagger \)
  b) \( H \sim H \otimes H_{\text{tri}}, \) with \( H_{\text{tri}} = S_1 \cdot S_2 + S_3 \cdot S_4 + \cdots \)
Symmetric gapped liquid phases w/o topo. order

- Gapped liquid phases with symmetry and no topological order
  → **symmetry protected topological/trivial (SPT)** order
  \((LU)H_{\text{SPT}}(LU)^\dagger \sim H_{\text{tri}}\) → trivial topological order
  \((\text{symm}.LU)H_{\text{SPT}}(\text{symm}.LU)^\dagger \sim H_{\text{tri}}\) → non-trivial SPT order

- **Examples:**
  1. Haldane phase of spin-1 chain (bosonic) \textit{Haldane 83}
  2. Topological insulators (fermionic).
     - 2D: Kane-Mele 05; Bernevig-Zhang 06; 3D: Moore-Balents 07; Fu-Kane-Mele 07

An SO(3) SPT state in spin-1 chain (Haldane phase)

- If we do not project out the spin-0 state and consider spin-1, 0 chain, it ideal SO(3) SPT state (RG fixed point).

• Degenerate spin-1/2 doublet at each boundary, if we do not break the SO(3) symmetry.

• The above is a different gapped symmetric phase than the local singlet state of spin-1, 0 chain.
A $\mathbb{Z}_2$ SPT state on square lattice

- **Haldane phase with $SO(3)$ symm.:**

  spin-$1/2$ is not a rep. of $SO(3)$

  \[
  \begin{array}{cccc}
  \downarrow & \downarrow & \downarrow & \downarrow \\
  \text{one site} & \text{spin-$1/2$} & \text{one site} & \text{spin-$1/2$}
  \end{array}
  \]

- **2D SPT phase with $\mathbb{Z}_2$ symm.:**

  - Physical states on each site:
    \[
    (\text{spin-$1/2$})^4 = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle
    \]
  
  - The ground state wave function:
    \[
    |\Psi_{\text{CZX}}\rangle = \otimes_{\text{all squares}} (|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)
    \]
A $\mathbb{Z}_2$ SPT state on square lattice

- **Haldane phase with $SO(3)$ symm.**:
  spin-1/2 is not a rep. of $SO(3)$

  \[
  \begin{array}{ccc}
  \text{spin}-0 & \text{spin}-1 & \text{spin}-1/2 \times \text{spin}-1/2 \\
  \bullet & + & \rightarrow \downarrow & \uparrow & \downarrow & \uparrow & \downarrow \\
  \text{one site} & \text{one site} & \text{spin}-1/2 & \text{one site} & \text{spin}-1/2 \\
  \end{array}
  \]

- **2D SPT phase with $\mathbb{Z}_2$ symm.**:

  - Physical states on each site:
    \[(\text{spin-}\frac{1}{2})^4 = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle\]
  - The ground state wave function:
    \[|\Psi_{CZX}\rangle = \bigotimes_{\text{all squares}}(|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)\]
  - The on-site $\mathbb{Z}_2$ symmetry: (acting on each site $|\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle$):
    \[U_{CZX} = U_{CZ} U_X, \quad U_X = X_1 X_2 X_3 X_4, \quad U_{CZ} = CZ_{12} CZ_{23} CZ_{34} CZ_{41}\]
    \[CZ_{ij}: |\uparrow\uparrow\rangle \rightarrow |\uparrow\uparrow\rangle, \quad |\uparrow\downarrow\rangle \rightarrow |\uparrow\downarrow\rangle, \quad |\downarrow\uparrow\rangle \rightarrow |\downarrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle \rightarrow -|\downarrow\downarrow\rangle\]
A $\mathbb{Z}_2$ SPT state on square lattice

Chen-Liu-Wen arXiv:1106.4752

- **Haldane phase with $SO(3)$ symm.:**
  - spin-1/2 is not a rep. of $SO(3)$
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    - $(\text{spin-1/2})^4 = |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle$
  - The ground state wave function:
    - $|\Psi_{CZX}\rangle = \otimes_{\text{all squares}} (|\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\rangle)$
  - The on-site $\mathbb{Z}_2$ symmetry:
    - $(acting\ on\ each\ site\ |\alpha\rangle \otimes |\beta\rangle \otimes |\gamma\rangle \otimes |\lambda\rangle)$:
      - $U_{CZX} = U_{CZ} U_X$, $U_X = X_1 X_2 X_3 X_4$, $U_{CZ} = CZ_{12} CZ_{23} CZ_{34} CZ_{41}$
      - $CZ_{ij}: |\uparrow\uparrow\rangle \rightarrow |\uparrow\uparrow\rangle, \ |\uparrow\down\rangle \rightarrow |\uparrow\down\rangle, \ |\down\up\rangle \rightarrow |\down\up\rangle, \ |\down\down\rangle \rightarrow -|\down\down\rangle$
  - $\mathbb{Z}_2$ symm. Hamiltonian $H = \sum_{\text{square}} H_p$, $H_p = -X_{abcd} P_{ef} P_{gh} P_{ij} P_{kl}$,

- **2D SPT phase with $\mathbb{Z}_2$ symm.:**
  - $\mathbb{Z}_2$ symm. Hamiltonian $H = \sum_{\text{square}} H_p$, $H_p = -X_{abcd} P_{ef} P_{gh} P_{ij} P_{kl}$,
Edge excitations for the 2D $Z_2$ SPT state

- **Bulk Hamiltonian** \( H = \sum H_p \), \( H_p = -X_{abcd} P_{ef} P_{gh} P_{ij}, P_{kl} \), \( X_{abcd} = |↓↓↓↓\rangle\langle↑↑↑↑| + |↑↑↑↑\rangle\langle↓↓↓↓| \), \( P = |↑↑\rangle\langle↑↑| + |↓↓\rangle\langle↓↓| \).

- **Edge excitations**: gapless or break the $Z_2$ symmetry, robust against any perturbations that do not break the $Z_2$ symmetry.

- **Edge effective spin** \( \tilde{\uparrow} \) and \( \tilde{\downarrow} \).

- **Edge eff. $Z_2$ symm.** : $\tilde{U}_{Z_2} = \prod_i CZ_{i,i+1} \prod_i \tilde{X}_i$ which cannot be written as $U_{Z_2} = \prod_i O_i$, such as $U_{Z_2} = \prod \tilde{X}_i$.

  *Not an on-site symmetry!*

- **Edge effective Ham.** (\( c = 1 \) gapless if the $Z_2$ is not broken)

  \[
  H = \sum \left( -J \tilde{Z}_i \tilde{Z}_{i+1} + B_x [\tilde{X}_i + \tilde{Z}_{i-1} \tilde{X}_i \tilde{Z}_{i+1}] + B_y [\tilde{Y}_i - \tilde{Z}_{i-1} \tilde{Y}_i \tilde{Z}_{i+1}] \right)
  \]

  \[
  H = \sum (\tilde{X}_i + \tilde{Z}_{i-1} \tilde{X}_i \tilde{Z}_{i+1}) \text{ dual to } H = - \sum (X_i X_{i+1} + Y_i Y_{i+1})
  \]

- **Non-on-site $\tilde{U}_{Z_2} \rightarrow$ anomalous symmetry.** Not gaugable.

  **On-site $U_{Z_2} \rightarrow$ anomaly-free symm.** – the usual global symm.
Field theory for edge of the 2+1D $\mathbb{Z}_2$ SPT state

- The primary field of $U(1)$ current algebra $V_{l,m}$ has dimensions $(h_R, h_L) = \left(\frac{(l+2m)^2}{8}, \frac{(l-2m)^2}{8}\right)$.
- The $\mathbb{Z}_2$ symmetry action $V_{l,m} \rightarrow (-)^{l+m} V_{l,m}$
- Edge effective (chiral boson) theory: Chen-Wen arXiv:1206.3117

$$\mathcal{L}_{\text{edge}} = -\frac{1}{4\pi} (\partial_x \phi_1 \partial_t \phi_2 + \partial_x \phi_2 \partial_t \phi_1) - \frac{1}{4\pi} V(\partial_x \phi_1 \partial_x \phi_1 + \partial_x \phi_2 \partial_x \phi_2)$$

where $\phi_1 \sim \phi_1 + 2\pi$, $\phi_2 \sim \phi_2 + 2\pi$.

- $\mathbb{Z}_2$ symmetry = $(\phi_1, \phi_2) \rightarrow (\phi_2, \phi_1)$
- Choose $\phi_{\pm} = \phi_1 \pm \phi_2$ (Under $\mathbb{Z}_2$: $\phi_{\pm} \rightarrow \pm \phi_{\pm}$)

$$\mathcal{L}_{\text{edge}} = -\frac{1}{8\pi} (\partial_x \phi_+ \partial_t \phi_+ - \partial_x \phi_- \partial_t \phi_-) - \frac{1}{8\pi} V[(\partial_x \phi_+)^2 + (\partial_x \phi_-)^2]$$

$\phi_+$ right movers with no $\mathbb{Z}_2$ charge. $\phi_-$ left movers with $\mathbb{Z}_2$ charge.

- The 1+1D theory has a $\mathbb{Z}_2$ anomaly
  → It is the edge of a 2+1D $\mathbb{Z}_2$ SPT state.
Non-linear $\sigma$-model (NL$\sigma$M) for generic SPT states

- Consider an $d + 1$D system: $S = \int d^d x dt \frac{1}{2\lambda} |g^{-1}(x) \partial g(x)|^2$.
  Symmetry $g(x) \rightarrow hg(x)$. $h, g$ are unitary matrices in $G$.
- $\lambda$ is small $\rightarrow$ the ground state is ordered $\langle g(x, t) \rangle = g_0$.
  $\lambda$ is large $\rightarrow$ the ground state is disordered $\langle g(x, t) \rangle = 0$.
- The fixed point action of the disordered phases:
  Under RG, $\lambda \rightarrow \infty \rightarrow$ the disordered symmetric phase is described by a fixed point theory $S_{\text{fixed}} = 0$ or $e^{-S_{\text{fixed}}} = 1$.
  All correlations are short ranged. All excitations are gapped.

- We thought that the disordered ground states are trivial, no excitations at low energies.
  But the disordered ground states can be non-trivial: they can belong to different phases, even without symmetry breaking.
  $\rightarrow$ the notions of topological orders and SPT orders.

Chen-Gu-Liu-Wen arXiv:1106.4772
Different disordered phases can arise from NL$\sigma$M with different topological terms.
Different disordered phases can arise from NL$\sigma$M with different topological terms.

- Another $G$ symmetric system

\[ S = \int d^d x d t \left( \frac{1}{2\lambda} |g^{-1}\partial g|^2 + 2\pi i W_{G\text{-top}} \right) \]

where $W_{G\text{-top}}[g(x^i, t)]$ is a topological term.

- The topo. term $W_{G\text{-top}}[g(x^i, t)]$ can appear if $\pi_{d+1}(G) \neq 0$.

**Example 1:** $0 + 1$D with $G = U_1 = S^1$ ($g = e^{i \theta}$), $\pi_1(U_1) = \mathbb{Z}$

\[ W_{G\text{-top}} = -k \frac{i}{2\pi} g^{-1}\partial_t g = k \frac{\dot{\theta}}{2\pi}, \quad k \in \mathbb{Z}. \]

\[ \int dt W_{G\text{-top}} = k \times \text{winding number}. \]

**Example 2:** $2 + 1$D with $G = SU_2 = S^3$, $\pi_3(SU_2) = \mathbb{Z}$

\[ W_{G\text{-top}} = k \frac{1}{24\pi^2} \epsilon^{\mu\nu\lambda} (ig^{-1}\partial_\mu g)(ig^{-1}\partial_\nu g)(ig^{-1}\partial_\lambda g), \quad k \in \mathbb{Z}. \]

\[ \int dt d^2x W = k \times \text{winding number}. \]

- The topological term has no dynamical effect $e^{-S_{\text{fixed}}} = 1$, but can give rise to different symmetric phases classified by $k$. 

Xiao-Gang Wen, MIT PiTP, IAS, July., 2014

A theory for symmetry protected topological order
Fixed-point eff. Lagrangian for symmetric phases

- If $\lambda$ is large $\rightarrow$ disordered state.
  Under RG, $\lambda \rightarrow \infty \rightarrow$ disordered symmetric ground state is described by a low energy fixed-point theory $S_{\text{fixed}} = 2\pi i \int W_{G\text{-top}}$
  - topological non-linear $\sigma$-model with pure topological term.

---

Ex. $S_{\text{fixed}} = 2\pi i \int W_{G\text{-top}} = k^2 \pi i^2 \int (i g^{-1} dg)^3, k \in \mathbb{Z}$.

$\text{Hom}(\pi^{d+1}(G), \mathbb{Z})$ is wrong. The right answer is $H^{d+1}(G, U_1)$.

In the $\lambda \rightarrow \infty$ limit, $g(x_i, t)$ is not a continuous function. The mapping classes $\pi^{d+1}(G)$ does not make sense. The above result is not valid. However, the idea is OK.

Can we define topological terms and topological non-linear $\sigma$-models when space-time is a discrete lattice?
Fixed-point eff. Lagrangian for symmetric phases

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  Under RG, $\lambda \rightarrow \infty \rightarrow$ disordered symmetric ground state is described by a low energy fixed-point theory $S_{\text{fixed}} = 2\pi i \int W_{G\text{-top}}$
  - \textit{topological non-linear $\sigma$-model with pure topological term}.

- Fixed point theories ($2\pi$-quantized topological terms) $\leftrightarrow$ symmetric phases:
  \textit{The symmetric phases are classified by different $2\pi$-quantized topological terms ($\text{Hom}(\pi_{d+1}(G), \mathbb{Z})$ linear maps $\pi_{d+1}(G) \rightarrow \mathbb{Z}$)}
  
  Ex. $S = 2\pi i \int W_{G\text{-top}} = k \frac{2\pi i}{24\pi^2} \int (ig^{-1}dg)^3, \ k \in \mathbb{Z}$
  $\text{Hom}(\pi_{d+1}(G), \mathbb{Z}) = \{k\}$. 

Fixed-point eff. Lagrangian for symmetric phases

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Fixed-point eff. Lagrangian for symmetric phases

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  Under RG, $\lambda \rightarrow \infty \rightarrow$ disordered symmetric ground state is described by a low energy fixed-point theory $S_{\text{fixed}} = 2\pi i \int W_{\text{G-top}}$ – topological non-linear $\sigma$-model with pure topological term.

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  Ex. $S = 2\pi i \int W_{\text{G-top}} = k \frac{2\pi i}{24\pi^2} \int (i g^{-1} dg)^3, \ k \in \mathbb{Z}$

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• Can we define topological terms and topological non-linear $\sigma$-models when space-time is a discrete lattice?
Path integral on 1+1D space-time lattice with branching structure:

\[ e^{-S} = \prod \nu_{ij}^{sijk}(g_i, g_j, g_k), \]

where \( \nu_{ij}^{sijk}(g_i, g_j, g_k) = e^{-\int_\Delta L} \) and \( s_{ijk} = 1, \ast \)

The above defines a LN\(\sigma\)M with target space \( G \) on 1+1D space-time lattice.

The \(\text{NL}\sigma\)M will have a symmetry \(G\) if \(g_i \in G\) and

\[ \nu_2(g_i, g_j, g_k) = \nu_2(hg_i, hg_j, hg_k), \quad h \in G \]

The above is the lattice version of \(\text{NL}\sigma\)M field theory:

\[ \mathcal{L} = \frac{1}{\lambda} |g^{-1} \partial g|^2, \quad \text{symm.} \quad h : g(x) \rightarrow hg(x) \]
Topo. term and topo. NLσM on space-time lattice

- \( \nu(g_i, g_j, g_k) \) give rise to a topological NLσM if 
  \( e^{-S_{\text{fixed}}} = \prod \nu^{s_{ijk}}(g_i, g_j, g_k) = 1 \) on any sphere, 
  including a tetrahedron (simplest sphere).

- \( \nu(g_i, g_j, g_k) \in U_1 \)

- On a tetrahedron \( \rightarrow \) 2-cocycle condition
  \[
  \nu_2(g_1, g_2, g_3)\nu_2(g_0, g_1, g_3)\nu_2^{-1}(g_0, g_2, g_3)\nu_2^{-1}(g_0, g_1, g_2) = 1
  \]
  The solutions of the above equation are called group cocycle.

- The 2-cocycle condition has many solutions: 
  \( \nu_2(g_0, g_1, g_2) \) and \( \tilde{\nu}_2(g_0, g_1, g_2) = \nu_2(g_0, g_1, g_2) \frac{\beta_1(g_1, g_2)}{\beta_1(g_0, g_1)} \) are both cocycles. We say \( \nu_2 \sim \tilde{\nu}_2 \) (equivalent).

- The set of the equivalent classes of \( \nu_2 \) is denoted as 
  \( \mathcal{H}^2(G, U_1) = \pi_0(\text{space of the solutions}) \).

- \( \mathcal{H}^2(G, U_1) \) (\( \rightarrow \) topo. terms) describes 1+1D SPT phases 
  protected by \( G \).
Group cohomology $\mathcal{H}^d[G, U_1]$ in any dimensions

- **$d$-Cochain:** $U_1$ valued function of $d + 1$ variables
  \[ \nu_d(g_0, \ldots, g_d) = \nu_d(gg_0, \ldots, gg_d) \in U_1, \quad \text{on-site } G \text{-symmetry} \]

- **$\delta$-map:** $\nu_d$ with $d + 1$ variables $\rightarrow (\delta \nu_d)$ with $d + 2$ variables
  \[ (\delta \nu_d)(g_0, \ldots, g_{d+1}) = \prod_i (\nu_d)^{(-i)}(g_0, \ldots, \hat{g}_i, \ldots, g_{d+1}) \]

- **Cocycles = cochains that satisfy**
  \[ (\delta \nu_d)(g_0, \ldots, g_{d+1}) = 1. \]

- **Equivalence relation generated by any $d - 1$-cochain:**
  \[ \nu_d(g_0, \ldots, g_d) \sim \nu_d(g_0, \ldots, g_d)(\delta \beta_{d-1})(g_0, \ldots, g_d) \]

- $\mathcal{H}^{d+1}(G, U_1)$ is the equivalence class of cocycles $\nu_d$.

$d + 1$D lattice topological NL$\sigma$Ms with symmetry $G$ in are classified by $\mathcal{H}^{d+1}(G, U_1)$:

\[ e^{-S} = \prod \nu_{d+1}^{s(i,j,\ldots)}(g_i, g_j, \ldots), \quad \nu_{d+1}(g_0, g_1, \ldots, g_{d+1}) \in \mathcal{H}^{d+1}(G, U_1) \]
As we change the space-time lattice, the action amplitude \( e^{-S} \) does not change:

\[
\nu_2 (g_0, g_1, g_2) \nu_2^{-1} (g_1, g_2, g_3) = \nu_2 (g_0, g_1, g_3) \nu_2^{-1} (g_0, g_2, g_3)
\]

\[
\nu_2 (g_0, g_1, g_2) \nu_2^{-1} (g_1, g_2, g_3) \nu_2 (g_0, g_2, g_3) = \nu_2 (g_0, g_1, g_3)
\]

as implied by the cocycle condition:

\[
\nu_2 (g_1, g_2, g_3) \nu_2 (g_0, g_1, g_3) \nu_2^{-1} (g_0, g_2, g_3) \nu_2^{-1} (g_0, g_1, g_2) = 1
\]

**The topological NL\(\sigma\)M is a RG fixed-point.**

- What is the ground state wave function of the topological NL\(\sigma\)M?
The NLσM ground state is short-range entangled

The ground state wave function $\Psi(\{g_i\}) = \prod_i \nu_2(g_i, g_{i+1}, g^*)$
The NLσM ground state is short-range entangled

\[ \Psi(\{g_i\}) = \prod \nu_2(g_i, g_{i+1}, g^*) \]

- It is symmetric under the \( G \)-transformation

Xiao-Gang Wen, MIT PiTP, IAS, July., 2014

A theory for symmetry protected topological order
The NLσM ground state is short-range entangled

The ground state wave function \( \Psi(\{g_i\}) = \prod_i \nu_2(g_i, g_{i+1}, g^*) \)

- It is symmetric under the \( G \)-transformation
- It is equivalent to a product state \( |\Psi_0\rangle = \bigotimes_i \sum_{g_i} |g_i\rangle \) under a LU transformation (note that \( \Psi_0(\{g_i\}) = 1 \))

\[
\Psi(\{g_i\}) = \prod_{i=\text{even}} \nu_2(g_i, g_{i+1}, g^*) \prod_{i=\text{odd}} \nu_2(g_i, g_{i+1}, g^*) \Psi_0(\{g_i\})
\]

\( \Psi_0(\{g_i\}) \rightarrow \text{Short-range entangled} \)

The ground state is symmetric with a trivial topo. order
The NL$\sigma$M defined by $\nu_{d+1}$ has no topological order

Does the partition function $Z[M^{d+1}]$ have any nontrivial dependence on the “shape” or topology of the space-time manifold $M^{d+1}$?

$$Z[M^{d+1}] = \sum_{\{g_i\}} \prod \nu_{d+1}^{s_0 s_1 s_2 \cdots} (g_0, g_1, g_2, \cdots) = |G|^{N_\nu}$$

for any space-time manifold $M^{d+1}$ obtained by gluing $S^{d+1}$’s.

No topological order (?)
### SPT phases from $H^{d+1}(G, U_1)$

<table>
<thead>
<tr>
<th>Symmetry $G$</th>
<th>$d = 0$</th>
<th>$d = 1$</th>
<th>$d = 2$</th>
<th>$d = 3$</th>
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<tbody>
<tr>
<td>$U_1 \times Z_2^T$ (top. ins.)</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2 (0)$</td>
<td>$\mathbb{Z}_2 (\mathbb{Z}_2)$</td>
<td>$\mathbb{Z}_2^2 (\mathbb{Z}_2)$</td>
</tr>
<tr>
<td>$U_1 \times Z_2^T \times \text{trans}$</td>
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<td>$\mathbb{Z} \times \mathbb{Z}_2$</td>
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</tr>
<tr>
<td>$U_1 \times Z_2^T$ (spin sys.)</td>
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<td>$\mathbb{Z}_2^2$</td>
<td>0</td>
<td>$\mathbb{Z}_2^3$</td>
</tr>
<tr>
<td>$U_1 \times Z_2^T \times \text{trans}$</td>
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<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_2^4$</td>
<td>$\mathbb{Z}_2^9$</td>
</tr>
<tr>
<td>$Z_2^T$ (top. SC)</td>
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<td>$U_1 \times \text{trans}$</td>
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“$Z_2^T$”: time reversal, “trans”: translation, 0 → only trivial phase. ($\mathbb{Z}_2$) → free fermion result.
• How do you know the NLσM’s with different cocycles produce different SPT orders? Why “seemsly-the-same” path integrals can produce different SPT phases? How do you measure SPT orders? 

**Universal probe** = one probe to detect all possible orders.
Universal probe for SPT orders

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- Universal probe for crystal order
  
  = X-ray diffraction:

  ![X-ray diffraction diagram](image-url)
Universal probe for SPT orders

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**Universal probe** = one probe to detect all possible orders.

- Universal probe for crystal order
  = X-ray diffraction:

- Partitional function as an universal probe, but $Z_{\text{top}}^{SPT}(M^d) = 1 \rightarrow$ does not work.

- Twist the symmetry by “gauging” the symmetry on $M^d$
  $\rightarrow A - G$ gauge field.
  $\rightarrow Z_{\text{top}}^{SPT}(A, M^d) \neq 1$.


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A theory for symmetry protected topological order
Symmetry twist and “gauging” NL$\sigma$M

- In the NL$\sigma$M path integral

\[ Z(M^d) = \int D[g(x)] e^{-\int d^d x \ L(g, g^{-1}dg)}, \]

we sum over all the cross-sections of a trivial bundle $G \times M^d$.

- In the “gauged” NL$\sigma$M (with symmetry twist), we sum over all the cross-sections of a flat bundle $G \times M^d$ with a flat connection.

“Gauging” (adding symmetry twist) in more details

- Change variable $g(x) \rightarrow h(x)g(x)$:

\[ \mathcal{L}(hg, (hg)^{-1}d(hg)) = \mathcal{L}(g, g^{-1}dg + A), \quad A = h^{-1}dh \rightarrow \]

\[ Z(M^d) = \int D[g(x)] e^{-\int d^d x \ L(g, g^{-1}(d - iA)g)}, \quad A = i h^{-1}dh. \]

“Gauged” partition function (with symmetry twist)

\[ Z(A, M^d) = \int D[g(x)] e^{-\int d^d x \ L(g, g^{-1}(d - iA)g)}, \quad F = dA + i[A, A] = 0 \]

- For continuous group $G$, we can generalize the above to non-flat connection and non-flat bundle.
Examples of symmetry twist (gauge configuration)

- **$U(1)$** symmetry twist: closed one-form $A$ with
  \[ \oint_{S^1_x} A = \phi_x, \]
  \[ \oint_{S^1_y} A = \phi_y. \]

- **$\mathbb{Z}_2$** symmetry twist: closed quantized one-form $A$ with
  \[ \oint_{S^1_x} A = 0, \pi, \oint_{S^1_y} A = 0, \pi. \]

- We can choose the one-form $A$ to be non-zero only on some codimension-1 closed sub-manifolds.

- Contractable loop $\rightarrow$ exact one-form $A = df$

  (pure gauge or coboundary)

- $\mathbb{Z}_2$ symmetry twist $\leftrightarrow$ codimension-1 sub-manifolds

  (Poincaré duality).
Universal topo. inv.: “gauged” partition function

\[
\frac{Z(A, M^d)}{Z(0, M^d)} = \int Dg e^{-\int \mathcal{L}(g^{-1}(d-iA)g)} = e^{-i2\pi \int W_{A-top}(A)}
\]

- \(W_{A-top}(A)\) and \(W'_{A-top}(A)\) are equivalent if
  \[
  W'_{A-top}(A) - W_{A-top}(A) = \frac{1}{\lambda_g} \text{Tr}(F^2) + \cdots
  \]
- The equivalent class of the gauge-topological term \(W_{A-top}(A)\) is the topological invariant that probe different SPT state.
- The topological invariant \(W_{A-top}(A)\) are Chern-Simons terms or Chern-Simons-like terms.
- Such Chern-Simons-like terms are classified by
  \[
  H^{d+1}(BG, \mathbb{Z}) = \mathcal{H}^d[G, U(1)]
  \]

*The topological invariant \(W_{A-top}(A)\) can probe all the NL\(\sigma\)M SPT states*
- \(W_{A-top}(A)\) can be viewed as a Lagrangian that defines a gauge theory:
  **Dijkgraaf-Witten gauge theory**

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Calculate the SPT invariants for SPT phases

- Topological NL$\sigma$M, as a fixed-point theory, contain only the pure topological term, and it is easy to calculate $W_{A\text{-top}}(A)$:

- **Lattice**: cocycle $\nu_d(\{g_i\}) \rightarrow$ DW action $W_{A\text{-top}}(A)$
- NL$\sigma$M: $Z = \sum_{\{g_i\}} \prod \nu^3_{ijk}(g_i, g_j, g_k) = \int D[g] e^{i2\pi W_{G\text{-top}}(g^{-1}dg)}$
- DW-gauge theory $Z = \sum_{\{g_{ij}\}} \prod \omega^3_{ijk}(g_{ij}, g_{jk}) = \int D[A] e^{i2\pi W_{A\text{-top}}(A)}$
  where the amplitude $e^{i2\pi W_{A\text{-top}}(A)}$ is non-zero only for flat connections: $g_{ik} = g_{ij}g_{jk}$.
- Connection: $\nu_3(g_i, g_j, g_k) = \nu_3(hg_i, hg_j, hg_k) = \omega_3(g_i^{-1}g_j, g_j^{-1}g_k)$

- **Continuum**: $G$-topo. term $W_{G\text{-top}}(g^{-1}dg)g^{-1}dg \rightarrow A \rightarrow W_{A\text{-top}}(A)$

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An example: $SU_2$ SPT state

- Topo. term for $SU_2$ SPT state:
- In 2+1D $\pi_3(SU_2) = \mathbb{Z}$:
  \[
  W^3_{G\text{-top}} = k \frac{\text{Tr}(ig^{-1}dg)^3}{24\pi^2}.
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- In 2 + 1D $\pi_3(SU_2) = \mathbb{Z}$: $A = 2\times2$ matrix;
  
  \[
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  \]
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A theory for symmetry protected topological order

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\]

**SPT inv. → phys. measurement:**
- Spin quantum Hall conductance $\sigma^\text{spin}_{xy} = \frac{k}{4\pi}$
- Gapless state if the $SU_2$ symm. is not broken.

(No topo. order, need symm. protection) Liu-Wen arXiv:1205.7024

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The edge of the $SU(2)$ SPT state must be gapless

Bulk fixed-point action:  

$$S_{\text{bulk}} = -i \frac{k}{12\pi} \int_{M^3} \text{Tr}(g^{-1} dg)^3, \quad k \in \mathbb{Z}, \quad g \in SU(2)$$

The $SU(2)$ symmetry $g(x) \rightarrow hg(x), \quad h, g(x) \in SU(2)$
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The $SU(2)$ symmetry $g(x) \to hg(x), \quad h, g(x) \in SU(2)$

The edge excitations on $\partial M^3$ described by fixed-point WZW:

$$S_{\text{edge}} = \int_{\partial M^3} \frac{k}{8\pi} \text{Tr}(\partial g^{-1} \partial g) - i \int_{M^3} \frac{k}{12\pi} \text{Tr}(g^{-1}dg)^3,$$

At the fixed point, we have an equation of motion

$$\partial_z [(\partial_z g)g^{-1}] = 0, \quad \partial_z [(\partial_{\bar{z}} g^{-1})g] = 0, \quad z = x + i t.$$

Right movers $[(\partial_z g)g^{-1}](z) \to SU(2)$-charges
Left movers $[(\partial_{\bar{z}} g^{-1})g](\bar{z}) \to SU_L(2)$-charges, $g(x) \to g(x)h_L$

$Level-k$ Kac-Moody algebra  
Witten NPB 223, 422 (83)

The $SU(2)$ symmetry is anomalous at the edge.
In general, $G$ SPT state has anomalous $G$-symmetry at the boundary – a defining property of SPT phases.

Theorem: The boundary of any 2+1D SPT states must be gapless or symmetry breaking.  
Chen-Liu-Wen arXiv:1106.4752;
**U(1) SPT phases and their physical properties**

- Topo. terms for $U_1$ SPT state:
  - In 0 + 1D, $W_{A\text{-top}}^1 = k \frac{A}{2\pi}$. 

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**U(1) SPT phases and their physical properties**

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$$Z[A] = \text{Tr}(U_\theta^{\text{twist}} e^{-H}) = e^{ik \oint_{S^1} A} = e^{ik\theta}$$

**SPT inv. → phys. measurement:**
→ ground state carries charge $k$

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Lu-Vishwanath arXiv:1205.3156
Xiao-Gang Wen, MIT PiTP, IAS, July., 2014

A theory for symmetry protected topological order
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  - $Z[A] = \text{Tr}(U_{\theta}^{\text{twist}} e^{-H}) = e^{i k \oint_{S_1} A} = e^{i k \theta}$

**SPT inv. $\rightarrow$ phys. measurement:**
- $\rightarrow$ ground state carries charge $k$

- In $2 + 1D$, $W_{A-top}^3 = k \frac{AF}{(2\pi)^2}$ (?)

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$U(1)$ SPT phases and their physical properties

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- In $2+1$D, $W_{A\text{-top}}^3 = k \frac{AF}{(2\pi)^2}$ (?)

SPT inv. $\rightarrow$ phys. measurement:
  $\rightarrow$ Hall conductance $\sigma_{xy} = 2k \frac{e^2}{h}$
  $\rightarrow$ The edge of $U_1$ SPT phase must be gapless with left/right movers and has anomalous $U(1)$ symm.
  $\rightarrow$ Choose space-time $S^1 \times M^2$ and put $2\pi m$ flux through $M^2$.
  $\mathcal{L}^{2+1D} = k \text{AdA}/2\pi \rightarrow \mathcal{L}^{0+1D} = k \int_{M^2} \text{AdA}/2\pi = 2kmA$.

- The $2+1$D $U_1$ SPT state labeled by $k$ reduces to a $0+1$D $U_1$ SPT state labeled by $2km$ (with charge $2km$ in ground state).
- $2\pi m$ flux in space $M^2$ induces $2km$ unit of charge $\rightarrow$ Hall conductance $\sigma_{xy} = 2ke^2/h$.

Lu-Vishwanath arXiv:1205.3156
From probe to mechanism of SPT states

- **To probe:**
  
  \[2\pi \text{ flux inducing } 2k \text{ charge probes } U(1) \text{ SPT state.}\]

- **To create:**
  
  Attaching \(2k\) charges to a \(U(1)\) vortex makes \(U(1)\) SPT state.

---

Xiao-Gang Wen, MIT PiTP, IAS, July, 2014

A theory for symmetry protected topological order
From probe to mechanism of SPT states

• To probe:
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• To create:
  \[ \text{Attaching } 2k \text{ charges to a } U(1) \text{ vortex makes } U(1) \text{ SPT state.} \]

• Start with 2+1D bosonic superfluid:
  proliferate vortices \(\rightarrow\) trivial Mott insulator.
  proliferate vortex+2k-charge \(\rightarrow\) \(U(1)\) SPT state labeled by \(k\).

- Why? \(U(1) \text{ flux is the } U(1) \text{ symmetry twist. A vortex in } U(1) \text{ superfluid is}
  a } 2\pi \text{ } U(1) \text{ symmetry twist } = 2\pi \text{ flux.}

  The vortex condensed state (or the vortex proliferated state)
  remembers the binding of vortex and 2k-charge:
  a } 2\pi \text{ } U(1) \text{ symmetry twist (2}\pi \text{ flux) carries } 2k \text{ charges.}
Can we bind charge-1 to a vortex, to make a new $U(1)$ SPT state beyond group cohomology, which has $\sigma_{xy} = \frac{e^2}{h}$?
Can we bind charge-1 to a vortex, to make a new $U(1)$ SPT state beyond group cohomology, which has $\sigma_{xy} = \frac{e^2}{h}$? No!

(1) Inserting $2\pi$ flux will always create a quasiparticle. Such a quasiparticle would carry a unit $U(1)$ charge. The bound state of charge-1 + $2\pi$-flux is a fermion, which implies that the $\sigma_{xy} = \frac{e^2}{h}$ SPT state must carry a non-trivial topological order.
Why bind even charge to vortex?

- Can we bind charge-1 to a vortex, to make a new $U(1)$ SPT state beyond group cohomology, which has $\sigma_{xy} = \frac{e^2}{h}$? **No!**

  (1) Inserting $2\pi$ flux will always create a quasiparticle. Such a quasiparticle would carry a unit $U(1)$ charge. The bound state of charge-1 $+ 2\pi$-flux is a fermion, which implies that the $\sigma_{xy} = \frac{e^2}{h}$ SPT state must carry a non-trivial topological order.

  (2) The bound state of charge-1 $+ v$ortex is a fermion. They cannot condense to make the superfluid into an insulator. But a (charge-1 $+ v$ortex)-pair is a boson. Proliferate/condensing such (charge-1 $+ v$ortex)-pairs can make the superfluid into an insulator with non-trivial $\mathbb{Z}_2$ topological order described by $\mathbb{Z}_2$ gauge theory.

The duality between the probe and the mechanism is a general phenomenon which also appears for other SPT orders.
Electromagnetic response in state with no topological order

- Bosons with no topo. order: $\frac{2k}{4\pi} \int_{M^3} A dA$
- Fermions with no topo. order: $\frac{k}{4\pi} \int_{M^3} A dA$

An understanding via algebraic topology:

- The Chern-Simons term is better defined by going to one higher dimension:
  
  \[ \frac{\sigma_{xy}}{4\pi} \int_{M^3=\partial M^4} A dA = \frac{\sigma_{xy}}{2} \frac{2\pi}{4\pi} \int_{M^4} \left( \frac{dA}{2\pi} \right)^2 \]

  which well defined only when $\frac{\sigma_{xy}}{2} \int_{M^4} \left( \frac{dA}{2\pi} \right)^2$ is always integer for closed $M^4$.

- Two math relations:
  1) $Sq^2(x_2) = x_2 \cup x_2$ for any 2-cocycle $x_2 \in H^2(M^4, \mathbb{Z}_2)$
  2) $Sq^2(x_2) = u_2 \cup x_2 = (w_2 + w_1 \cup w_1) \cup x_2$ in 4-dimensions

  Choose $x_2 = \frac{dA}{2\pi} \mod 2 \rightarrow (\frac{dA}{2\pi})^2 = (w_2 + w_1 \cup w_1) \cup \frac{dA}{2\pi} \mod 2$.

- On $CP^2$, $\int_{M^4} \left( \frac{dA}{2\pi} \right)^2 = 1 \rightarrow \frac{\sigma_{xy}}{2} = \text{int. for bosons}$.

- On spin manifold, $w_1, w_2 = 0$ and $\int_{M^4} \left( \frac{dA}{2\pi} \right)^2 = 0 \mod 2$

  $\rightarrow \sigma_{xy} = \text{int. for fermions}$.
The boundary of the 2+1D bosonic $U(1)$ SPT state has a 1+1D bosonic $U(1)$ gauge anomaly

The boundary of the 2+1D $U(1)$ SPT state must be gapless.
• Partition function of $U(1)$-SPT state on space-time with boundary:
\[
Z(A, M^3) = e^{-\epsilon \text{Vol} e^{i \frac{2k}{4\pi} A dA + \int_{\partial M} dt dx \mathcal{L}_{\text{edge}}}}
\]

• The total $Z(A, M^3)$ is gauge invariant under $A \rightarrow A + df$, but the bulk CS-term and the edge action separately are not gauge invariant if $2k \neq 0$. We need a $U(1)$ anomalous edge described by $\mathcal{L}_{\text{edge}}$ to cancell the gauge non-invariance of the CS-term.

• Such an edge must be gapless.  

Wen PRB 43, 11025 (89)
A mechanism for 2+1D $U_1 \rtimes Z_2^T$ SPT state

- 2+1D boson superfluid + gas of vortex
  $\rightarrow$ boson Mott insulator.
- 2+1D boson superfluid + gas of $S^z$-vortex
  $\rightarrow$ boson topological insulator ($U_1 \rtimes Z_2^T$ SPT state)

- The boson superfluid + spin-1 system

  $S^z$-vortex $=\,$
  vortex $+$ ($S_z = +1$)-spin
  anti $S^z$-vortex $=\,$
  anti-vortex $+$ ($S_z = -1$)-spin

Probing 2+1D $U_1 \rtimes Z_2^T$ SPT state

Let $\Phi_{\text{vortex}}$ be the creation operator of the vortex. Then

$$T^{-1} \Phi_{\text{vortex}} T = \Phi_{\text{vortex}}^\dagger, \quad \Phi_{S^z\text{-vortex}} = S^+ \Phi_{\text{vortex}}, \quad T^{-1} \Phi_{S^z\text{-vortex}} T = -\Phi_{S^z\text{-vortex}}^\dagger.$$

The $\pi$-flux in the $U_1 \rtimes Z_2^T$ SPT state is Kramer doublet:

$\Phi_{S^z\text{-vortex}} \begin{pmatrix} -\pi \end{pmatrix} = \begin{pmatrix} \pi \end{pmatrix}, \quad \Phi_{S^z\text{-vortex}}^\dagger \begin{pmatrix} \pi \end{pmatrix} = \begin{pmatrix} -\pi \end{pmatrix}$,
\[ \oint A_{\mathbb{Z}_2} = 0, \pi; \quad a \equiv \frac{A_{\mathbb{Z}_2}}{\pi}; \]

<table>
<thead>
<tr>
<th>(d)</th>
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Wen arXiv:1410.8477
**Z₂ SPT phases and their physical properties**

- **Topological terms:**
  \[ \oint A Z^2_2 = 0, \pi; a \equiv \frac{A Z^2_2}{\pi}; \]

- **In 0 + 1D**, \( W^1_{A\text{-top}} = k \frac{A Z^2_2}{2\pi} = ka. \)

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Wen arXiv:1410.8477
Z₂ SPT phases and their physical properties

- Topological terms:
  \[ \oint A Z₂^2 = 0, \pi; a \equiv \frac{AZ₂}{\pi}; \]

- In 0 + 1D, \( W_{A\text{-top}}^1 = k \frac{AZ₂}{2\pi} = ka. \)
  \[ Z[a] = \text{Tr}(U^{\text{twist}}_\pi e^{-H}) = e^{2\pi i \oint S_1 W_{A\text{-top}}} = e^{ik\pi} \oint a = e^{ik\pi} = \pm 1, \ k = 0, 1 \]

SPT inv. → phys. measurement:
→ ground state \( Z_2 \)-charge = \( k = 0, 1 \)

\[ \begin{array}{|c|c|c|}
\hline
 d & \mathcal{H}^d[Z₂] & W_{A\text{-top}}^d \\
\hline
 0 + 1 & \mathbb{Z}_2 & \frac{1}{2}a \\
 1 + 1 & 0 & \frac{1}{2}a^3 \\
 2 + 1 & \mathbb{Z}_2 & \frac{1}{2}a^3 \\
 3 + 1 & 0 & \\
\hline
\end{array} \]

Wen arXiv:1410.8477
Z₂ SPT phases and their physical properties

- Topological terms:

- In 0 + 1D, \( W^1_{A\text{-top}} = k \frac{A_{Z_2}}{2\pi} = ka \).
  
  \[
  Z[a] = \text{Tr}(U^{\text{twist}} e^{-H}) = e^{2\pi i \oint_{S^1} W_{A\text{-top}}} = e^{i k \pi} = \pm 1, \ k = 0, 1
  \]
  
  **SPT inv. \( \rightarrow \) phys. measurement:**
  
  \( \rightarrow \) ground state \( Z_2 \)-charge = \( k = 0, 1 \)

- In 2 + 1D, \( \int_{M^3} W^3_{A\text{-top}} = \int_{M^3} \frac{1}{2} a^3 \).

Here we do not view \( a \) as 1-form

but as 1-cocycle \( a \in H^1(M^3, \mathbb{Z}_2) \), and \( a^3 \equiv a \cup a \cup a \):

\[
\int_{M^3} a \cup a \cup a = 0 \ or \ 1 \ \rightarrow \ e^{2\pi i \oint_{M^3} W_{A\text{-top}}} = e^{\pi i \oint_{M^3} a^3} = \pm 1
\]

\[
\oint A_{Z_2} = 0, \pi; \ a \equiv \frac{A_{Z_2}}{\pi};
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Z₂ SPT phases and their physical properties

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  \[ \oint A = 0, \pi; a \equiv \frac{A_{Z_2}}{\pi}; \]

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  \[ Z[a] = \text{Tr}(U_{\pi}^{\text{twist}} e^{-H}) = e^{2\pi i \oint S_1 W_{A\text{-top}}} = e^{ik\pi \oint S_1 a} = e^{ik\pi} = \pm 1, \quad k = 0, 1 \]

  **SPT inv. → phys. measurement:**
  \[ \rightarrow \text{ground state } Z_2\text{-charge} = k = 0, 1 \]

- **In 2 + 1D,** \( \int_{M^3} W_{A\text{-top}}^{3} = \int_{M^3} \frac{1}{2} a^3. \)

  Here we do not view \( a \) as 1-form but as 1-cocycle \( a \in H^1(M^3, \mathbb{Z}_2), \) and \( a^3 \equiv a \cup a \cup a: \)

  \[ \int_{M^3} a \cup a \cup a = 0 \text{ or } 1 \rightarrow e^{2\pi i \oint_{M^3} W_{A\text{-top}}} = e^{\pi i \oint_{M^3} a^3} = \pm 1 \]

- **Poincaré duality:** 1-cocycle \( a \leftrightarrow 2\text{-cycle } N^2 \) (2D submanifold)

  \( N^2 \) is the surface across which we do the \( \mathbb{Z}_2 \) symmetry twist.

  Give \( M^3 \) an time and space slices, as we evolve in time:

  (a) \( \rightarrow \) (b) \( \rightarrow \) (c) \( \rightarrow \)

  \[ \int_{M^3} a^3 = \# \text{ of loop creation/annihilation} + \# \text{ of line reconnection} \]

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Wen arXiv:1410.8477

Xiao-Gang Wen, MIT PiTP, IAS, July, 2014
A theory for symmetry protected topological order
How calculate $\int_{M^3} a^3$ (which can be 0 or 1 mod 2)

- $\int_{M^3} a_1 \cup a_2 \cup a_3 = \#$ of intersections of $N_1, N_2, N_3$ mod 2, where $a_i \rightarrow N_i$.

- $\int_{M^3} a \cup a \cup a = \#$ of intersections of $N, N', N''$ mod 2, where $a \rightarrow N, N', N''$.
How to probe $Z_2$ SPT phase; How to measure $\int_{M^3} \frac{1}{2} a^3$

$$\frac{Z[a, M^3]}{Z[0, M^3]} = e^{i2\pi \int_{M^3} \frac{1}{2} a^3}$$

How to design $(a, M^3)$ such that $\frac{Z[a, M^3]}{Z[0, M^3]} = e^{i2\pi \int_{M^3} \frac{1}{2} a^3} = -1$

- If we choose $M^3 = T^3$, $e^{i2\pi \int_{M^3} \frac{1}{2} a^3} = 1$ no matter how we choose the $Z_2$ symmetry twists $a \in H^1(T^3, \mathbb{Z}_2)$.

- Let us choose $M^3 = T^2 \rtimes_{\text{Dehn}^2} S^1$, then, we can have a $Z_2$ symmetry twist to make $e^{i2\pi \int_{M^3} \frac{1}{2} a^3} = -1$.

*The $Z_2$ symmetry twist $a$ is represented by a 2D surface in space-time $M^3$, which is a curve in space.*

Hung-Wen arXiv:1311.5539

It is hard to probe the $Z_2$ SPT order (or $Z_2$ SPT inv.) using bulk measurement.
How to create 2+1D $\mathbb{Z}_2$ SPT phase?

- $H = \sum H_p, \quad H_p = -X_{abcd} P_{ef} P_{gh} P_{ij} P_{kl}$,
  
  $X_{abcd} = \langle \uparrow\uparrow\uparrow\uparrow \rangle \langle \downarrow\downarrow\downarrow\downarrow \rangle + \langle \downarrow\downarrow\downarrow\downarrow \rangle \langle \uparrow\uparrow\uparrow\uparrow \rangle$,
  
  $P = \langle \uparrow\uparrow \rangle \langle \uparrow\uparrow \rangle + \langle \downarrow\down\rangle \langle \downarrow\down\rangle$.

Chen-Liu-Wen arXiv:1106.4752;

- Start with a $\mathbb{Z}_2$ symmetry breaking state, then proliferate the symmetry breaking domain walls to restore the $\mathbb{Z}_2$-symmetry.
  - Domain wall quantum liquid = disordered $\mathbb{Z}_2$-symmetric state.
  - If Domain wall quantum liquid $= \sum |\ldots\rangle \langle \ldots |$, then the $\mathbb{Z}_2$-symmetric state is the trivial $\mathbb{Z}_2$ SPT state.
  - If Domain wall quantum liquid $= \sum (-)^{\# \text{ of loops}} |\ldots\rangle \langle \ldots |$, then the $\mathbb{Z}_2$-symmetric state is the non-trivial $\mathbb{Z}_2$ SPT state.

(a) (b) (c)

Levin-Gu arXiv:1202.3120
Why there is no non-trivial 1+1D $\mathbb{Z}_2$ SPT phase?

• Because $H^2(\mathbb{Z}_2, U_1) = 0$.
  But this only implies that our NL$\sigma$M construction fails to produce a non-trivial 1+1D $\mathbb{Z}_2$ SPT state.

• May be non-trivial 1+1D $\mathbb{Z}_2$ SPT state exists since we have a potential $\mathbb{Z}_2$ SPT invariant in 1+1D $W_{A \text{-top}}^2(a) = \frac{1}{2} a \cup a$. 

Xiao-Gang Wen, MIT PiTP, IAS, July., 2014
Why there is no non-trivial 1+1D $\mathbb{Z}_2$ SPT phase?

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- May be non-trivial 1+1D $\mathbb{Z}_2$ SPT state exists since we have a potential $\mathbb{Z}_2$ SPT invariant in 1+1D $W^2_{\text{A-top}}(a) = \frac{1}{2} a \cup a$.
  - However, $\int a \cup a = 0$ mod 2 on oriented manifold. There is not even non-trivial potential $\mathbb{Z}_2$ SPT invariant in 1+1D $\rightarrow$ there is no non-trivial 1+1D $\mathbb{Z}_2$ SPT phase.

Proof:
$Sq^1(a) = a \cup a$ and $Sq^1(a) = u_1 \cup a = w_1 \cup a$
$w_1 = 0$ for oriented manifold, and thus $a \cup a = 0$ mod 2.
A $1+1$D $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ SPT state

\[ H^3[\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}, U(1)] = \mathbb{Z}_{N_{12}} = \{0, 1, \cdots, k, \cdots, N_{12} - 1\} \]

where $N_{12} = \gcd(N_1, N_2)$.

- We consider a SPT state labeled by $k \in \mathbb{Z}_{N_{12}}$ and assume $N_1 = N_2 = N$.
- **What is the SPT invariant?**
A 1+1D $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ SPT state

$$H^3[\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}, U(1)] = \mathbb{Z}_{N_{12}} = \{0, 1, \ldots, k, \ldots, N_{12} - 1\}$$

where $N_{12} = \text{gcd}(N_1, N_2)$.

- We consider a SPT state labeled by $k \in \mathbb{Z}_{N_{12}}$ and assume $N_1 = N_2 = N$.
- **What is the SPT invariant?**

The fixed-point partition function on space-time $T^2 = S^1 \times S^1$ with symmetry twists in $x, t$ directions:

$$\frac{Z[a_1, a_2, T^2]}{Z[0, 0, T^2]} = e^{i k \frac{2\pi}{N_{12}} \oint a_1 a_2},$$

$$\oint a_1 \in \mathbb{Z}_{N_1}; \oint a_2 \in \mathbb{Z}_{N_2}.$$
A symmetry twist of $Z_{N_1}$ carries $Z_{N_2}$-charge $k$.

**Example:** 1D $Z_2 \times Z_2 = D_2$ SPT state (spin-1 Haldane chain) $Z_2 \times Z_2 = D_2 = 180^\circ$ spin rotations in $S^x, S^z$.

Untwisted case:

$$H_{D_2} = \sum_i J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z$$

$$+ J_x S_1^x S_1^x + J_y S_1^y S_1^y + J_z S_1^z S_1^z$$

The ground state has $e^{i\pi \sum S_i^z} = 1$.

Twisted case (by $e^{i\pi \sum S_i^x}$):

$$H_{D_2}^{\text{twist}} = \sum_i J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z$$

$$+ J_x S_1^x S_1^x - J_y S_1^y S_1^y - J_z S_1^z S_1^z$$

The ground state has $e^{i\pi \sum S_i^z} = -1$.  

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Xiao-Gang Wen, MIT PiTP, IAS, July., 2014  
A theory for symmetry protected topological order
For a 1D $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ SPT state

- SPT invariant: a symmetry twist of $\mathbb{Z}_{N_1}$ carries a “charge” of $\mathbb{Z}_{N_2}$

Since the symmetry twist of $\mathbb{Z}_{N_1}$ = the domain wall of $\mathbb{Z}_{N_1}$

- Bind $k \mathbb{Z}_{N_2}$-charge to the domain wall of $\mathbb{Z}_{N_1}$

$\rightarrow$ 1D $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ SPT state labeled by $k \in \mathcal{H}^2[\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}, U(1)]$
Example: A $\mathbb{Z}_2^x \times \mathbb{Z}_2^z$ spin-1 chain, & its symmetric phases

- $|x\rangle, |y\rangle, |z\rangle$ basis:

$$
|\uparrow z\rangle = \frac{|x\rangle + i|y\rangle}{\sqrt{2}},
|0_z\rangle = |z\rangle,
|\downarrow z\rangle = \frac{|x\rangle - i|y\rangle}{\sqrt{2}}
$$

$$
S^x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},
S^y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},
S^z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

- $\mathbb{Z}_2^x \times \mathbb{Z}_2^z$ symmetry: $U^x: (|x\rangle, |y\rangle, |z\rangle) \rightarrow (-|x\rangle, |y\rangle, |z\rangle)$

$U^z: (|x\rangle, |y\rangle, |z\rangle) \rightarrow (|x\rangle, |y\rangle, -|z\rangle)$

- $H^0 = \sum_i -J_z S^z_i S^z_{i+1} \rightarrow \mathbb{Z}_2^x$ breaking

Two kinds of domain walls with the same energy, but different $\mathbb{Z}_2^z$-charges and different hopping operators:

$$
H^{\text{hop}}_1 = \sum_i -K[(S^+_i)^2 + \text{h.c.}],
H^{\text{hop}}_2 = \sum_i -J_{xy}(S^+_i S^+_ {i+1} + \text{h.c.}).
$$

- $H^0 + H^{\text{hop}}_1$ & $H^0 + H^{\text{hop}}_2 \rightarrow$ different symm. ground states
A $2+1$D $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ SPT state

$$H^3[\prod_{i=1}^3 Z_{N_i}, U(1)] = \mathbb{Z}_{N_1} \oplus \mathbb{Z}_{N_2} \oplus \mathbb{Z}_{N_3} \oplus \mathbb{Z}_{N_{12}} \oplus \mathbb{Z}_{N_{23}} \oplus \mathbb{Z}_{N_{13}} \oplus \mathbb{Z}_{N_{123}}$$

where $N_{123} = \gcd(N_1, N_2, N_3)$.

- We consider a SPT state labeled by $k \in \mathbb{Z}_{N_{123}}$ and assume $N_1 = N_2 = N_3 = N$.


$$\frac{Z[a_1, a_2, a_3]}{Z[0, 0, 0]} = e^{i k \frac{2\pi}{N_{123}} \int a_1 a_2 a_3}$$

- SPT inv. $\rightarrow$ physical measurement: The intersection of the symmetry twists in $Z_{N_1}$ and $Z_{N_2}$ carries $Z_{N_3}$-charge $k$.

- A mechanism for such a SPT state: Bind $k$ $Z_{N_3}$-charge to the intersection of the domain walls of $Z_{N_1}$ and $Z_{N_2}$.
Dimension reduction: $T^3 = T^2_{x,t} \times S^1_y$ and $\oint_{S^1_y} a_3 = 1$:

$$\frac{Z[a_1, a_2, T^2]}{Z[0, 0, T^2]} = e^{i \frac{kN_{12}}{N_{123}} \frac{2\pi}{N_{12}} \int a_1 a_2}$$

→ A 1+1D SPT state with $\frac{kN_{12}}{N_{123}} \in H^2[Z_{N_1} \times Z_{N_2}, U(1)] = \mathbb{Z}_{N_{123}}$.
→ degenerated states at the end of 1D chain that form a projective representation of $Z_{N_1} \times Z_{N_2}$.

• A $Z_{N_3}$ “vortex” (end of $Z_{N_3}$ symmetry twist) carries degenerated states that form a projective representation of $Z_{N_1} \times Z_{N_2}$.

• **How to make $Z_3$-vortex:**
  1) Consider $U(1)$ symm. break down to $Z_3$ symm.
  2) A vortex of the order parameter = $Z_3$-vortex.

• Another mechanism for the 2+1D $Z_{N_1} \times Z_{N_2} \times Z_{N_3}$ SPT state:
  bind the 1+1D $Z_{N_1} \times Z_{N_2}$ SPT state to the domain wall of $Z_{N_3}$.
\( \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \) does not produce all the SPT phases with symm. \( G \): Topological states and anomalies

SPT order from \( \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \)

- SPT state
- theory with on-site symmetry
- with gauge anomaly (symm.)

Topological order

- Topologically ordered state
- effective theory with gravitational anomaly

Pure SPT order within \( \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \):

\[
W_{A-top}^d = \frac{AF}{(2\pi)^2}, \quad \frac{1}{2} a^3
\]


Invertible topological order:

\[
W_{A-top}^d = \omega^3, \quad \frac{1}{2} w_2 w_3
\]

\( p_1 \) is the first Pontryagin class, \( d \omega^3 = p_1 \), and \( w_i \) is the Stiefel-Whitney classes.
$\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ does not produce all the SPT phases with symm. $G$: Topological states and anomalies

SPT order from $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$

- SPT state with on-site symmetry
- theory with gauge (symm.) anomaly
- theory with mixed gauge–grav anomaly

Topological order

- Topologically ordered state
- effective theory with gravitational anomaly

SPT order beyond $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$

Pure SPT order within $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$: $W_{A\text{-top}}^d = \frac{AF}{(2\pi)^2}, \frac{1}{2} a^3$

Mixed SPT order beyond $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$: $W_{A\text{-top}}^d = \frac{F}{2\pi} \omega_3, \frac{1}{2} a p_1$

Invertible topological order: $W_{A\text{-top}}^d = \omega_3, \frac{1}{2} w_2 w_3$

$p_1$ is the first Pontryagin class, $d\omega_3 = p_1$, and $w_i$ is the Stiefel-Whitney classes.
A general theory for bosonic pure STP orders, mixed SPT orders, and invertible topological orders

Wen arXiv:1410.8477

- **$\text{NL}$-$\sigma$-$M$** (group cohomology) approach to pure SPT phases:
  1. $\text{NL}$-$\sigma$-$M$+topo. term: $\frac{1}{2\lambda}|\partial g|^2 + 2\pi i W(g^{-1}\partial g), \ g \in G$
  2. Add symm. twist: $\frac{1}{2\lambda}|(\partial - iA)g|^2 + 2\pi i W[(\partial - iA)g]$
  3. Integrate out matter field: $Z_{\text{fixed}} = e^{2\pi i \int W_{\text{A-top}}(A)}$

- **$G \times SO_\infty$** $\text{NL}$-$\sigma$-$M$ (group cohomology) approach:
  1. $\text{NL}$-$\sigma$-$M$: $\frac{1}{2\lambda}|\partial g|^2 + 2\pi i W(g^{-1}\partial g), \ g \in G \times SO$
  2. Add twist: $\frac{1}{2\lambda}|(\partial - iA - i\Gamma)g|^2 + 2\pi i W[(\partial - iA - i\Gamma)g]$
  3. Integrate out matter field: $Z_{\text{fixed}} = e^{2\pi i \int W_{\text{A-top}}(A,\Gamma)}$

**Pure STP orders, mixed SPT orders, and invertible topological orders are classified by**

$\mathcal{H}^d(G \times SO, \mathbb{R}/\mathbb{Z})$

$= \mathcal{H}^d(G, \mathbb{R}/\mathbb{Z}) \oplus_{k=1}^{d-1} \mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})] \oplus \mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})$

*after quotient out something $\Gamma^d(G)$.*
Trying to classify bosonic pure STP orders, mixed SPT orders, and invertible topological orders

- **Pure STP orders**: $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$
- **mixed SPT orders**: $\bigoplus_{k=1}^{d-1} \mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})]$
- **iTO’s**: $\mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})$

The above are one-to-one description of pure SPT orders, but only many-to-one description of mixed SPT orders and iTO’s

- $G$-symmetry twists $A \rightarrow W^d_{A-top}(A)$ can fully detect/distinguish all elements of $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$.
- $SO$-symmetry twists $\Gamma_{SO} \rightarrow W^d_{A-top}(\Gamma_{SO})$ can fully detect/distinguish all elements of $\mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})$.
- $SO$-symmetry twists $\Gamma$ from the tangent bundle of $M^d$ are only special $SO$-symmetry twists (which are arbitrary $SO$-bundles on $M^d$) $\rightarrow W^d_{A-top}(\Gamma)$ cannot fully detect/distinguish all elements of $\mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})$. $\rightarrow$ $\text{iTO}^d = \mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z})/\Gamma^d$
Trying to classify bosonic pure STP orders, mixed SPT orders, and invertible topological orders

- **Pure STP orders**: $\mathcal{H}^d(G, \mathbb{R}/\mathbb{Z})$ (the black entries below)
- **mixed SPT order**
  \[ \bigoplus_{k=1}^{d-1} \mathcal{H}^k(G, \text{iTO}^{d-k}) = \bigoplus \mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})] / \Gamma^d(G) \]
- **iTO’s**: $\text{iTO}^d = \mathcal{H}^d(SO, \mathbb{R}/\mathbb{Z}) / \Gamma^d$
- Probe mixed SPT order described by $\mathcal{H}^k[G, \mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})]$:
  - put the state on $M^d = M^k \times M^{d-k}$ and add a $G$-symmetry twist on $M^k$
  - Induce a state on $M^{d-k}$ described by $\mathcal{H}^{d-k}(SO, \mathbb{R}/\mathbb{Z})$ → a iTO state in $\text{iTO}^{d-k}$

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Wen arXiv:1410.8477

Xiao-Gang Wen, MIT PiTP, IAS, July., 2014