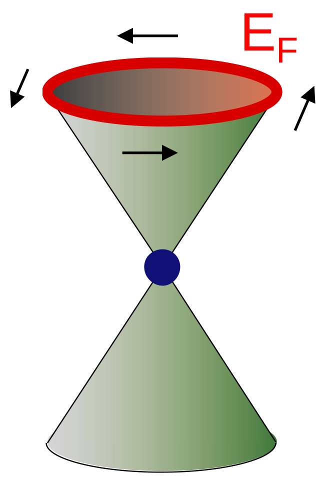
Topological Superconductors, Majorana Fermions and Topological Quantum Computation

0. … from last time: The surface of a topological insulator
1. Bogoliubov de Gennes Theory
2. Majorana bound states, Kitaev model
3. Topological superconductor
4. Periodic Table of topological insulators and superconductors
5. Topological quantum computation
6. Proximity effect devices

Unique Properties of Topological Insulator Surface States

"Half" an ordinary 2DEG ; ¼ Graphene



Spin polarized Fermi surface

- Charge Current ~ Spin Density
- Spin Current ~ Charge Density

π Berry's phase

- Robust to disorder
- Weak Antilocalization
- Impossible to localize

Exotic States when broken symmetry leads to surface energy gap:

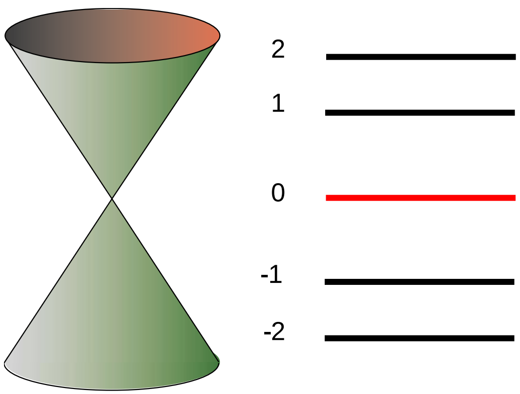
- Quantum Hall state, topological magnetoelectric effect
- Superconducting state

Even more exotic states if surface is gapped without breaking symmetry

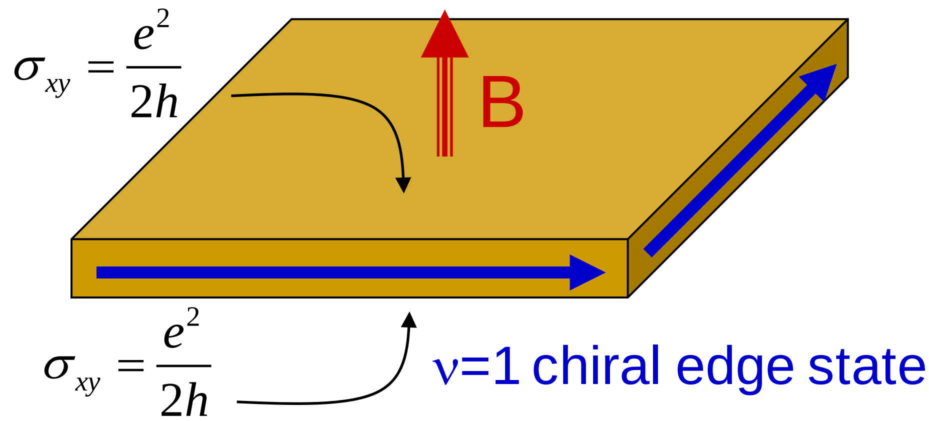
- Requires intrinsic topological order like non-Abelian FQHE

Surface Quantum Hall Effect

Orbital QHE : E=0 Landau Level for Dirac fermions. "Fractional" IQHE



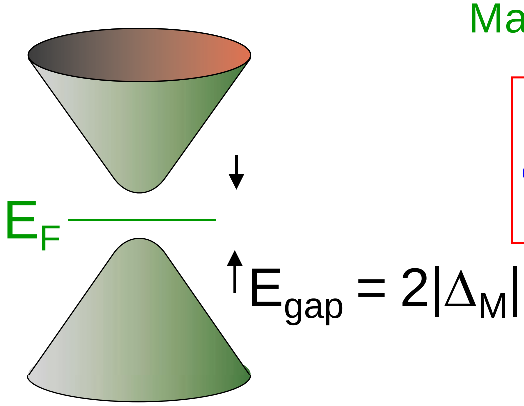
$$\sigma_{xy} = \frac{e^2}{h}\left(n + \frac{1}{2}\right)$$



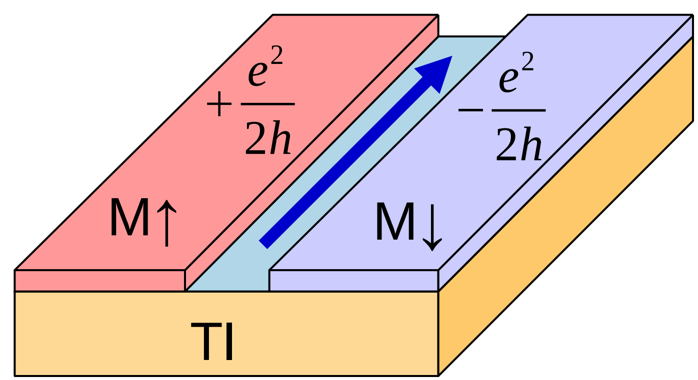
Anomalous QHE : Induce a surface gap by depositing magnetic material

$$H_0 = \psi^\dagger(-iv\vec{\sigma}\cdot\vec{\nabla} - \mu + \Delta_M \sigma_z)\psi$$

Mass due to Exchange field



$$\sigma_{xy} = \mathrm{sgn}(\Delta_M)\frac{e^2}{2h}$$

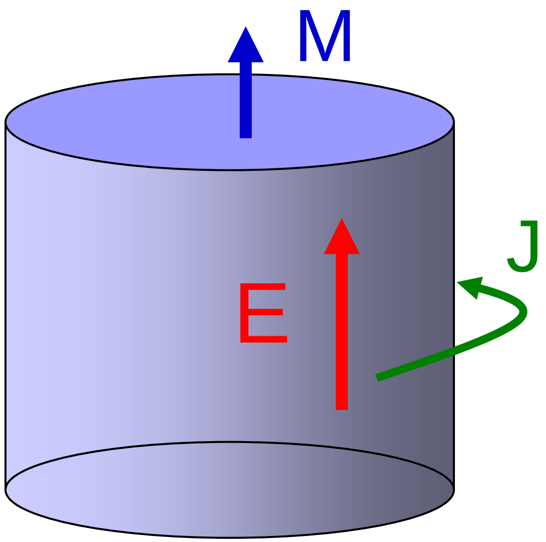


Chiral Edge State at Domain Wall : $\Delta_M \leftrightarrow -\Delta_M$

Topological Magnetoelectric Effect

Qi, Hughes, Zhang ’08; Essin, Moore, Vanderbilt ‘09

Consider a solid cylinder of TI with a magnetically gapped surface



$$J = \sigma_{xy} E = \frac{e^2}{h}\left(n + \frac{1}{2}\right) E = M$$

Magnetoelectric Polarizability

$$M = \alpha E \qquad \alpha = \frac{e^2}{h}\left(n + \frac{1}{2}\right)$$

topological “θ term”

$$\Delta L = \alpha \mathbf{E} \cdot \mathbf{B}$$

$$\alpha = \theta \frac{e^2}{2\pi h}$$

TR sym. : $\theta = 0$ or π mod 2π

The fractional part of the magnetoelectric polarizability is determined by the bulk, and independent of the surface (provided there is a gap)

Analogous to the electric polarization, P, in 1D.

	ΔL	formula	“uncertainty quantum”
d=1 : Polarization P	$P \cdot \mathbf{E}$	$\frac{e}{2\pi}\int_{BZ} \mathrm{Tr}[\mathbf{A}]$	e (extra end electron)
d=3 : Magnetoelectric poliarizability α	$\alpha \mathbf{E} \cdot \mathbf{B}$	$\frac{e^2}{4\pi^2 h}\int_{BZ} \mathrm{Tr}[\mathbf{A} \wedge d\mathbf{A} + \frac{2}{3}\mathbf{A} \wedge \mathbf{A} \wedge \mathbf{A}]$	e^2/h (extra surface quantum Hall layer)

BCS Theory of Superconductivity

mean field theory : $\Psi^\dagger\Psi\Psi^\dagger\Psi \Rightarrow \langle\Psi^\dagger\Psi^\dagger\rangle\Psi\Psi = \Delta^*\Psi\Psi$

$$H = \frac{1}{2}\sum_{\mathbf{k}}\begin{pmatrix}\Psi^\dagger & \Psi\end{pmatrix} H_{BdG} \begin{pmatrix}\Psi \\ \Psi^\dagger\end{pmatrix}$$

Bogoliubov de Gennes Hamiltonian

$$H_{BdG} = \begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0 \end{pmatrix}$$

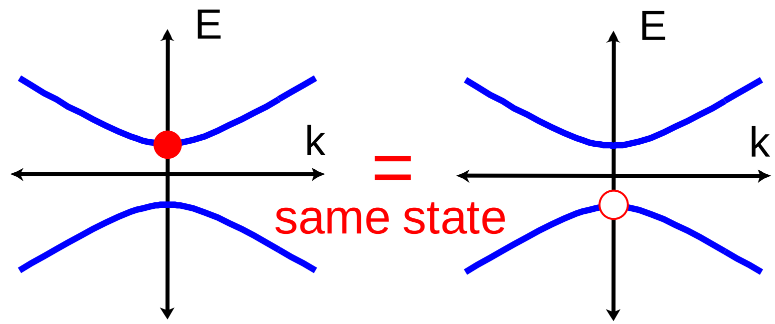
Intrinsic anti-unitary particle – hole symmetry

$$\Xi H_{BdG} \Xi^{-1} = -H_{BdG} \qquad \Xi\varphi = \tau_x \varphi^* \qquad \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Xi^2 = +1$$

Particle – hole redundancy

$$\varphi_{-E} = \Xi\varphi_E \quad \Rightarrow \quad \gamma_E^\dagger = \gamma_{-E}$$



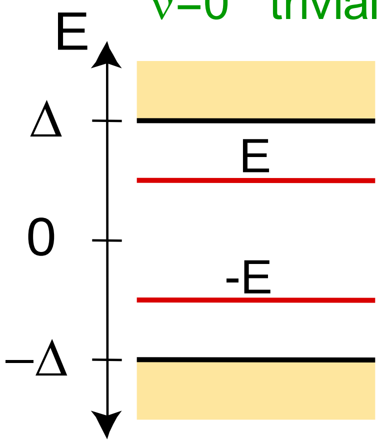
Bloch - BdG Hamiltonians satisfy $\Xi H_{BdG}(\mathbf{k})\Xi^{-1} = -H_{BdG}(-\mathbf{k})$

Topological classification problem similar to time reversal symmetry

1D $\mathbb{Z}_2$ Topological Superconductor : $\nu = 0,1$ (Kitaev, 2000)

Bulk-Boundary correspondence : Discrete end state spectrum

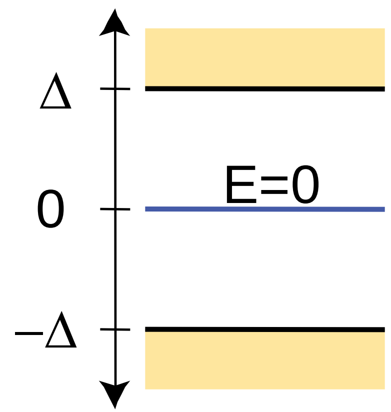
ν=0 "trivial"



Γ_E

$\Gamma_{-E} = \Gamma_E^\dagger$

ν=1 "topological"



Zero mode

$$\Gamma_{E=0}^\dagger = \Gamma_{E=0} \equiv \gamma$$

Majorana fermion bound state

Majorana Fermion : Particle = Antiparticle $\gamma = \gamma^\dagger$

Real part of a Dirac fermion :

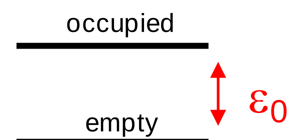
$$\begin{cases} \gamma_1 = \Psi + \Psi^\dagger & ; \quad \Psi = \gamma_1 + i\gamma_2 \\ \gamma_2 = -i(\Psi - \Psi^\dagger) & ; \quad \Psi^\dagger = \gamma_1 - i\gamma_2 \end{cases}$$

$$\gamma_i^2 = 1 \qquad \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

"Half a state"

Two Majorana fermions define a single two level system

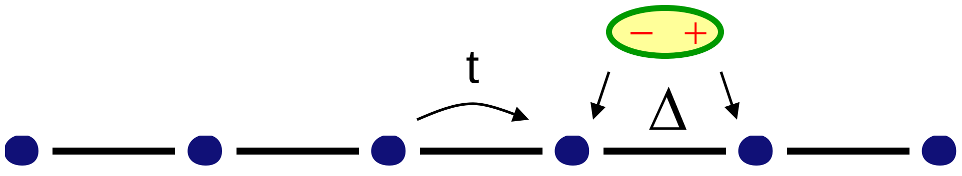
$$H = 2i\varepsilon_0 \gamma_1 \gamma_2 = \varepsilon_0 \Psi^\dagger \Psi$$



Kitaev Model for 1D p wave superconductor

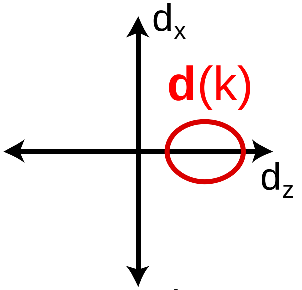
$$H-\mu N=\sum_i t(c_i^\dagger c_{i+1}+c_{i+1}^\dagger c_i)-\mu c_i^\dagger c_i+\Delta(c_i c_{i+1}+c_{i+1}^\dagger c_i^\dagger)$$

$$=\sum_k \begin{pmatrix} c_k^\dagger & c_{-k}\end{pmatrix} H_{BdG}(k)\begin{pmatrix} c_k \\ c_{-k}^\dagger\end{pmatrix}$$

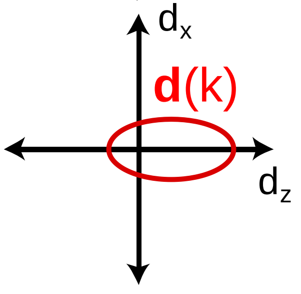


$$H_{BdG}(k)=\tau_z(2t\cos k-\mu)+\tau_x\Delta\sin k=\mathbf{d}(k)\cdot\vec{\tau}$$

$|\mu|>2t$: Strong pairing phase
trivial superconductor



$|\mu|<2t$: Weak pairing phase
topological superconductor



Similar to SSH model, except different symmetry : $(d_x,d_y,d_z)\big|_k=(-d_x,-d_y,d_z)\big|_{-k}$

Majorana Chain

$$c_i \rightarrow \gamma_{1i} + i\gamma_{2i}$$

$$H = 2i\sum_i t_1\gamma_{1i}\gamma_{2i} + t_2\gamma_{2i}\gamma_{1i+1}$$

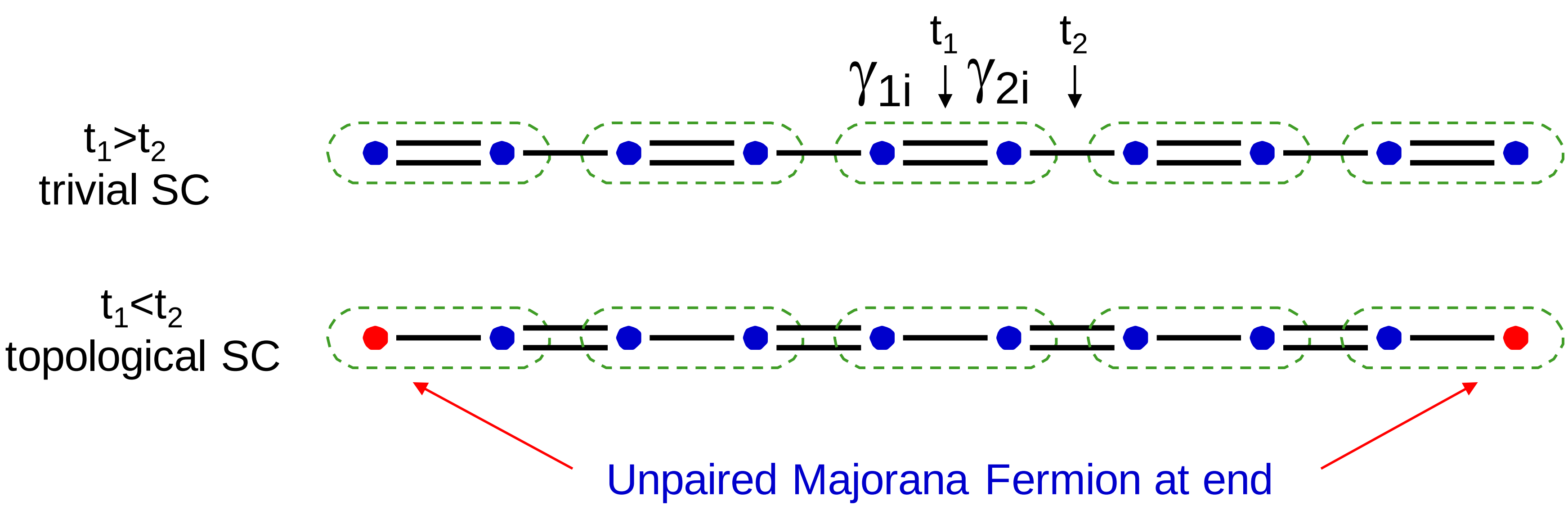
$$\mu c_i^\dagger c_i \rightarrow 2i\mu\gamma_{1i}\gamma_{2i}$$

$$t\left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i\right) \rightarrow 2it\left(\gamma_{1i}\gamma_{2i+1} - \gamma_{2i}\gamma_{1i+1}\right)$$

$$\Delta\left(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger\right) \rightarrow 2i\Delta\left(\gamma_{1i}\gamma_{2i+1} + \gamma_{2i}\gamma_{1i+1}\right)$$

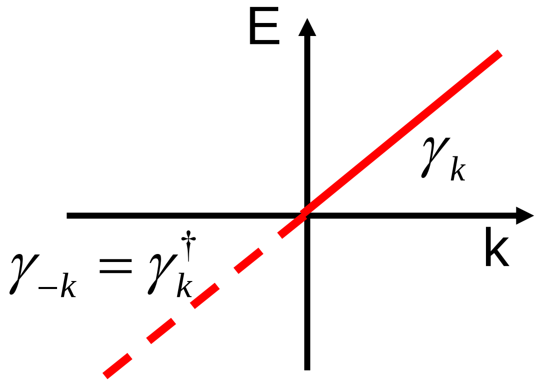
For $\Delta = t$: nearest neighbor Majorana chain

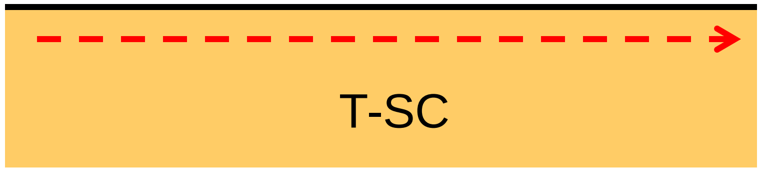
$$t_1 = \mu,\ \ t_2 = 2t$$



2D $\mathbb{Z}$ topological superconductor (broken T symmetry)

Bulk-Boundary correspondence: n = # Chiral Majorana Fermion edge states





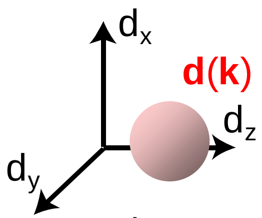
Examples

- Spinless p_x+ip_y superconductor (n=1)
- Chiral triplet p wave superconductor (eg Sr_2RuO_4) (n=2)

Read Green model : $H = \sum_{\mathbf{k}} \left(\frac{\mathbf{k}^2}{2m} - \mu \right) c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \left(\Delta(\mathbf{k}) c_{\mathbf{k}} c_{-\mathbf{k}} + c.c. \right)$ $\quad \Delta(\mathbf{k}) = \Delta_0 \left(k_x + i k_y \right)$

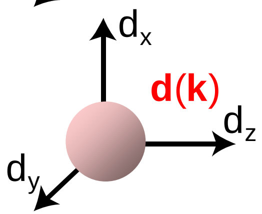
Lattice BdG model : $H_{BdG}(\mathbf{k}) = \tau_z \left(2t \left[\cos k_x + \cos k_y \right] - \mu \right) + \Delta \left(\tau_x \sin k_x + \tau_y \sin k_y \right) = \mathbf{d}(k) \cdot \vec{\tau}$

$|\mu| > 4t$: Strong pairing phase
trivial superconductor



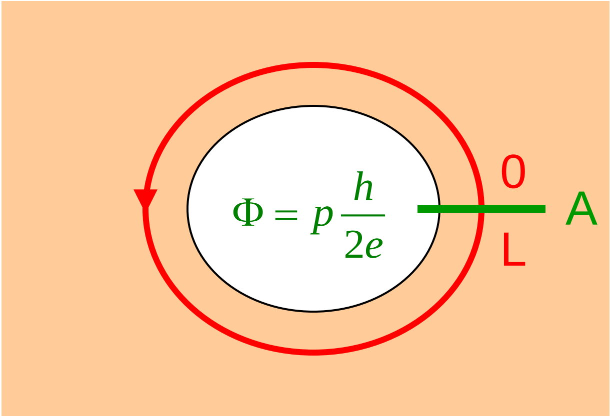
Chern number 0

$|\mu| < 4t$: Weak pairing phase
topological superconductor



Chern number 1

Majorana zero mode at a vortex

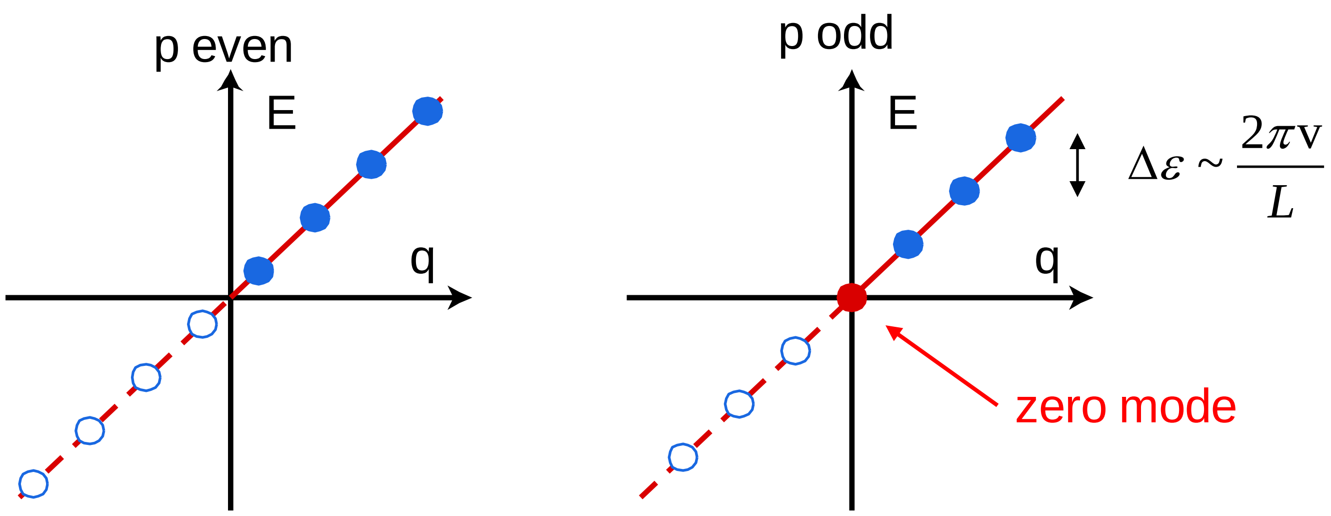


Hole in a topological superconductor threaded by flux

Boundary condition on fermion wavefunction

$$\psi(L) = (-1)^{p+1}\psi(0)$$

$$\psi(x) \propto e^{iq_m x} \quad ; \quad q_m = \frac{\pi}{L}(2m+1+p)$$



Without the hole : Caroli, de Gennes, Matricon theory ('64) $\Delta\varepsilon \sim \frac{\Delta^2}{E_F}$

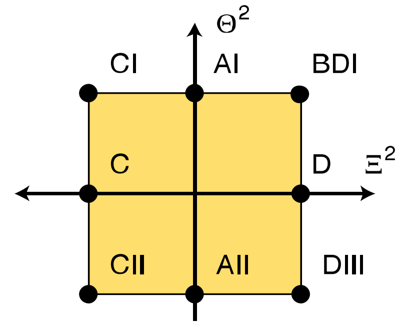
Periodic Table of Topological Insulators and Superconductors

Anti-Unitary Symmetries :

- Time Reversal : $\Theta H(\mathbf{k})\Theta^{-1} = +H(-\mathbf{k})\ ; \quad \Theta^2 = \pm 1$

- Particle - Hole : $\Xi H(\mathbf{k})\Xi^{-1} = -H(-\mathbf{k})\ ; \quad \Xi^2 = \pm 1$

Unitary (chiral) symmetry : $\Pi H(\mathbf{k})\Pi^{-1} = -H(\mathbf{k})\ ; \quad \Pi \propto \Theta\Xi$



8 antiunitary symmetry classes

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	−1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	−1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	−1	−1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	−1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	−1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Complex K-theory

Real K-theory

Altland-Zirnbauer Random Matrix Classes

Bott Periodicity $d \rightarrow d+8$

Kitaev, 2008
Schnyder, Ryu, Furusaki, Ludwig 2008

Majorana Fermions and Topological Quantum Computing

(Kitaev '03)

The degenerate states associated with Majorana zero modes define a topologically protected quantum memory

- 2 Majorana separated bound states = 1 fermion $\Psi = \gamma_1 + i\gamma_2$
  - 2 degenerate states (full/empty) = 1 qubit
- 2N separated Majoranas = N qubits
- Quantum Information is stored non locally
  - Immune from local decoherence

Braiding performs unitary operations

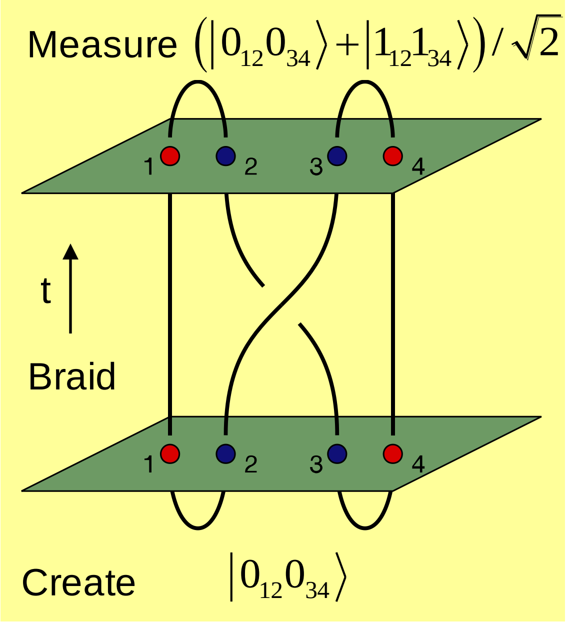
Non-Abelian statistics

Interchange rule (Ivanov 03)

$$\gamma_i \rightarrow \gamma_j$$

$$\gamma_j \rightarrow -\gamma_i$$

These operations, however, are not sufficient to make a universal quantum computer



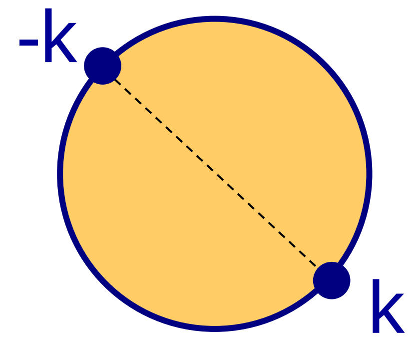
Potential condensed matter hosts for topological superconductivity

- Quasiparticles in fractional Quantum Hall effect at $\nu=5/2$ Moore Read '91
- Unconventional superconductors
  - Sr_2RuO_4 Das Sarma, Nayak, Tewari '06
  - Fermionic atoms near feshbach resonance Gurarie '05
  - $Cu_xBi_2Se_3$?
- Proximity Effect Devices using ordinary superconductors
  - Topological Insulator devices Fu, Kane '08
  - 2D Semiconductor/Magnet devices Sau, Lutchyn, Tewari, Das Sarma '09, Lee '09
  - 1D Semiconductor devices:
    eg In As quantum wires Oreg, von Oppen, Alicea, Fisher '10
    Lutchyn, Sau, Das Sarma '10
    Expt : Maurik et al. (Kouwenhoven) '12
  - 1D Ferromagnetic atomic chains on superconductors
    Expt : Nadj-Perg et al. (Yazdani) '14

Topological Superconductors

Spinless p-wave superconductor:

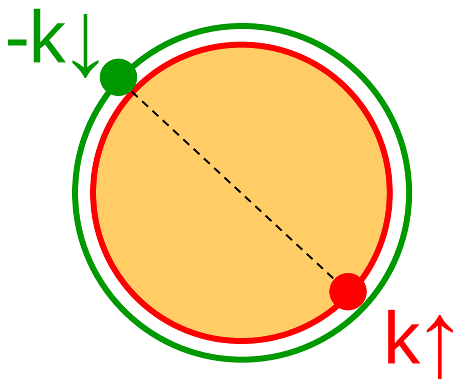
$$\left\langle c_k^\dagger c_{-k}^\dagger \right\rangle \propto \Delta e^{i\varphi}(k_x + ik_y)$$



Ordinary Superconductor :

$$\left\langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \right\rangle \propto \Delta e^{i\varphi}$$

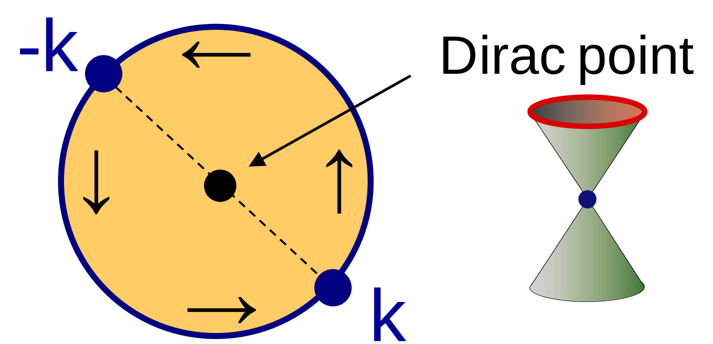
(s-wave, singlet pairing)



Surface of topological insulator

$$\left\langle c_k^\dagger c_{-k}^\dagger \right\rangle \propto \Delta_{\text{surface}} e^{i\varphi}$$

(s-wave, singlet pairing)

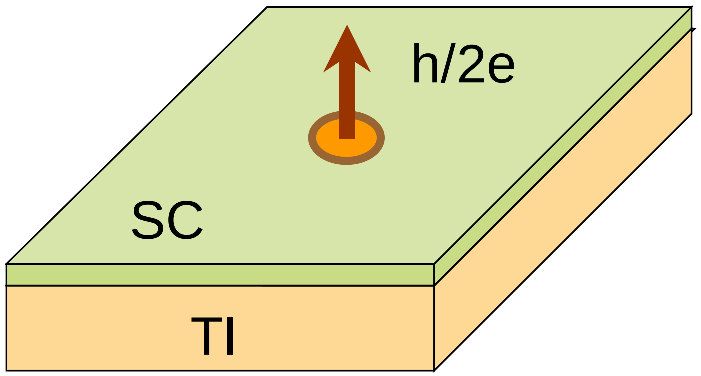


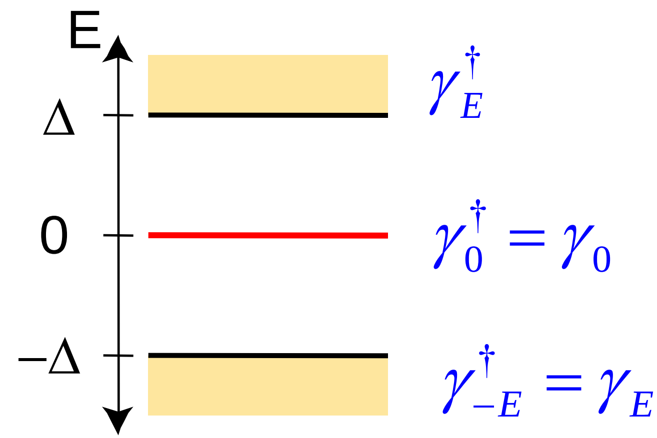
Half an ordinary superconductor

Nontrivial ground state supports Majorana fermions at vortices

Majorana Bound States on Topological Insulators

1. h/2e vortex in 2D superconducting state

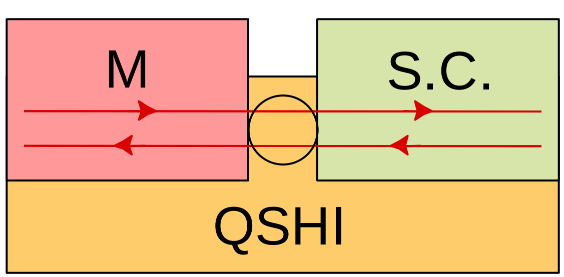




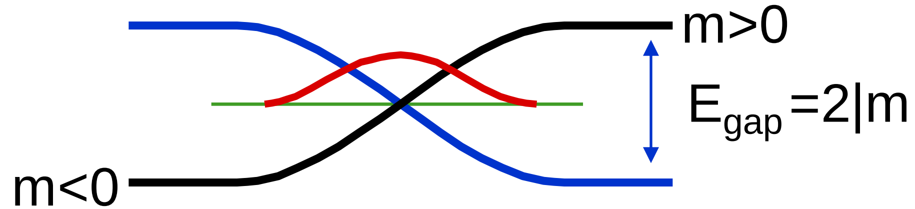
Quasiparticle Bound state at E=0

Majorana Fermion γ_0 "Half a State"

2. Superconductor-magnet interface at edge of 2D QSHI



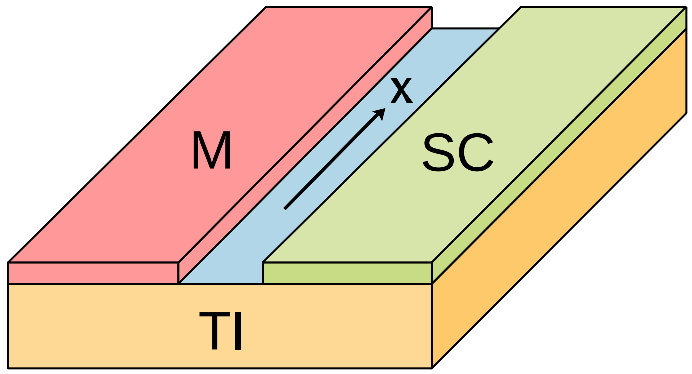
$$m = |\Delta_S| - |\Delta_M|$$

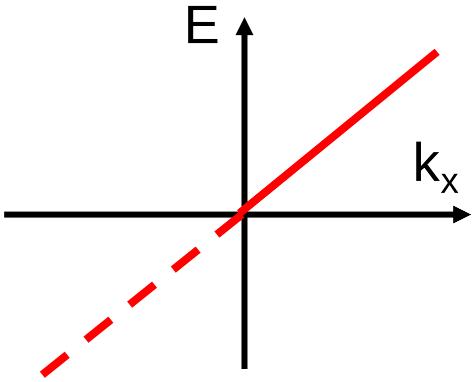


Domain wall bound state γ_0

1D Majorana Fermions on Topological Insulators

1. 1D Chiral Majorana mode at superconductor-magnet interface

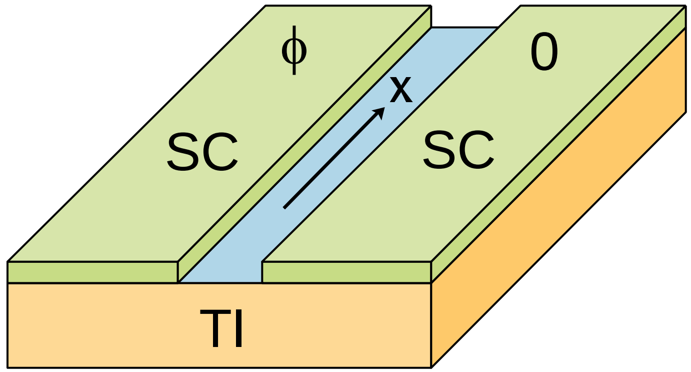


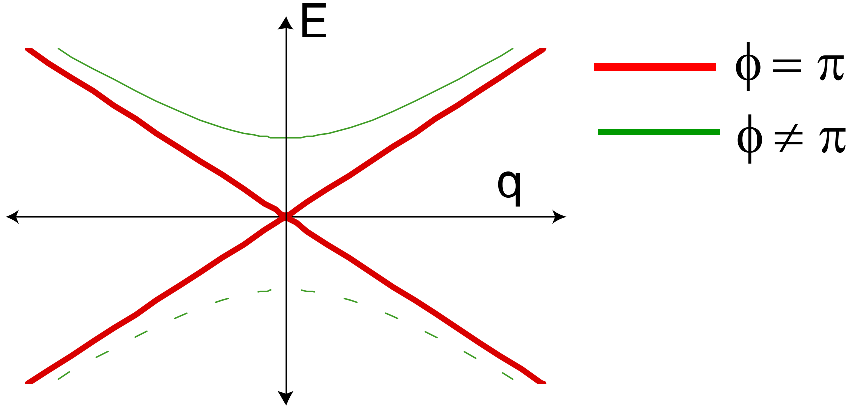


$\gamma_k = \gamma^\dagger_{-k}$: "Half" a 1D chiral Dirac fermion

$$H = -i\hbar v_F \; \gamma \partial_x \gamma$$

2. S-TI-S Josephson Junction





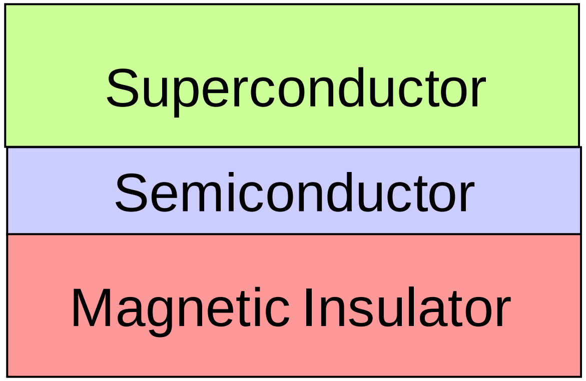
Gapless non-chiral Majorana fermion for phase difference $\phi = \pi$

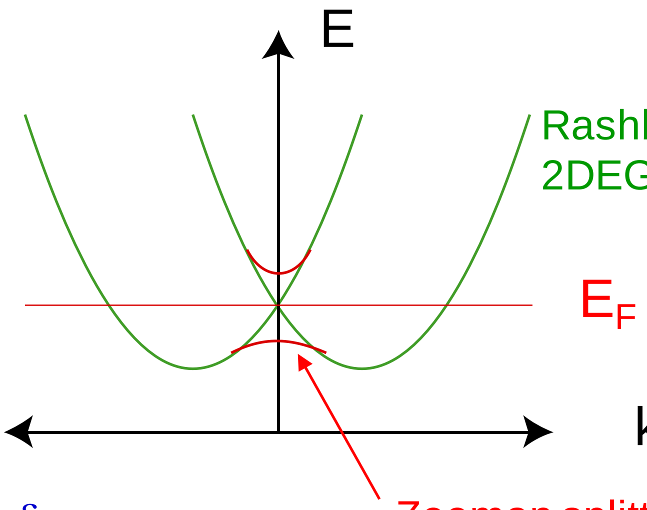
$$H = -i\hbar v_F \; \left(\gamma_L \partial_x \gamma_L - \gamma_R \partial_x \gamma_R\right) + i\Delta \cos(\phi/2)\gamma_L \gamma_R$$

Another route to the 2D p+ip superconductor

Semiconductor - Magnet - Superconductor structure

Sau, Lutchyn, Tewari, Das Sarma '09





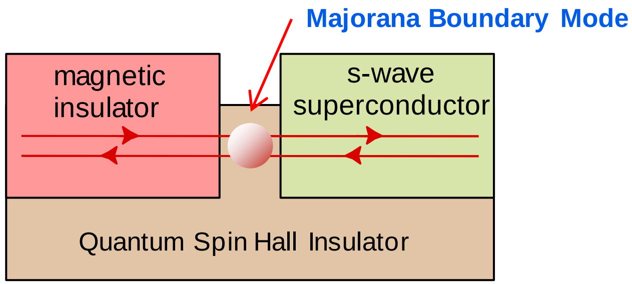
- Single Fermi circle with Berry phase $\pi-\varepsilon$
- Topological superconductor with Majorana edge states and Majorana bound states at vortices.
- Variants :
  - use applied magnetic field to lift Kramers degeneracy (Alicea '10)
  - Use 1D quantum wire (eg InSb). A route to 1D p wave superconductor with Majorana end states. (Oreg, von Oppen, Alicea, Fisher '10

Experiments on Proximity Devices

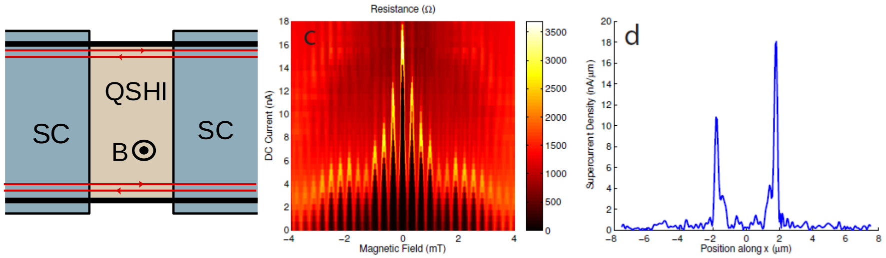
Superconducting Proximity Effect: Use ordinary superconductors and topological materials to engineer topological superconductivity

Superconductor - Topological Insulator Devices

Theory: Fu, Kane '07, '08



Expt: Hart, … Yacoby '14 (HgTe);
Pribiag, … Kouwenhoven '14 (InAs/GaSb)

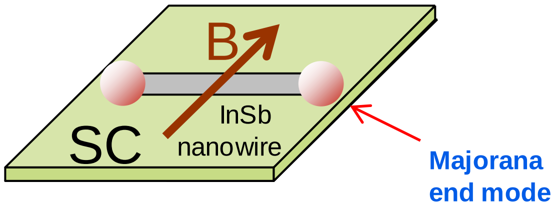


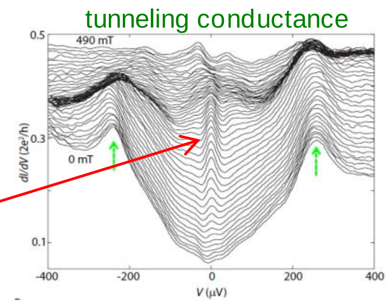
"Two slit" interference pattern in a S-TI-S Josephson Junction
Demonstrates edge superconductivity

Superconductor - Semiconductor Nanowire Devices

Theory:
Lutchyn, Sau , Das Sarma '10
Oreg, Refael, von Oppen '10

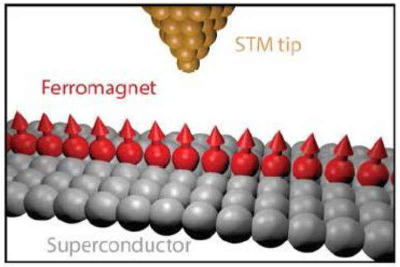
Expt: Mourik, …Kouwenhoven '12

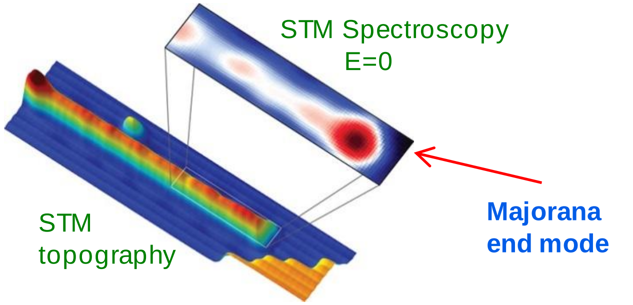




Ferromagnetic Atomic Chains on Superconductors

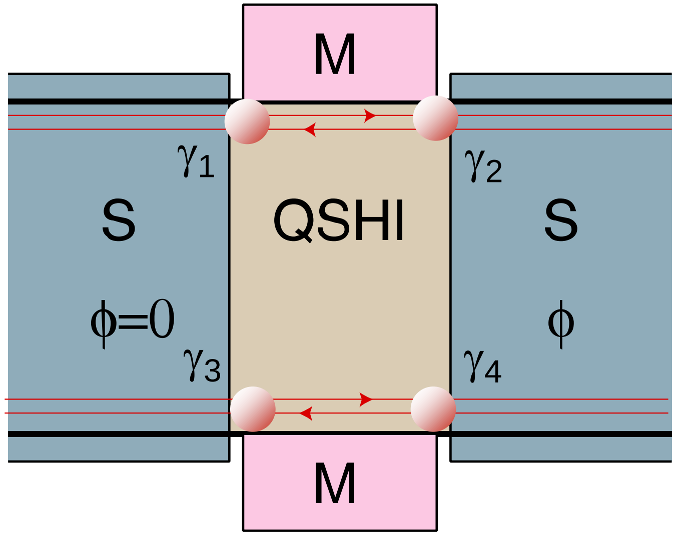
Nadj-Perg, …, Yazdani '14 (Fe on Pb)

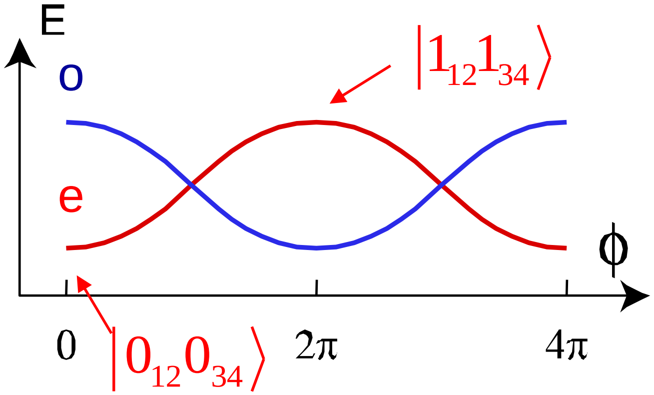




Fractional Josephson Effect

Kitaev '01
Kwon, Sengupta, Yakovenko '04
Fu, Kane '08

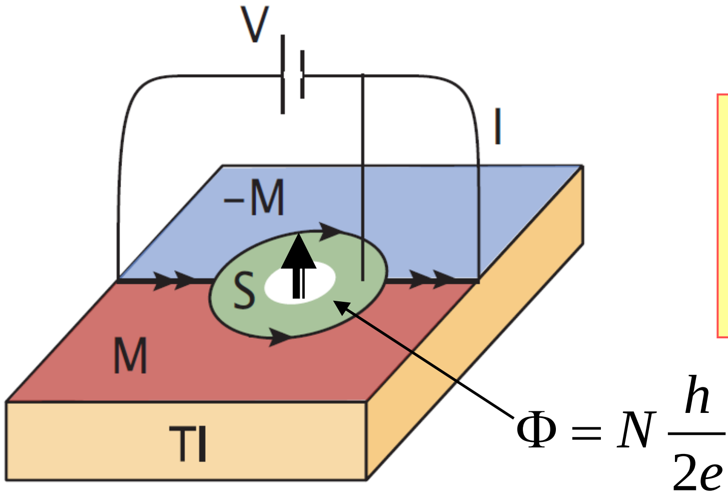




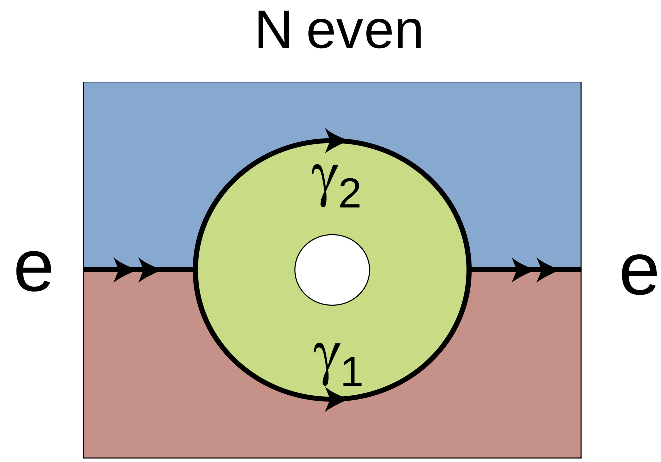
- 4π perioidicity of E(ϕ) protected by local conservation of fermion parity.
- AC Josephson effect with half the usual frequency: f = eV/h

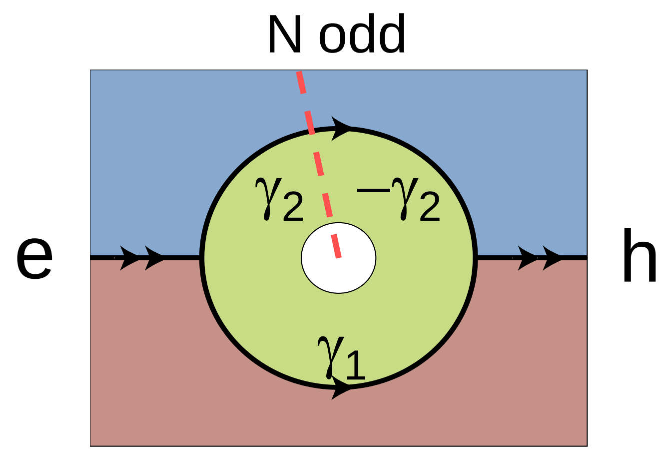
A Z_2 Interferometer for Majorana Fermions

A signature for neutral Majorana fermions probed with charge transport

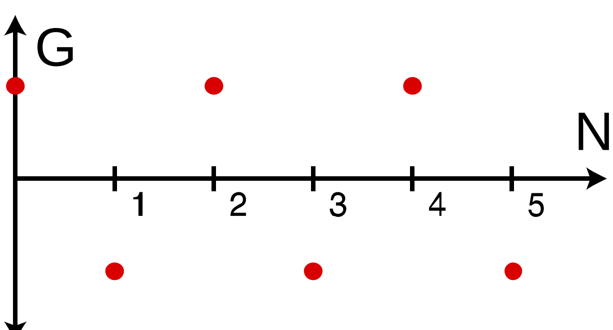


$$c^{\dagger} = \gamma_1 - i\gamma_2$$
$$c = \gamma_1 + i\gamma_2$$





- Chiral electrons on magnetic domain wall split into a pair of chiral Majorana fermions
- "Z_2 Aharonov Bohm phase" converts an electron into a hole



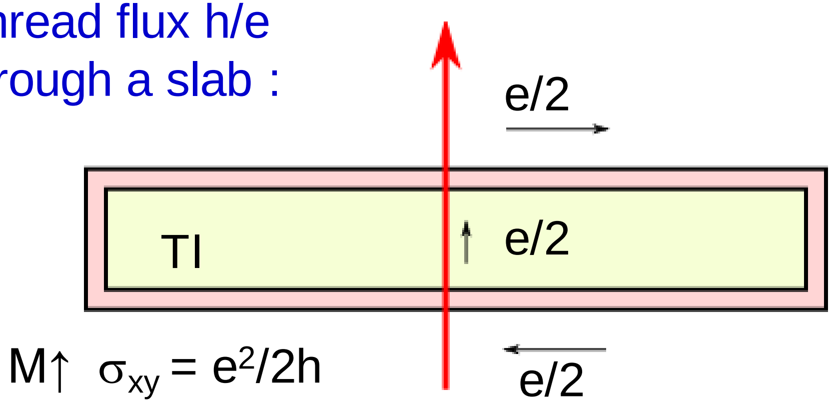
Akhmerov, Nilsson, Beenakker, PRL '09
Fu and Kane, PRL '09

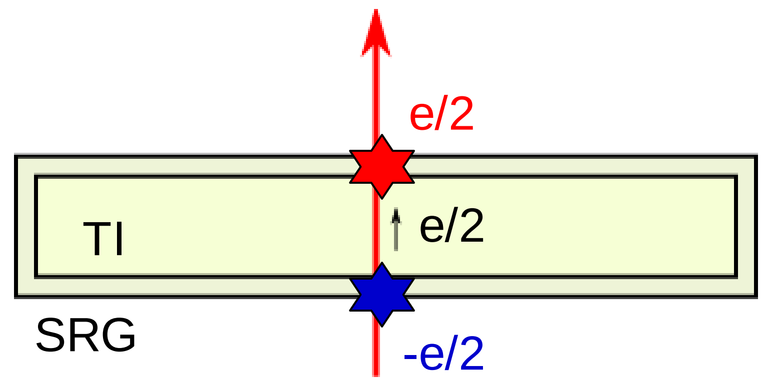
Symmetry Respecting Gapped (SRG) State of a TI Surface

Without interactions such as state is impossible. With interactions a gapped state with intrinsic topological order is possible.

Laughlin argument: $\Phi = h/e$

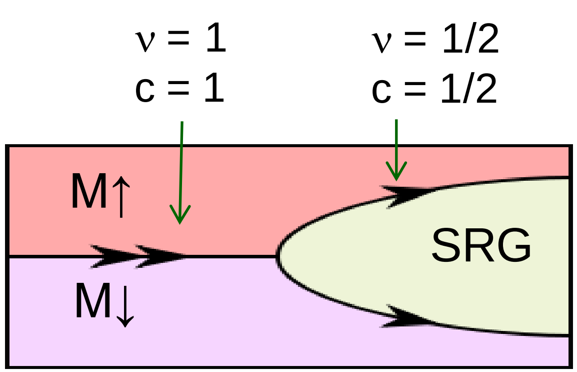
Thread flux h/e through a slab :





SRG State must support charge e/2 excitations

Split a chiral Dirac fermion :



SRG State must have non-Abelian topological order

Much recent progress :

Metlitski, Kane, Fisher	arXiv: 1306:3286
Bonderson, Nayak, Qi	arXiv: 1306: 3230
Chen, Fidkowski, Vishwanath	arXiv: 1306: 3250
Wang, Potter, Senthil	arXiv: 1306: 3223
Mross, Essen, Alicea	arXiv: 1410.4201
Metlitski, Vishwanath	arXiv:1505.05142
Wang, Senthil	arXiv:1505.05141