Introduction to Topological and Conformal Field Theory

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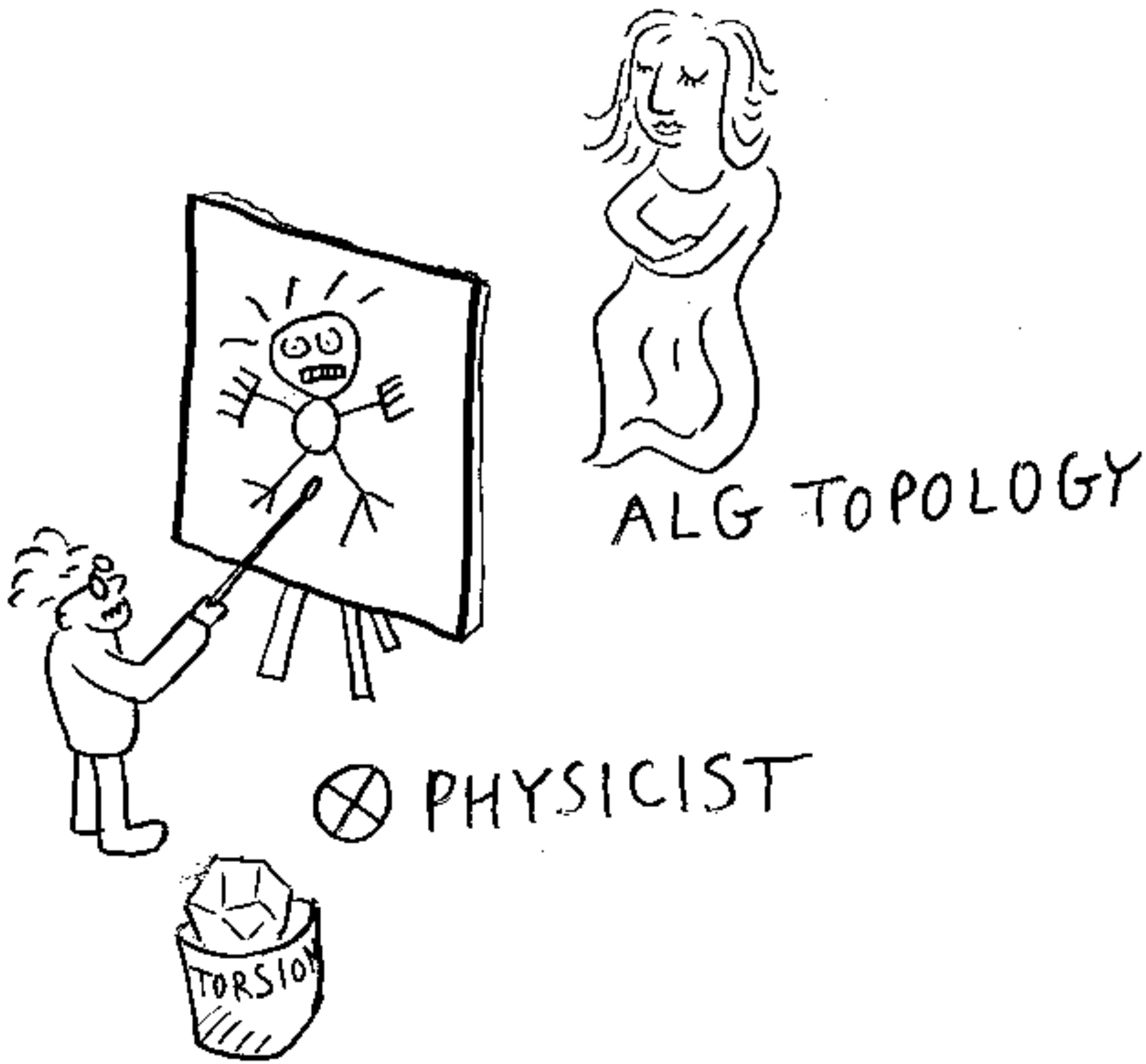
Progress in Theoretical Physics 2015
New Insights Into Quantum Matter

Topological Field Theory

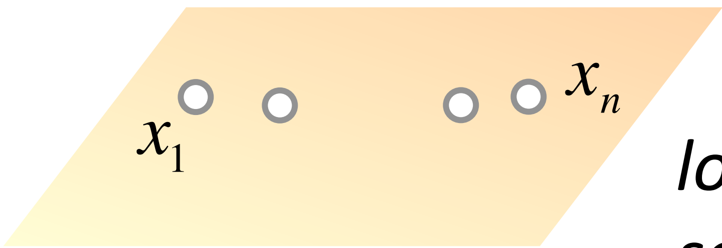
- Describes the (approximate) ground states of quantum matter.
- "Skeleton" of more physical quantum field theories such as 2d CFT.
- Exact results for sectors of (susy) QFT.
- Toy model to study geometrical properties of QFT.
- Connects to deep mathematics.

A Physicist's Apology



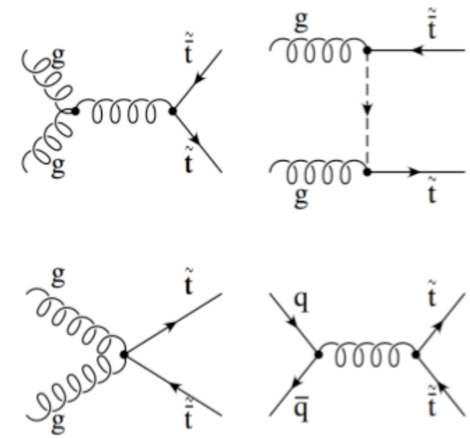
Defining Quantum Field Theory

Algebra



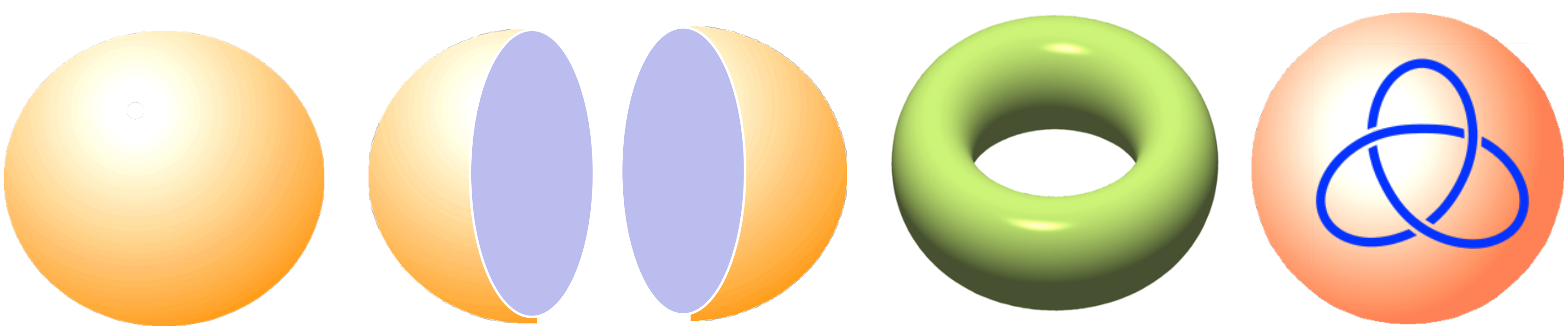
$$\langle O_1(x_1)....O_n(x_n)\rangle$$

local operators
scattering amplitudes



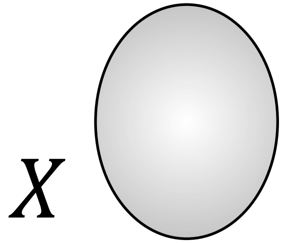
Geometry

cut & paste
topological indices
defect operators



QFT in dimension *n + 1*

Closed space manifold of dim *n*: Hilbert space of states



$$X^n \to \mathcal{H}_X$$

Space of wave functions $\Psi[\varphi_X]$ on field space

Examples

$$X = \mathbb{R}^n, \;\; T^n, \;\; S^n, \;\; \mathbb{R}^{n-1} \times S^1, \ldots$$

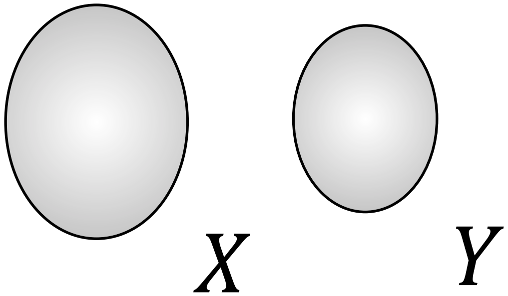
Hilbert Space

Hilbert space of states satisfy certain axioms

$$\mathcal{H}_{\varnothing} = \mathbb{C}$$

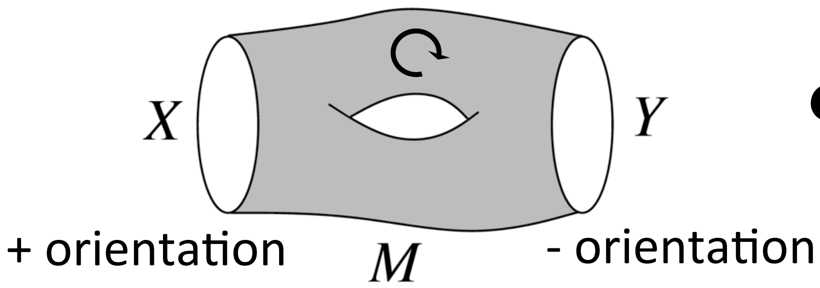
$$\mathcal{H}_{-X} = \mathcal{H}_X^*$$

$$\mathcal{H}_{X \cup Y} = \mathcal{H}_X \otimes \mathcal{H}_Y$$



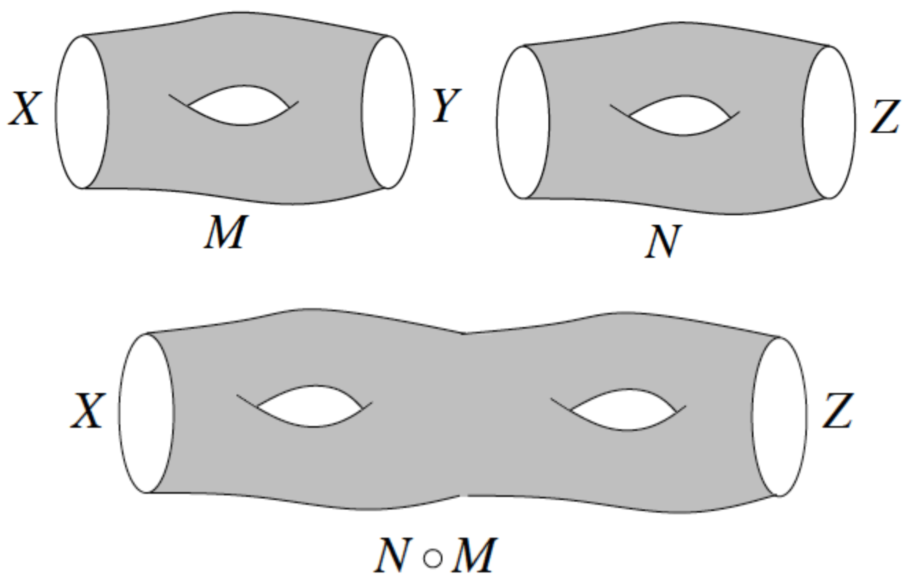
QFT in dimension *n + 1*

Space-time manifold of dim *n+1*: evolution operator



$$\Phi_M : \mathcal{H}_X \to \mathcal{H}_Y$$

Composition law (cutting and gluing)

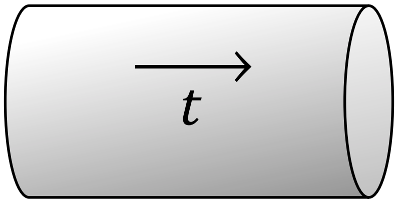


$$\Phi_{N \circ M} = \Phi_N \circ \Phi_M$$

Hamiltonian picture

Standard space time

$$M = X \times [0,t]$$

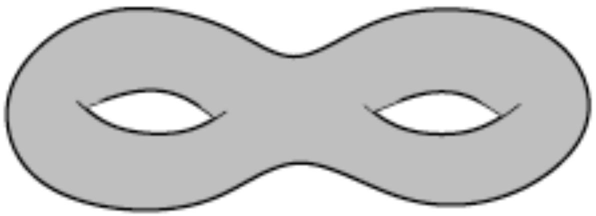
X 

$$\Phi_t = e^{-tH} : \mathcal{H}_X \to \mathcal{H}_X$$

$$\Phi_{t_1 + t_2} = \Phi_{t_1} \Phi_{t_2}$$

Partition functions

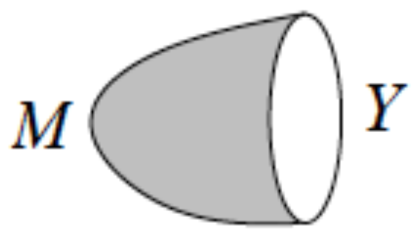
Closed manifold $\Phi_M : \mathbb{C} \to \mathbb{C}$



$$\Phi_M \in \mathbb{C}$$

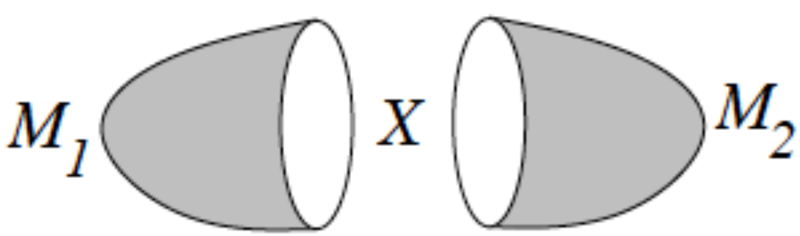
$$\Phi_M = \int d\varphi \, e^{-S[\varphi]}$$

Boundary states $\Phi_M : \mathbb{C} \to \mathcal{H}_Y$



$$\Phi_M = \left| M \right\rangle \in \mathcal{H}_Y$$

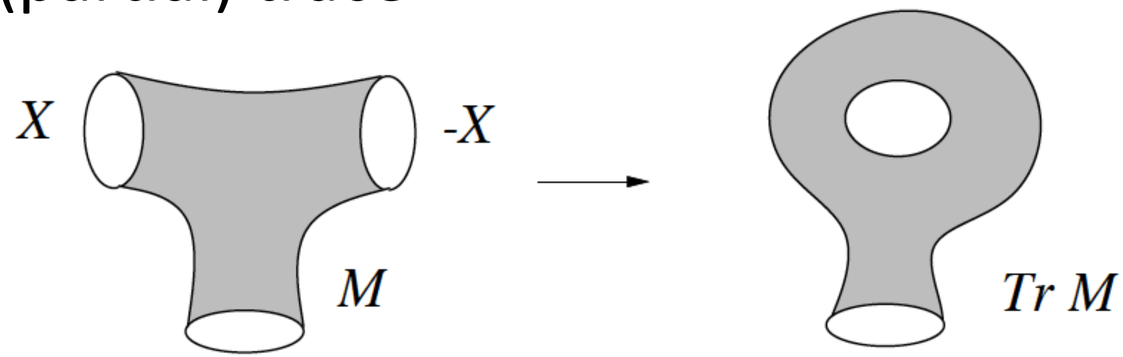
$$\Phi_M[\varphi_Y] = \int_{\varphi_Y} d\varphi \, e^{-S}$$



$$\Phi_M = \left\langle M_2 \middle\| M_1 \right\rangle$$

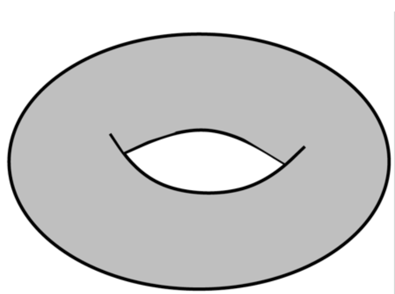
Trace

(partial) trace



$$\mathrm{Tr}_{\mathcal{H}_X}\Phi_M$$

Partition function



$$\Phi_M = \mathrm{Tr}\, e^{-tH}$$

$$M = X \times S^1$$

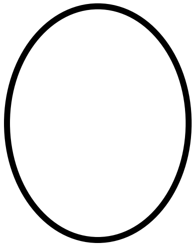
Quantum Mechanics

Hilbert space of a point

$$\Phi_t = e^{-tH} : \mathcal{H} \to \mathcal{H}$$

Partition function

S^1



$$\Phi_{S^1} = \mathrm{Tr}\, e^{-tH}$$

Supersymmetric QM

Maps $x^{\mu}(t)$ coordinates on Σ + fermions

Wave functions = differential forms

$$\Psi = \alpha_{\mu_1 \ldots \mu_k}(x) dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}$$

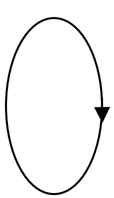
$$\mathcal{H}_{\Sigma} = \Omega^{*}(\Sigma)$$

$$H = -\Delta = -(dd^{*} + d^{*}d)$$

Ground states = harmonic forms, topological QM

$$\mathrm{Harm}^{*}(\Sigma) \cong H^{*}(\Sigma)$$

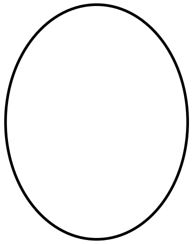
Partition function = Witten index



$$\mathrm{Tr}\,(-1)^{F} = \mathrm{Tr}\left((-1)^{\deg} e^{-tH}\right) = \mathrm{Euler}(\Sigma)$$

Topological Field Theory in 1+1

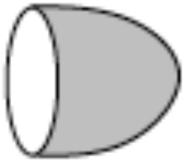
Hilbert space of circle: finite dimensional



$S^1 \rightarrow \mathcal{H}$, basis ϕ_i, $i = 1,\ldots,N$

Special states

$\phi_0 = |0\rangle = 1 \in \mathcal{H}$



$\langle 0| = \langle \cdots \rangle_0 \in \mathcal{H}^*$

Topological Field Theory in 1+1

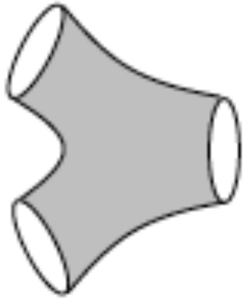
Bilinear form

$$\eta \;:\; \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$$

$$\eta_{ij} = \eta\left(\phi_i, \phi_j\right)$$

Non-degenerate, inverse $\eta^{-1} =$

Pair of pants: multiplication

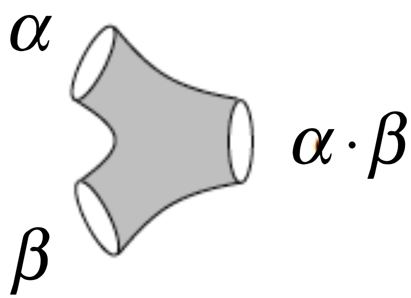


$$c : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$$

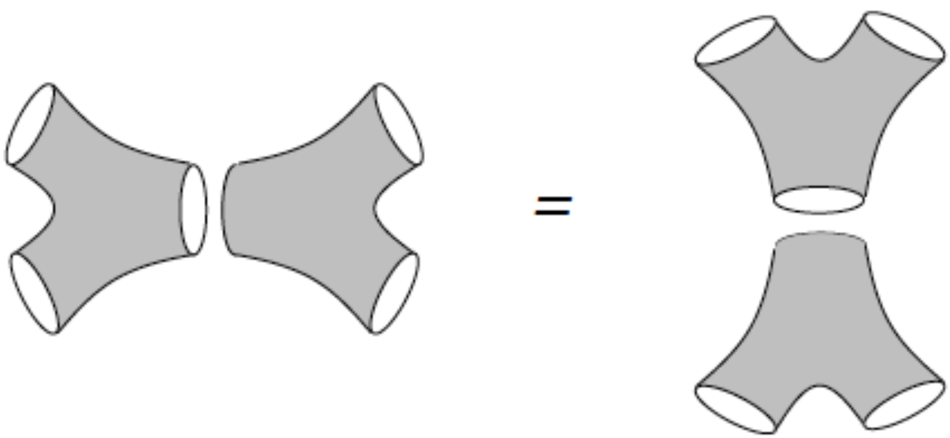
$$\phi_i \cdot \phi_j = c_{ij}{}^{k} \phi_k$$

Algebra of states

Hilbert space = commutative, associative algebra



$$c_{ij}^{\ \ k} = c_{ji}^{\ \ k}$$



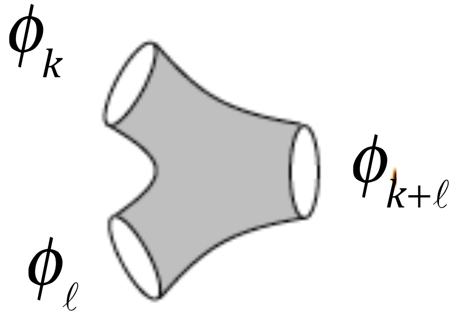
$$\left(\alpha \cdot \beta\right) \cdot \gamma = \alpha \cdot \left(\beta \cdot \gamma\right)$$

$$c_{ij}^{\ \ n} c_{nk}^{\ \ \ell} = c_{jk}^{\ \ n} c_{ni}^{\ \ \ell}$$

Example: Z_N Gauge Field

Holonomy around circle

$$\phi_k = e^{2\pi i k/N}, \quad k = 0,1,\ldots,N-1$$



Frobenius Algebras

Unit  $1 \in \mathcal{H}, \quad 1 \cdot \alpha = \alpha$

Frobenius algebra $c_{ijk} = c_{ij}{}^{n} \eta_{nk} = c_{jki} = \ldots$

$$\eta(\alpha \cdot \beta, \gamma) = \eta(\alpha, \beta \cdot \gamma)$$

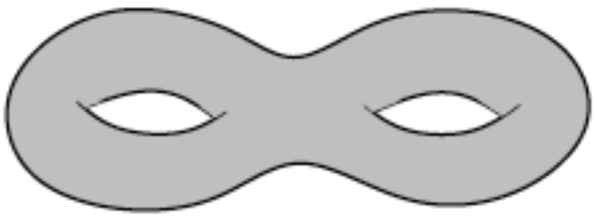
$$\eta(\alpha, \beta) = \langle \alpha \cdot \beta \rangle_0$$

If semi-simple (no idempotents $\alpha^n = 0$), basis e_i

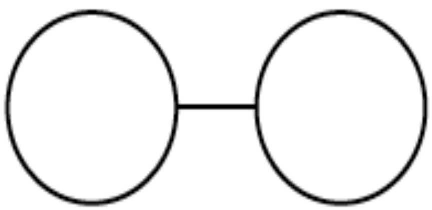
$$\eta(e_i, e_j) = \delta_{ij}, \quad e_i \cdot e_j = \lambda_i \delta_{ij} e_i$$

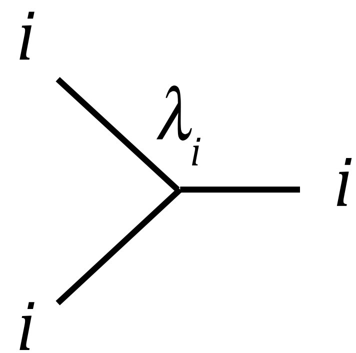
Partition Functions

Compute partition function in genus *g*

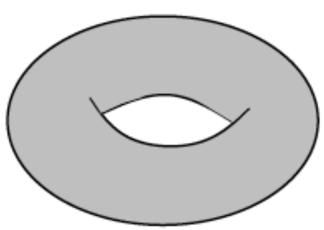


$$\Phi_g = \sum_i \lambda_i^{2g-2}$$





Partition function in genus *1*



$$S^1 \times S^1 : \quad \Phi_1 = \mathrm{Tr}\, 1 = \dim \mathcal{H}$$

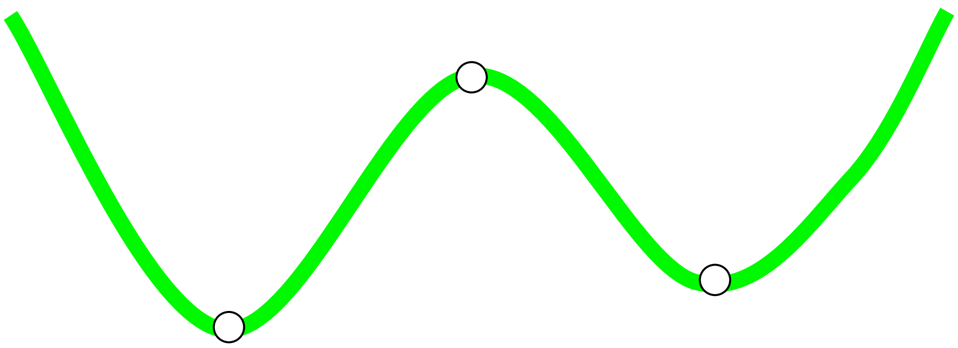
2d Landau-Ginzburg Models

2d sigma model with superpotential $W(x_1,\dots,x_n)$

$$\mathcal{H} = \mathbb{C}[x_1,\dots,x_n]/(dW)$$

$$\frac{\partial W}{\partial x_i} = 0$$

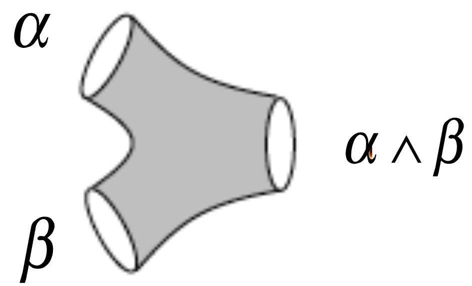
semi-simple = massive model = non-deg critical pts



Supersymmetric Sigma Model

$$\varphi : X \to \Sigma, \quad + \text{fermions}$$

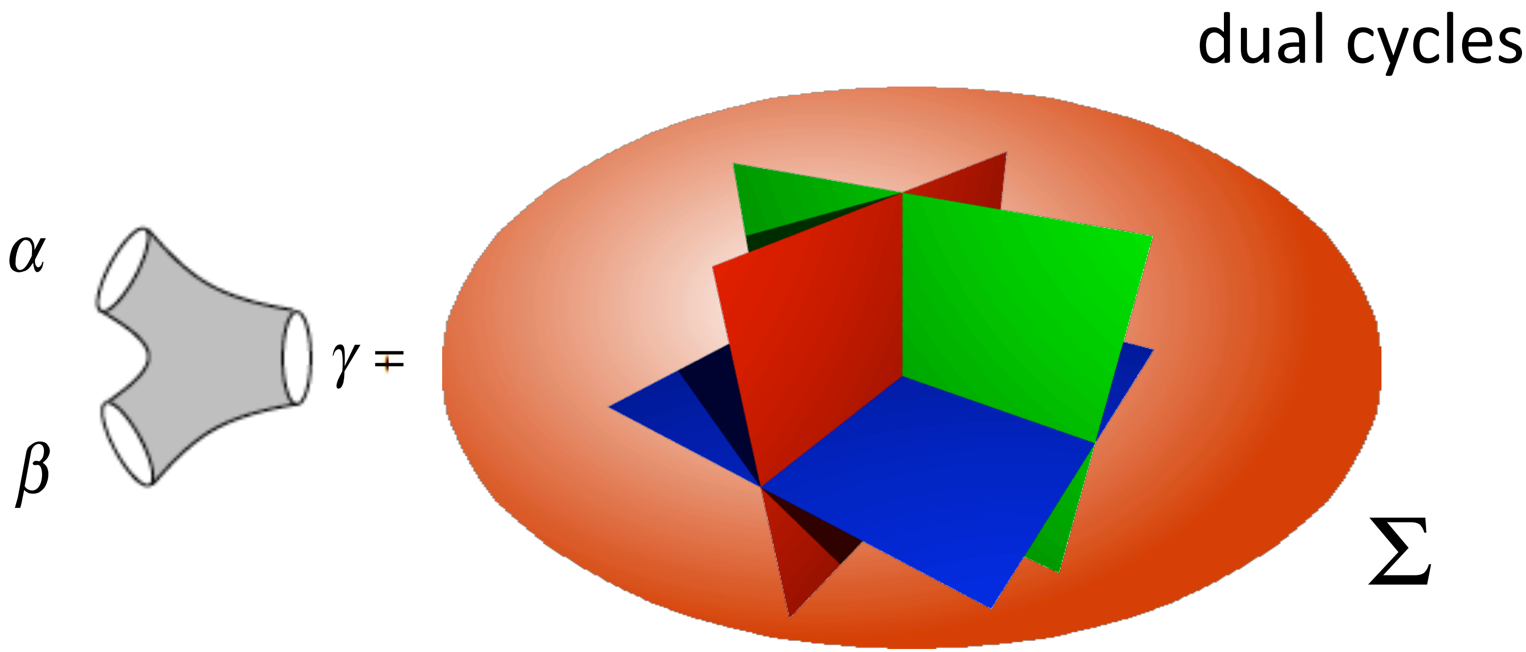
Ground states = cohomology classes $\alpha, \beta \in H^*(X)$



Not semi-simple. For example, ground states for CP^N are

$$1, x, x^2, \ldots, x^{N-1}, \quad x^N = 1$$

Classical Intersection Product

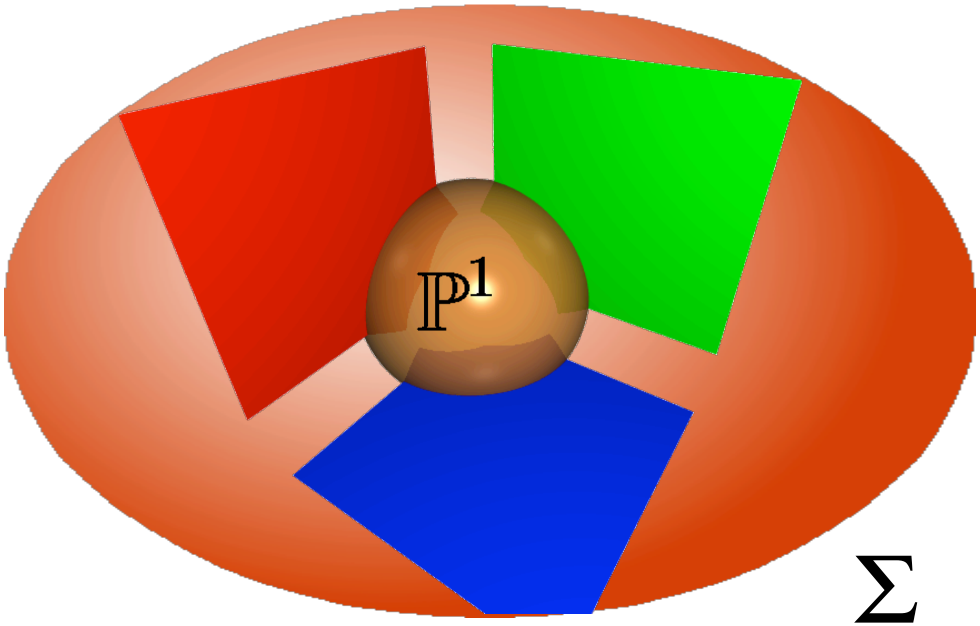


$$= \int_{\Sigma} \alpha \wedge \beta \wedge \gamma$$

Quantum Cohomology

Add two-dimensional instantons of degree *d*

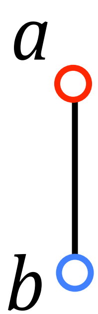
$$= \sum_{\substack{\text{rat curves} \\ \text{degree } d}} e^{-dt}$$



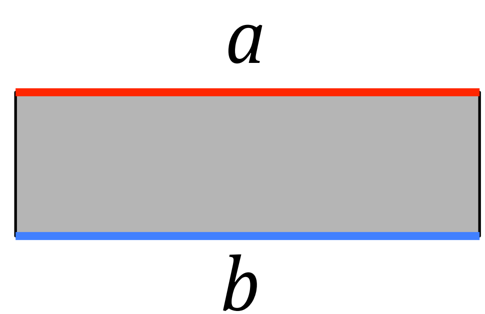
for $\mathbb{CP}^N$ $\quad x^{N+1} = 0 \quad \Rightarrow \quad x^{N+1} = e^{-t}$

Boundaries: Extended TFT

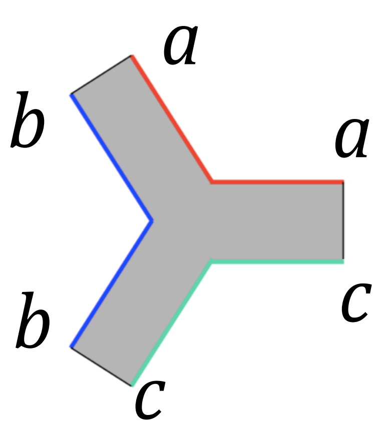
Consider spaces with boundaries, interval *I*



$$I \quad \rightarrow \quad \mathcal{H}_{a,b}, \quad a,b \in C$$



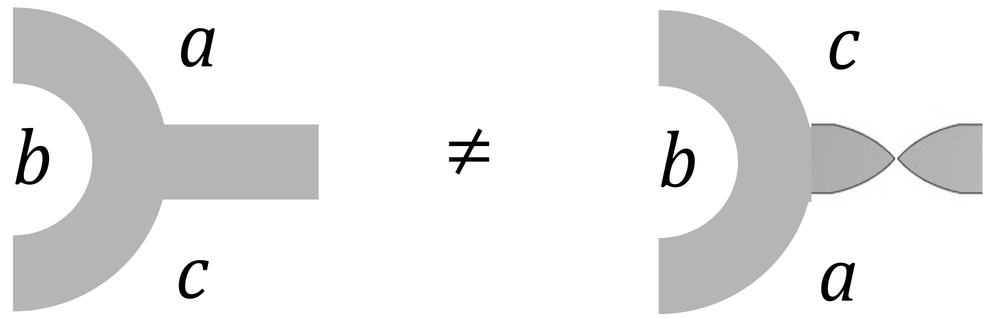
C = category of boundary conditions



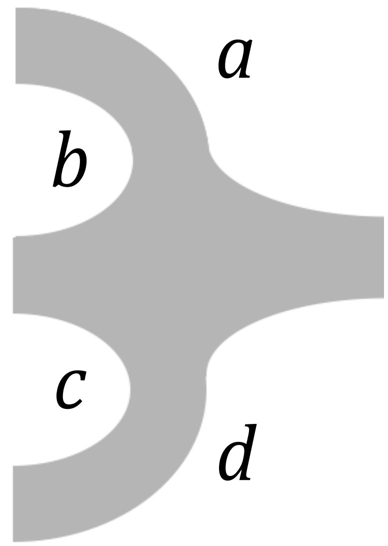
$$\mathcal{H}_{ab} \otimes \mathcal{H}_{bc} \rightarrow \mathcal{H}_{ac}$$

Boundaries: Extended TFT

No longer commutative



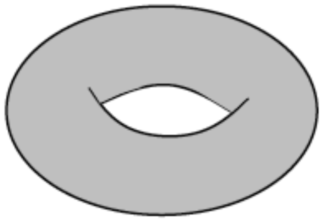
But still associative



Extended TFT in *n + 1* dim

Codim 0. closed space-time manifold: partition function

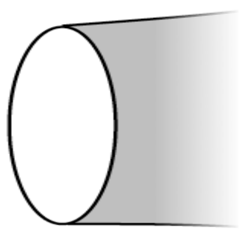
M



$$M^{n+1} \to \Phi_M \in \mathbb{C}$$

Codim 1. space-like boundary: Hilbert space

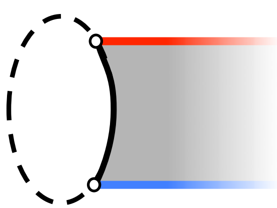
$X = \partial M$



$$X^n \to \mathcal{H}_X \in \text{Vect}$$

Codim 2. corners: category of boundary conditions

$B = \partial X$

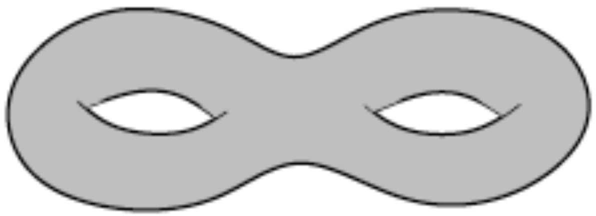


$$B^{n-1} \to C_B \in \text{Cat}$$

Codim 3. .....

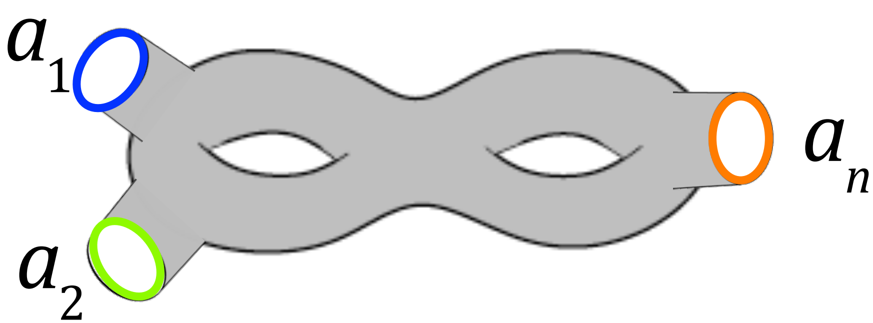
TFT in 2 + 1 dim

Space closed, genus *g*



$$X_g \to \mathcal{H}_g$$

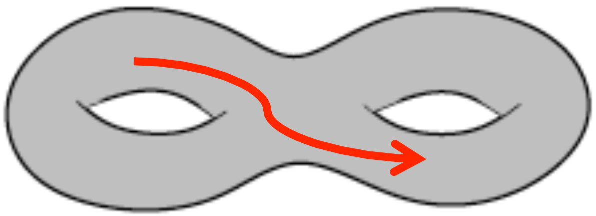
Boundaries $a_1,\ldots,a_n \in C_{S^1}$



$$X_{g,n} \to \mathcal{H}_g(a_1,\ldots,a_n)$$

TFT in 2 + 1 dim

Action of diffeomorphisms



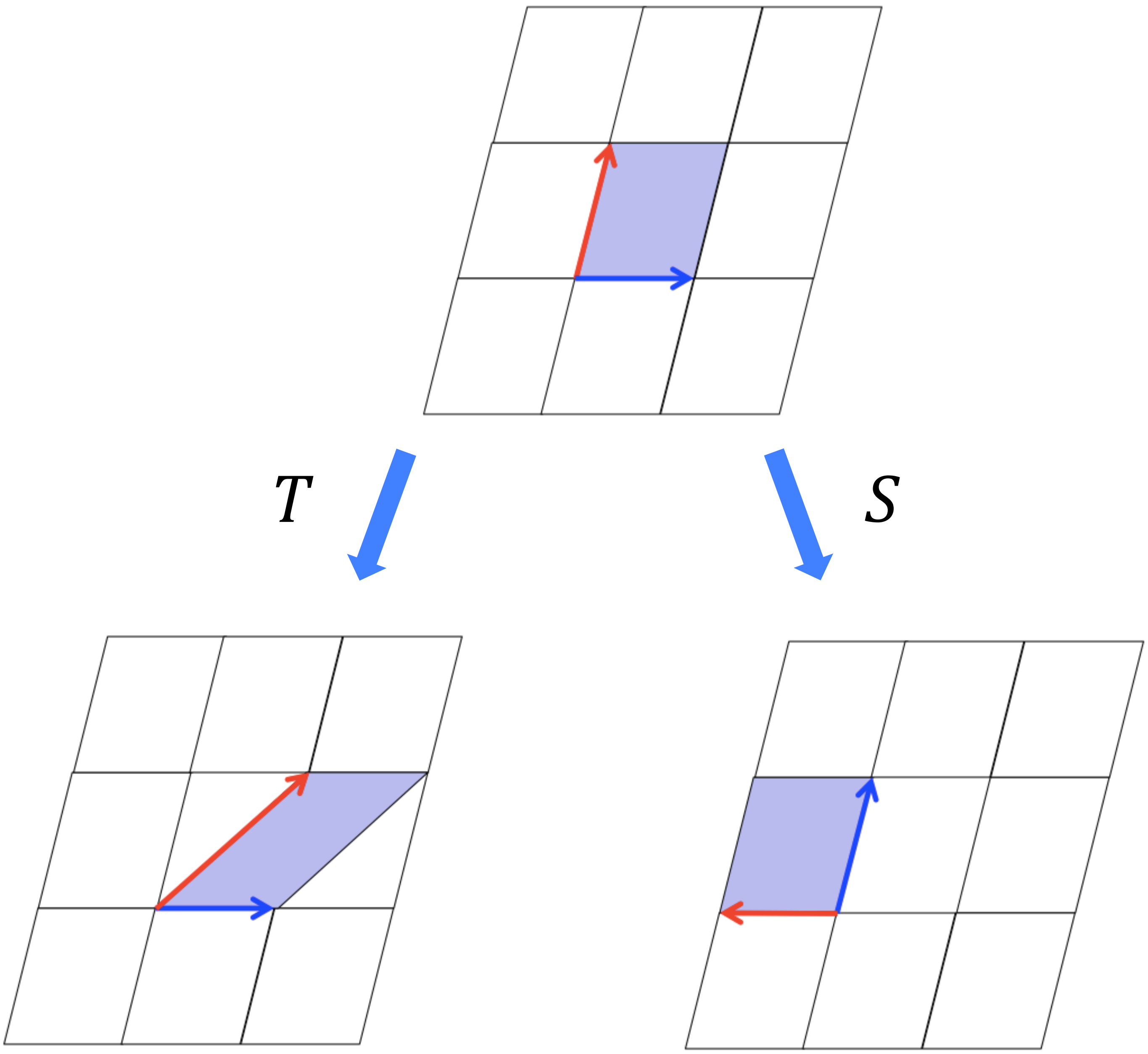
$$\varphi : X \to X, \quad \varphi \in Diff(X)$$

Representation on the Hilbert space

$$\hat{\varphi} : \mathcal{H}_X \to \mathcal{H}_X$$

Torus: action of $SL(2,\mathbb{Z})$

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$


T
S

2D Conformal Field Theory

Conformal invariance $g_{\mu\nu} \to e^{\lambda} g_{\mu\nu}$

Traceless stress tensor $T_{\mu}^{\ \mu} = 0$

Only sensitive to complex structure

$$g_{\mu\nu} dx^{\mu} dx^{\nu} = e^{\lambda} \left| dz \right|^2, \ \ z = x + iy$$

Chiral stress tensors: Virasoro algebra

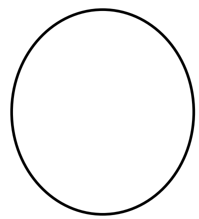
$$T(z) = T_{zz}, \quad \bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}, \quad \bar{\partial} T = \partial \bar{T} = 0$$

$$T(z) = \sum_n L_n z^{-n-2},$$

$$\left[L_m, L_n \right] = (n-m) L_{n+m} + \frac{c}{12}\left(m^3 - m \right) \delta_{m,-n}$$

Category point of view

Hilbert space of circle

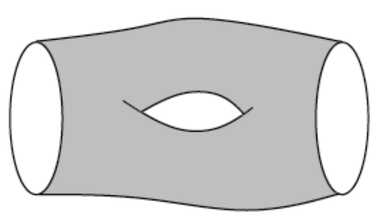


$$S^1 \to \mathcal{H}$$

infinite dimensional representation

$$Vir \otimes \overline{Vir} \to Diff(S^1)$$

Riemann surface of genus *g* with *n = p + q* punctures



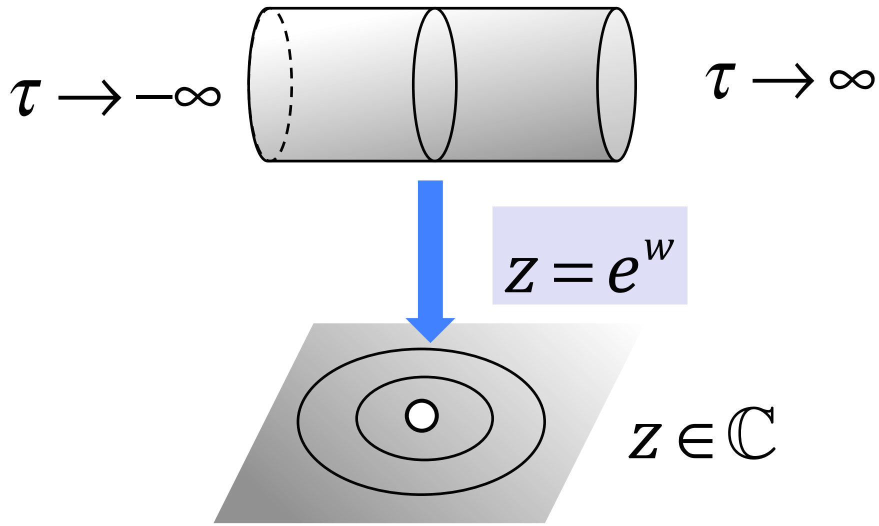
M

$$\Phi_M : \mathcal{H}^{\otimes p} \to \mathcal{H}^{\otimes q}$$

Finite number *3g – 3 + n* of moduli

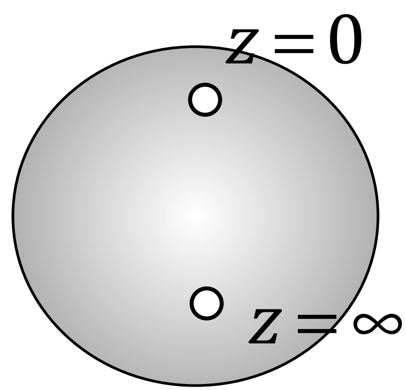
Cylinder, plane, sphere

cylinder $w = \tau + i\sigma, \quad \sigma \in [0, 2\pi]$



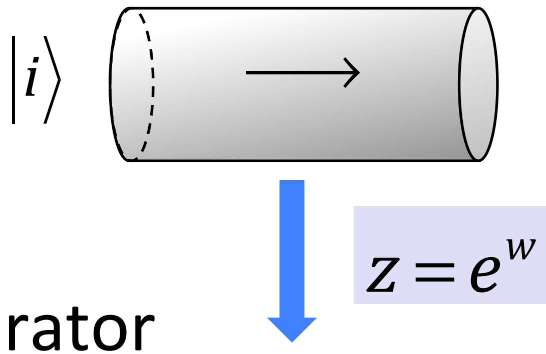
plane

Riemann sphere



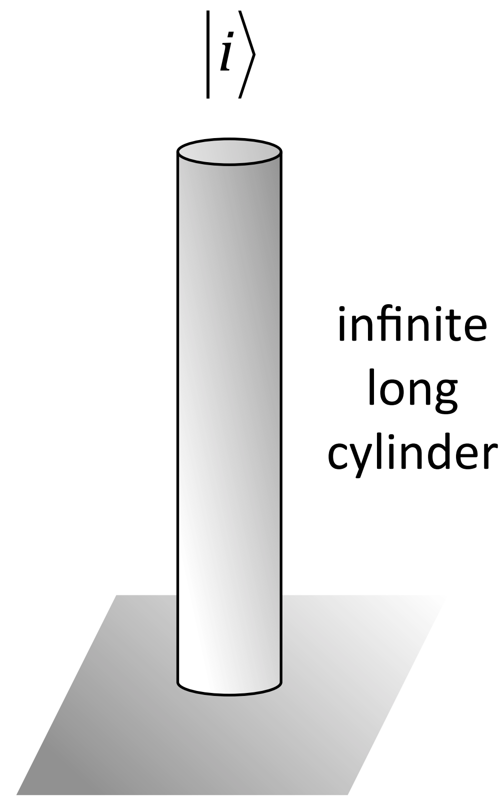
Operators and States

Hilbert space state $|i\rangle \in \mathcal{H}$

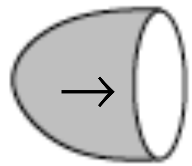


Local operator

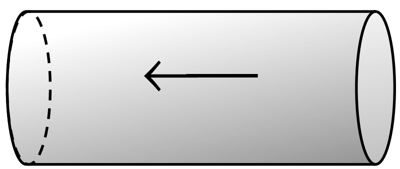
$$\lim_{z \to 0} O_i(z)|0\rangle = |i\rangle$$



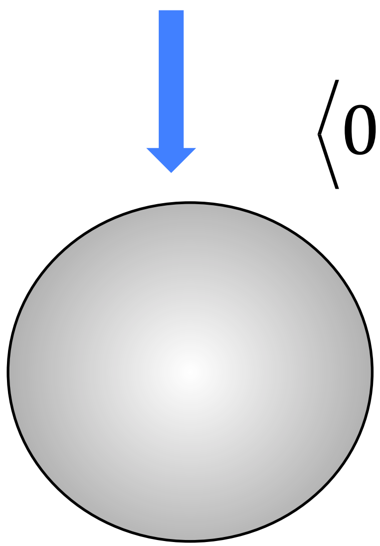
Vacuum State

Vacuum  $|0\rangle \in \mathcal{H}$

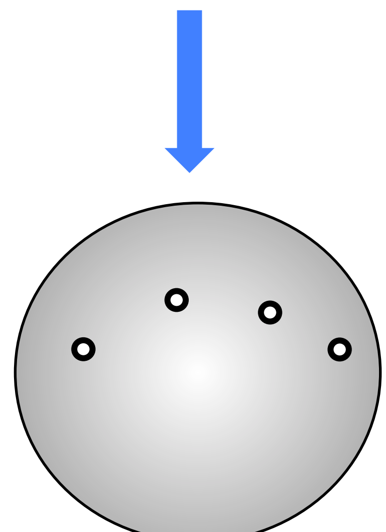
Vacuum amplitudes = correlations on the sphere

$\langle 0|$  $|0\rangle$

$\langle 0|O_{i_1}(z_1)\cdots O_{i_n}(z_n)|0\rangle$



$\langle 0||0\rangle = \Phi_{S^2}$



Free boson

Action $S=\frac{1}{4\pi}\int \partial\varphi\,\bar{\partial}\varphi\; d^2z$

Canonical quantization

$$\partial\bar{\partial}\varphi=0,\;\; \partial\varphi(z)=\sum_n \alpha_n z^{-n-1},\;\; \bar{\partial}\varphi(z)=\sum_n \bar{\alpha}_n \bar{z}^{-n-1},$$

$$[\alpha_n,\alpha_m]=n\delta_{n,-m}$$

Vacuum state $\alpha_n|0\rangle=\bar{\alpha}_n|0\rangle=0,\quad n\geq 0$

Fock space $\mathcal{F}_0:\;\; \alpha_{-n_1}\cdots\alpha_{-n_s}|0\rangle$

Chiral algebra

Stress tensor $T(z) = -\frac{1}{2}(\partial\varphi)^2$

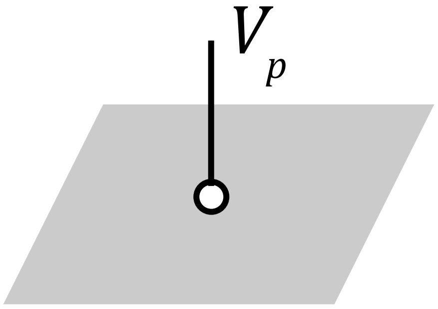
Chiral current $J(z) = \partial\varphi(z)$

Representations: charge/momentum states

$$\alpha_0|p\rangle = p|p\rangle, \quad \alpha_0 = \frac{1}{2\pi}\oint J(z)dz$$

Vertex operators

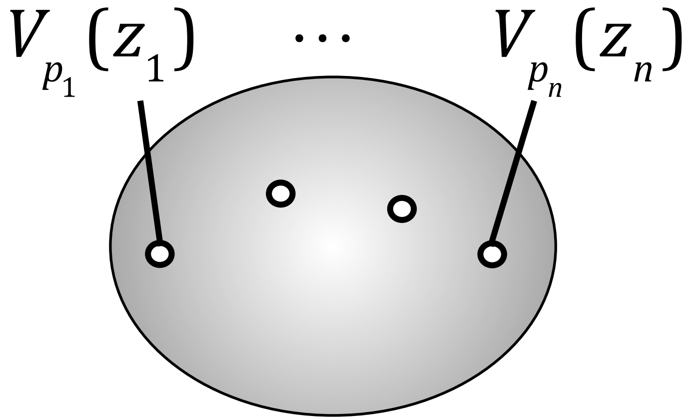
$$|p\rangle = V_p(0)|0\rangle, \quad V_p = e^{ip\varphi}$$



Fock space $\mathcal{F}_p : \alpha_{-n_1}\cdots\alpha_{-n_s}|p\rangle, \quad \mathcal{H} = \int dp\, \mathcal{F}_p \otimes \bar{\mathcal{F}}_p$

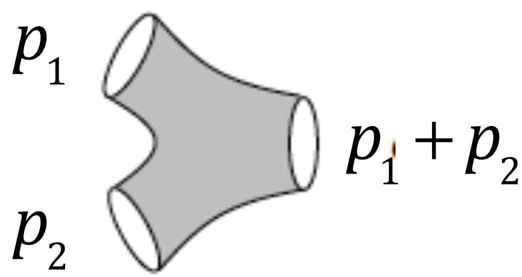
Correlators on the sphere

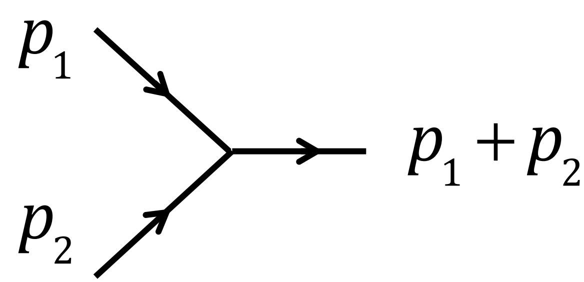
$$\sum_i p_i = 0$$



$$\left\langle V_{p_1}(z_1)\cdots V_{p_n}(z_n)\right\rangle = \prod_{i,j}\left(z_i - z_j\right)^{p_i p_j}$$

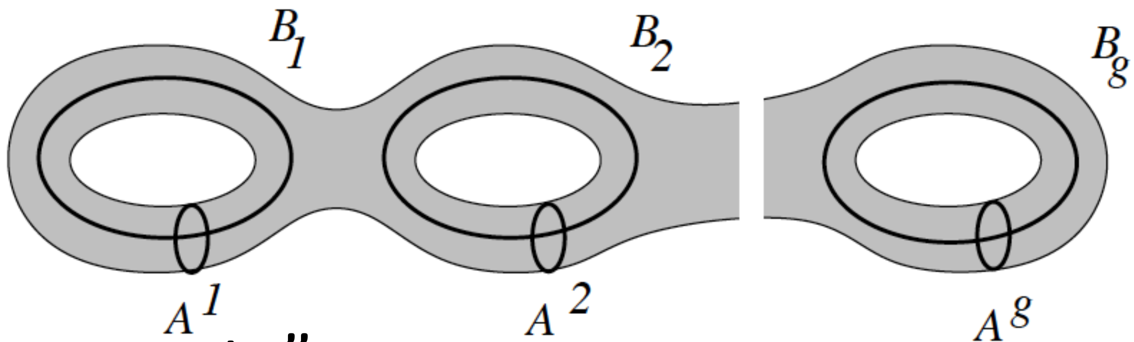
Charge conservation





Higher genus surfaces

Homology basis



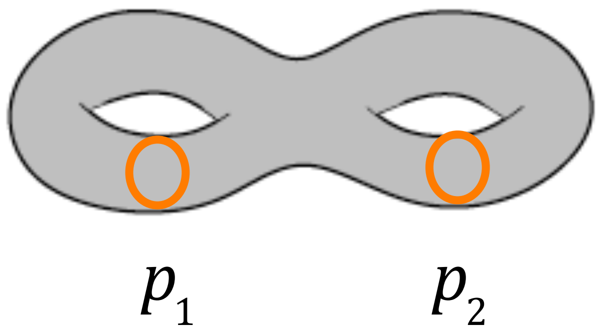
Fix "loop momenta"

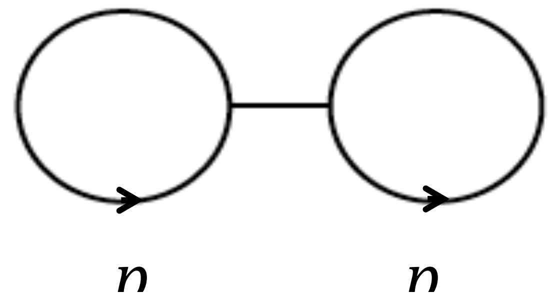
$$\frac{1}{2\pi}\oint_{A_I}\partial\varphi = p_I, \quad I = 1,\ldots,g$$

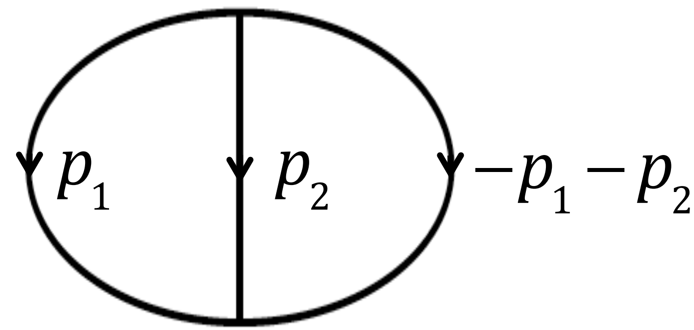
Holomorphic factorization (almost)

$$\Phi \approx \int d^g p \cdot \left|\Psi_p(\tau)\right|^2$$

Higher genus surfaces







Compactification

Compact scalar $\varphi \cong \varphi + 2\pi R$

Quantization of momentum $p = \dfrac{n}{R}$

Winding number

$$\frac{1}{2\pi}\oint d\varphi = mR$$

Left/right momenta $\varphi = q + p_L \log z + p_R \log \bar{z} + \ldots$

$$p_L = \frac{n}{R} + mR, \;\; p_R = \frac{n}{R} - mR$$

Rational Conformal Field Theory

If $p_R = \frac{n}{R} - mR = 0, \;\; R^2 = \frac{n}{m}$

Let's assume $(n,m) = (k,1), \;\; R^2 = k,$

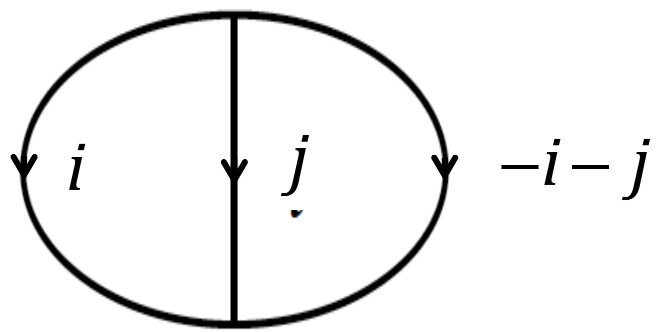
Chiral vertex operator $\;\; V_k(z) = e^{i\sqrt{k}\varphi}, \;\; d = k$

Finite number of representation of extended chiral algebra

$$V_n = e^{in\varphi/\sqrt{k}}, \;\; \mathrm{n} = 0,1,\ldots,k-1$$

$\mathbb{Z}_k$ structure $\;\; V_i V_j = V_{i+j}$

$i \mod k$



WZW Model

Sigma model on a group manifold

$$g(z,\bar{z}): M \rightarrow G$$

$$S = \frac{k}{8\pi} \int \mathrm{Tr}\left(g^{-1} \partial g g^{-1} \bar{\partial} g \right) + 2\pi k S_{WZ}$$

Representation of Kac-Moody algebra at level *k*

$$J_a(z), \quad a = 1, \ldots, \dim G$$

Finite number of integrable representations.
For example

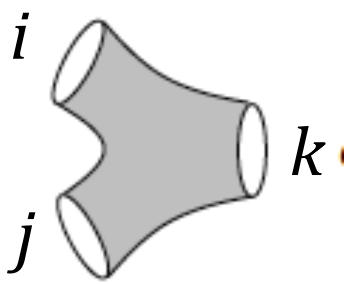
$$SU(2)_k: \quad \mathrm{spin}\, j = 0, \frac{1}{2}, \ldots, \frac{k}{2}$$

Modular Tensor Category

Hilbert space decomposition

$$\mathcal{H} = \oplus_{i,j} R_i \otimes \overline{R}_j$$

Fusion product



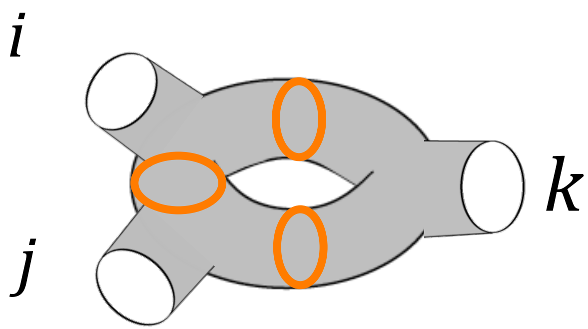
$N_{ij}^{\ k}$ number of invariant tensors

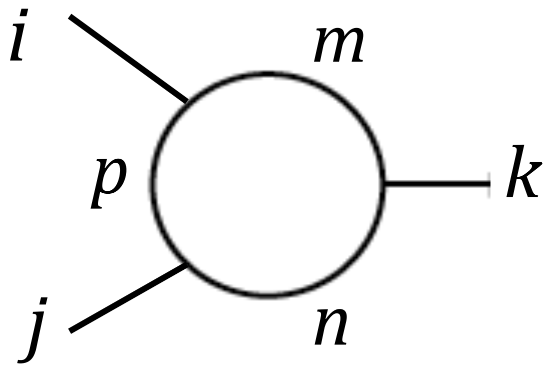
Verlinde algebra $\phi_i \cdot \phi_j = \sum_k N_{ij}^{\ k} \phi_k$

Defines a 2d TFT

Holomorphic blocks

Hilbert space decomposition





\# blocks =
partition function
of 2d TFT

Chern-Simons

Local action, compact gauge group *G*

$$S=\frac{k}{4\pi}\int \mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right),\quad k\in\mathbb{Z}$$

Field equation

$$\frac{\delta S}{\delta A}=\frac{k}{2\pi}F=0$$

Path integral

$$\Phi_M=\int_{\mathcal{A}/\mathcal{G}} DA\, e^{iS}$$

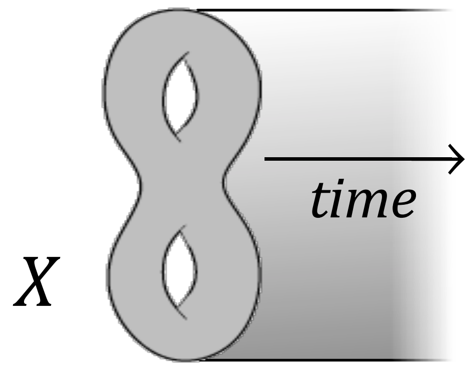
Large gauge transformations

$$S\rightarrow S+n\cdot 2\pi k$$

Quantization

Gauge choice, constraint

$$A_0 = 0, \quad F_{ij} = 0$$



Configurations space

$$C = \{ \mathit{flat\ connections} \} / \mathcal{G}$$

Polarization

$$S = \frac{k}{4\pi} \int d^2x\, dt \;\; \varepsilon^{ij} \; \mathrm{Tr}\left(A_i \frac{\partial A_j}{\partial t} \right)$$

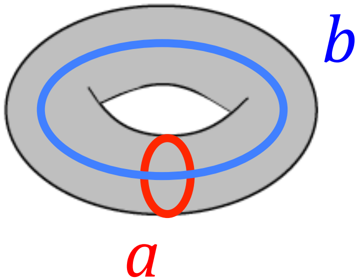
$$A_i = \hbar \varepsilon_{ij} \frac{\delta}{\delta A_j}, \quad \hbar = \frac{2\pi}{k}$$

U(1) Chern-Simons

Action for U(1) gauge group

$$S=\frac{k}{4\pi}\int A\wedge dA,\quad k\in\mathbb{Z}$$

Quantization on $X=T^2$



Holonomies

$$p=\oint_a A,\quad q=\oint_b A,\quad 0\le p,q\le 2\pi$$

Canonical quantization

$$[p,q]=i\hbar=\frac{2\pi i}{k},\quad p=i\hbar\frac{d}{dq}$$

U(1) Chern-Simons

But as q is periodic

$$p = n\hbar = \frac{2\pi n}{k}$$

But p itself is periodic mod 2π

Finite number of states

$$p\left|n\right\rangle = \frac{2\pi n}{k}\left|n\right\rangle, \qquad n = 0,1,\ldots,k-1$$

$$\mathcal{H}_{T^2} = \mathbb{C}^k$$