Fermion Path Integrals And Topological Phases

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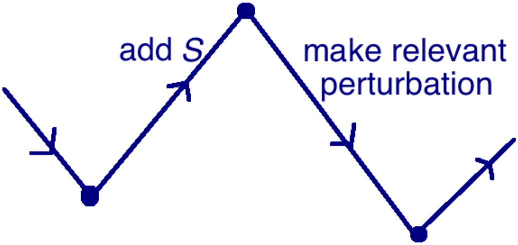
PiTP Lecture

One thing we are learning about this week is that SPT (symmetry-protected topological) phases of matter are associated, in some sense, to "anomalies." A d-dimensional SPT phase is gapped in bulk, but is not quite trivial. It is actually described at long distances by what is called an "invertible" topological quantum field theory $\mathcal{T}$. Its partition function Z_X on a closed $D = d + 1$-dimensional manifold X is a complex number of modulus 1. From this it follows that (a) there is an "inverse" theory $\mathcal{T}^{-1}$ whose partition function is Z_X^{-1}, and the tensor product $\mathcal{T} \otimes \mathcal{T}^{-1}$ is completely trivial; (b) on any spatial manifold M without boundary, the space of physical states of theory $\mathcal{T}$ is 1-dimensional. (This contrasts notably with a fractional quantum Hall system, for which the space of physical states on a compact manifold has dimension > 1.)

What makes theory $\mathcal{T}$ nontrivial is that on a D-manifold X with boundary, it is hard to make sense of Z_X. To be more precise, this cannot be done in a physically sensible way without adding some new degrees of freedom on $M = \partial X$. In my lecture today, the new degrees of freedom will usually be free fermions. But there are additional and more subtle possibilities (including the possibility that the boundary is gapped and topologically ordered).

If we are given a bulk theory $\mathcal{T}$, what has to exist on the boundary is not uniquely determined. The reason is that given any consistent construction, we could always add on the boundary a completely consistent d-dimensional theory $\mathcal{S}$. The very fact that $\mathcal{S}$ is consistent means that it can be added to any consistent theory without spoiling the consistency. Another operation that we can sometimes perform on a consistent theory is to add the the Hamiltonian a relevant operator, possibly coupling what we added in $\mathcal{S}$ to what we already had and removing some degrees of freedom.

Thus there are two basic operations that connect the possible boundary states of a given SPT phase $\mathcal{T}$:



So a given SPT phase $\mathcal{T}$ in d dimensions has a lot of possible boundary states. What they all have in common is that they are not completely consistent, or do not have the expected symmetries, by themselves. Rather, they are consistent and symmetry-preserving in conjunction with the bulk theory $\mathcal{T}$. In some sense, the boundary theory of an SPT phase is "anomalous," the anomaly being measured precisely by the bulk theory $\mathcal{T}$.

Although this qualitative picture is relatively well-established, I think that its implementation for some of the most basic SPT phases has not been completely well-understood. I have in mind the examples that can be constructed from free fermions using band theory, and whose boundary state consists of $d-1$-dimensional gapless fermions of some kind. The precise sense in which the boundary fermions in these theories are "anomalous" is rather subtle and has not been fully described. My aim today will be to do this, focusing for examples on topological insulators and superconductors in $d=2$ or $d=3$ dimensions.

We will have to describe "anomalies" in fermion path integrals in more detail than is usually done. In doing this, I will focus on three examples, which correspond to the three broad classes of fermion theories. The three classes of theory are distinguished by how the fermions transform under the symmetry group. By the "symmetry group," I mean the rotation group $\mathrm{SO}(D)$ in Euclidean signature, or rather its double cover $\mathrm{Spin}(D)$, supplemented possibly by time-reversal and/or reflection symmetries, and possibly by gauge and/or global symmetries. Let us just schematically write K for this full symmetry group.

Anyway there are three broad classes of theory, depending on whether fermions transform in a (1) pseudoreal, (2) real, or (3) complex representation of K. (Of course, it is also possible to consider mixtures, in which some fermions transform in one type and some in another.) The theories I will use to illustrate these possibilities are (1) a 3d topological insulator or superconductor, where we consider only orientable spacetimes; (2) a 2d topological insulator; (3) a 3d topological superconductor or insulator, on possibly unorientable spacetimes. As we will see, cases (1) and (2) are more elementary and the topological invariants that we have to use to understand those cases may be more familiar (at least in case (1)). In a sense, case (3) is universal: if one understands case (3), then (1) and (2) can be viewed as trivial special cases. However, I think that a discussion that would start with case (3) and view cases (1) and (2) as special cases would be rather delphic.

Here is another way to describe the three cases:

(1) A bare mass is possible for all fermions that is invariant under the connected part of the symmetry group K, but it may violate some discrete symmetry, such as $\mathbf{T}$. If one doubles the spectrum, a symmetry-preserving bare mass is possible.

(2) A fermion bare mass is not possible (even at the cost of breaking $\mathbf{T}$) unless one doubles the spectrum. If one doubles the spectrum, a symmetry-preserving bare mass is possible.

(3) No bare mass is possible even after doubling the spectrum (or taking any number of copies of it).

The example that I will consider first is the $d = 3$ topological insulator, which is a **T**-invariant system (**T** =time-reversal) that has on its boundary a $2+1$-dimensional massless Dirac fermion ψ:

$$I = \int \mathrm{d}^3 x \, \overline{\psi} i \not{D} \psi$$

that couples to electromagnetism with charge $-e$, just like an electron. (In fact, in band theory, ψ arises as a particular mode of the electron.) The reason that ψ is massless is that a mass term would violate **T**-symmetry. As I remarked last week, a *massive* 2-component fermion in 2 space dimensions

$$\left(i \not{D} - m \right) \psi = 0$$

describes a single spin state of mass $|m|$ and spin $\mathrm{sign}(m)/2$. So **T**-symmetry (or reflection symmetry) implies that $m = 0$. Hence if we find a **T**-invariant material that has a single massless Dirac-like mode ψ on its boundary, then this state of affairs is protected by time-reversal symmetry.

By contrast, if there are two such modes ψ, ψ', **T**-symmetry would allow mass terms of equal magnitude and opposite signs for the two modes, with **T**-symmetry exchanging them. So in $2+1$ dimensions, there are two kinds of **T**-invariant insulators: those that have an even number of massless fermions on the boundary (generically none) and those that have an odd number (generically 1). The first kind is generically gapped in both bulk and boundary, and so topologically trivial. The second kind is called the topological insulator; it is ungapped on the boundary. That is the case that we are going to study.

The partition function Z_ψ of the ψ field on a possibly curved three-manifold M, coupled to a background electromagnetic potential A, is formally the determinant of the Dirac operator:2

$$Z_\psi = \det\,\mathcal{D}, \qquad \mathcal{D} = i\not{D} = i\sum_{\mu=0}^{2}\gamma^\mu D_\mu.$$

The Dirac operator is hermitian, so its eigenvalues are real:

$$\mathcal{D}\psi_i = \lambda_i\psi_i, \qquad \lambda_i \in \mathbb{R}.$$

Formally the determinant is the product of eigenvalues:

$$Z_\psi = \prod_i \lambda_i.$$

If we had *two* Dirac fermions, we would be interested in

$$Z_\psi^2 = \prod_i \lambda_i^2.$$

This is formally positive, since every factor is positive. Of course, it needs some kind of regularization, such as Pauli-Villars

$$Z_{\psi,\mathrm{reg}}^2 = \prod_i \frac{\lambda_i^2}{\lambda_i^2 + m^2}$$

for large m. This regularization (which is part of a more elaborate procedure) preserves all symmetries and shows that the theory of two Dirac fermions is completely consistent and symmetry-preserving.

With one Dirac fermion, we have a bit of a problem, because Z_ψ is naturally real but not naturally positive:

$$Z_\psi = \prod_i \lambda_i.$$

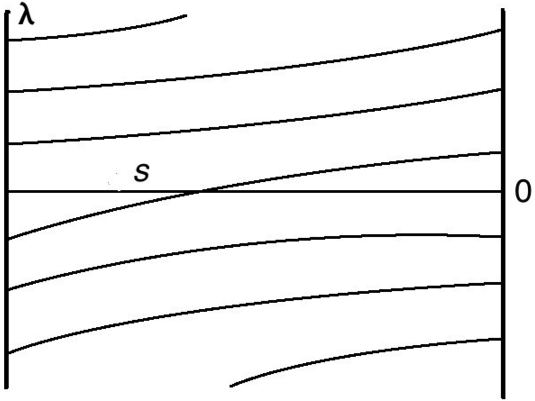
All the factors are real, so it is reasonable to claim that Z_ψ should be real, but there is no natural way to define its sign. The sign is roughly speaking the number of λ_i that are negative, mod 2, but there are infinitely many negative λ_i, and no way to decide if the number of negative ones is even or odd.

One could pick a particular metric and gauge field, say $g = g_0$ and $A = A_0$, and, after picking a sign of Z_ψ for $(A, g) = (A_0, g_0)$, evolve the sign of Z_ψ continuously as a function of A and g. The trouble is that if we do this, we run into a contradiction. Let ϕ be a gauge transformation (or a combination of a gauge transformation plus diffeomorphism). Let (A_0^ϕ, g_0^ϕ) be whatever (A_0, ϕ_0) transform into under ϕ. It is always possible to continuously interpolate from (A_0, g_0) to (A_0^ϕ, g_0^ϕ). One literally introduces a real parameter s and sets

$$A_s = (1-s)A_0 + sA_0^\phi, \quad g_s = (1-s)g_0 + sg_0^\phi.$$

Then one evolves (A, g) continuously in s from $s = 0$ to $s = 1$ and counts how many times Z_ψ changes sign. If it changes sign an odd number of times, that is our anomaly.

It is possible to have such an anomaly because when s is varied from 0 to 1, the spectrum of the Dirac operator $\mathcal{D} = i\not{D}$ can undergo a nontrivial "spectral flow."



Note that this is possible only because $\mathcal{D}$ has infinitely many positive and negative eigenvalues.

In the particular case of boundary fermions of the 3d topological insulator, there definitely is such an inconsistency in the sign of det $\mathcal{D}$, which therefore cannot be defined as a real number. I would like to mention though that even if there is no such inconsistency in the sign of det $\mathcal{D}$ on a particular M, we are not out of the woods. The absence of an anomaly in the sign of det $\mathcal{D}$ means that on a particular M, the sign is well-defined as a function of (A, g), up to an overall sign that depends on M but not on (A, g). However, we should certainly not expect to get a satisfactory theory if we define the sign of det $\mathcal{D}$ independently for each M. Physically, there must be a sensible behavior under various cutting and pasting operations in which various three-manifolds M_i are cut in pieces and glued together in different ways.

In examples relevant to topological states of matter – a good example being the 3d topological superconductor – even if there is no anomaly in the traditional sense of an inconsistency in defining the path integral on a specific M, there can be an anomaly in the more subtle sense that there is no satisfactory way to define overall signs or phases of the path integral on different M's. One needs to take these more subtle anomalies into account as part of the paradigm

"Anomalies in $d-1$ dimensions $\longleftrightarrow$ SPT phases in d dimensions."

Taking these more subtle anomalies into account means trying to give an absolute definition of the phase of the path integral for each M. That is what we will aim to do.

Anyway, going back to the $2+1$-dimensional charged Dirac fermion, the fact that there is a problem in defining the sign of $\det \mathcal{D}$ does not mean that the theory is inconsistent. It only means that the theory cannot be quantized in a **T**- and **R**-invariant way (**R** $=$reflection symmetry). After all, ψ could have a gauge-invariant bare mass, which violates **T**- and **R**-symmetry but otherwise is perfectly physically acceptable.

This means that at the cost of losing **T**- and **R**-symmetry, we can regularize the theory by adding a Pauli-Villars regulator field χ, which one can think of as a bosonic field that obeys a massive Dirac equation $(i\not{D} - m)\chi = 0$. In Euclidean signature, the regularized version of the path integral is

$$Z_{\psi,\mathrm{reg}} = \prod_i \frac{\lambda_i}{\lambda_i + im}.$$

(A more complete description may involve additional bosonic and fermionic regulator fields of different masses, to improve the convergence of this product. The regularized path integral then is well-defined for fixed m. One wants to then add local counterterms to the action – multiplying $Z_{\psi,\mathrm{reg}}$ by some $\exp(-W(m, A, g))$ such that the limit $m \to \infty$ exists. This limit is the renormalized $Z_{\psi,\mathrm{ren}}$ of the continuum quantum field theory. The necessary counterterms are **T**- and **R**-conserving and need not concern us.)

We are really only concerned with the phase of the path integral. From

$$Z_{\psi,\mathrm{reg}} = \prod_i \frac{\lambda_i}{\lambda_i + im}$$

we see that for large $m > 0$, each eigenvalue λ contributes i^{-1} or i to the phase of Z_ψ, depending on $\mathrm{sign}(\lambda)$, so formally

$$Z_\psi = |Z_\psi| \exp\left(-\frac{i\pi}{2}\sum_i \mathrm{sign}(\lambda_i)\right).$$

Thus

$$Z_\psi = |Z_\psi| \exp\left(-i\pi\eta/2\right)$$

where η (the Atiyah-Patodi-Singer or APS η-invariant) is a regularized version of the difference between the number of positive and negative eigenvalues of $\mathcal{D}$.

The precise regularization does not matter. The regularization used originally by APS was

$$\eta = \lim_{s \to 0} \sum_i \mathrm{sign}(\lambda_i) |\lambda_i|^{-s}.$$

The formula

$$Z_\psi = |Z_\psi| \exp\left(-i\pi\eta/2\right)$$

(Alvarez-Gaumé, Della Pietra, and Moore, 1985), together with any standard regularization of $|Z_\psi|$, gives a satisfactory definition of the partition function of the 3d Dirac fermion on any M, for any A, g, with all desireable physical properties except **T**- and **R**-symmetry. This failure of **T**- and **R**-symmetry is often called a "parity anomaly." The 3d charged Dirac fermion can be quantized, but not in a **T**- and **R**-invariant way. The quantization gives a gapless QFT that is perfectly unitary and Poincaré-invariant – and even conformally-invariant – but is not **T**- or **R**-invariant.

Let me describe one aspect of the claim that the formula

$$Z_\psi = |Z_\psi| \exp(-i\pi\eta/2)$$

is physically sensible. When an eigenvalue of $\mathcal{D}$ passes through 0, the fermion path integral is supposed to change sign. This happens in our formula because there is, of course, no sign change of $|Z_\psi|$, but

$$\eta = \lim_{s \to 0} \sum_i \mathrm{sign}(\lambda_i) |\lambda_i|^{-s}$$

jumps by ± 2 when an eigenvalue passes through 0, producing the desired sign change. Changing the sign of the regulator mass in this construction would have complex conjugated Z_ψ, so it is better to write

$$Z_\psi = |Z_\psi| \exp(\mp i\pi\eta/2)$$

with the sign depending on the choice of regulator.

In the theory of the $3+1$-dimensional topological insulator, it is shown that on the surface of such a material, there appears the $2+1$-dimensional Dirac fermion that we have been discussing. The $2+1$-dimensional Dirac fermion is not supposed to be **T**- and **R**-conserving by itself, but the combination of the bulk physics of the topological insulator with the boundary Dirac fermion is claimed to be **T**- and **R**-conserving.

So let us discuss the bulk physics a little bit. In vacuum, we describe the electromagnetic field by the Maxwell action

$$\frac{1}{2e^2}\int \mathrm{d}^3x\mathrm{d}t\left(\vec{E}^2-\vec{B}^2\right)=\frac{1}{4e^2}\int \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu}.$$

In the presence of a material, all sorts of additional interactions may be induced. For our present purposes, the important one is the "θ-term"

$$I_\theta=\theta P,\quad P=\frac{1}{32\pi^2}\int \mathrm{d}^4x\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}.$$

P is called the "instanton number"; on a compact four-manifold X without boundary, it is a topological invariant, in fact an integer. A typical example with nonzero P is $X=S^2\times S^2$ with one quantum of magnetic flux on each S^2 factor. Because P is always an integer and in quantum mechanics we only care about the value of the action mod 2π, θ is an angle

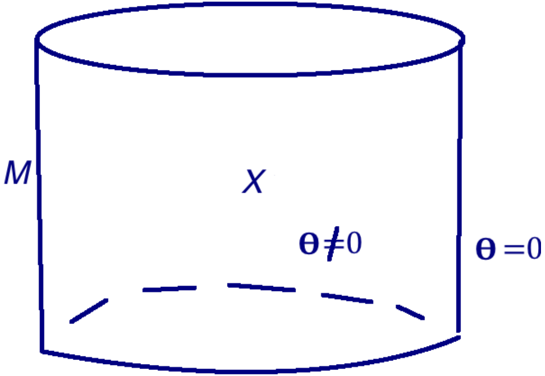
$$\theta\cong\theta+2\pi,$$

usually called the "theta-angle."

In the context of condensed matter physics, in a gapped system, the effective action for the electromagnetic field is in general an arbitrary linear combination of all possible gauge-invariant interactions, constrained only by symmetries. So in particular, we should expect I_θ to be present in the effective action whenever this is allowed by symmetries. What symmetries would forbid I_θ? I_θ is odd under $\mathbf{T}$ and $\mathbf{R}$ symmetry, so in a theory with neither $\mathbf{T}$ nor $\mathbf{R}$ symmetry, one should expect θ to appear with a completely generic coefficient.

What if **T** and/or **R** is a symmetry? These map $\theta \to -\theta$, so naively they would force $\theta = 0$. But since $\theta \cong \theta + 2\pi$, there are really 2 different **T**- and **R**-conserving values of θ, namely 0 and π. As shown by Hughes, Qi, and Zhang (2008), the 3d topological insulator is a **T**-conserving material with $\theta = \pi$.

Before specializing to the **T**-conserving case, let us discuss the generic case of a material with $\theta \neq 0$ and to begin with we assume that the boundary is gapped.



Topological considerations are not important for what I am about to say, and for the moment we can assume that everything is topologically trivial, in which case

$$
\begin{aligned}
P &= \frac{1}{32\pi^2}\int_X \mathrm{d}^4x\, \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}F_{\alpha\beta} = \frac{1}{8\pi^2}\int_X \mathrm{d}^4x \partial_\mu \left(\epsilon^{\mu\nu\alpha\beta} A_\nu \partial_\alpha A_\beta\right) \\
&= \frac{1}{8\pi^2}\int_M \mathrm{d}^3x\, \epsilon^{\nu\alpha\beta} A_\nu \partial_\alpha A_\beta .
\end{aligned}
$$

Stokes' Theorem was used in the last step. The right hand side is $\mathrm{CS}(A)/2\pi$, where

$$\mathrm{CS}(A) = \frac{1}{4\pi}\int \mathrm{d}^3x \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

is the $(2+1)$d Chern-Simons interaction that we discussed last week, so

$$I_\theta = \frac{\theta}{2\pi}\mathrm{CS}(A).$$

Last week, we considered $\mathrm{CS}(A)$ as a possible interaction in a purely 2d material, and we learned that its coefficient has to be an integer k, which moreover (for a gapped system without topological order) determines the Hall conductivity. In the present context, however, we are *not* considering an abstract 2d material, but the surface of a 3d material. We have just learned that in that context, there is no reason for k to be quantized; instead the effective value of k, namely

$$k_{\mathrm{eff}} = \frac{\theta}{2\pi}$$

is never an integer except in the trivial case that $\theta = 0 \bmod 2\pi\mathbb{Z}$. One can view this as a particularly elementary example of how the boundary of a 3d system can have a property that is impossible for a purely 2d system.

Now let us go back to the **T**- and/or **R**-invariant case. We are expecting or hoping that $\theta = \pi$ will be **T**- and/or **R**-invariant, but there is a problem: at $\theta = \pi$ the surface has a Hall conductivity of $1/2$ and at the **T**- or **R**-conjugate value $\theta = -\pi$, the surface Hall conductivity is $-1/2$. Neither of these values is **T**- or **R**-invariant.

One in fact has to be very careful about the meaning of the claim that $\theta = \pi$ is **T**- or **R**-invariant in the case that X has a boundary (the only real case in condensed matter physics). The claim means that the bulk physics of X is **T**- and **R**-invariant and that these symmetries can be maintained by a suitable boundary state. But a trivial gapped boundary state is not suitable. There has to be something on the boundary.

If we just include in the path integral measure a factor

$$\exp(\pm i\pi P) = \exp\left(\pm i\pi \frac{1}{32\pi^2}\int_X \mathrm{d}^4x\, \epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\right),$$

then this definitely does *not* maintain **T** or **R** symmetry. On the contrary, it leads to a Hall conductivity on the surface with $k_{\mathrm{eff}} = \pm 1/2$, as we have just discussed. In fact, **T** or **R** symmetry implies that the path integral measure must be real in Euclidean signature (but not necessarily positive) and the factor that I have written most definitely does not have that property.

The simple boundary state of the topological insulator has gapless Dirac fermions on the boundary and as I have explained they have a **T** and **R** anomaly. Their partition function is

$$Z_\psi = |Z_\psi| \exp\left(\mp i\pi\eta/2\right),$$

as we have discussed. This is not **T**- or **R**-invariant, just as the "bulk" factor $\exp(\pm i\pi P)$ is not. However, it turns out that, if one adds to P a gravitational corection that I will schematically call $\widehat{A}(R) \sim \int_X \mathrm{tr} R \wedge R$, the combination of these factors is real and thus **T**- and **R**- conserving. In fact, by a formula of Atiyah, Patodi, and Singer (APS):

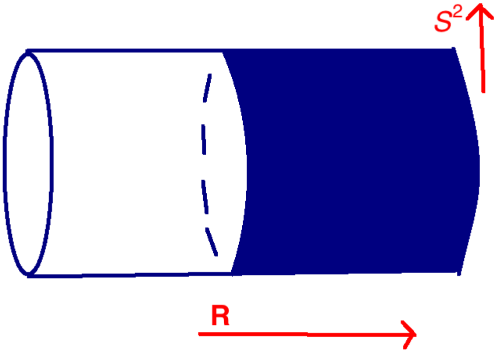
$$\exp\left(\mp i\pi\eta/2\right)\exp(\pm i\pi(P - \widehat{A}(R))) = (-1)^\iota,$$

where ι is an integer. Hence the complete path integral measure after integrating out the boundary fermions is just

$$|Z_\psi| \exp\left(\mp i\pi\eta/2\right)\exp(\pm i\pi(P - \widehat{A}(R))) = |Z_\psi|(-1)^\iota.$$

(In a related context involving string theory D-branes, this formula was described by V. Mikhaylov and EW (2014).)

Postponing for a moment an explanation of the APS formula that I used, let me just try to convey an idea of why the formula $|Z_\psi|(-1)^\iota$ for the path integral measure makes sense. An "instanton" is a localized field with $P = 1$. In condensed matter physics, we can, at least as a thought experiment, imagine a situation in which electromagnetic instantons exist. We take space to be $\mathbb{R} \times S^2$ with the topological insulator filling $\mathbb{R}_+ \times S^2$. (S^2 could be replaced by $S^1 \times S^1$ or any other compact 2-manifold.)



So spacetime is $\mathbb{R}_{\mathrm{time}} \times \mathbb{R} \times S^2$. We place one unit of magnetic flux on S^2, and choose a localized electric field on $\mathbb{R}_{\mathrm{time}} \times \mathbb{R}_+$ with

$$\int_{\mathbb{R}_{\mathrm{time}} \times \mathbb{R}_+} \frac{F_{01}}{2\pi} = 1.$$

Overall this gives

$$1 = \widehat{P} = \frac{1}{32\pi^2} \int_{\mathbb{R}_{\mathrm{time}} \times \mathbb{R} \times S^2} \mathrm{d}^4 x \, \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}.$$

Note that this is an integral over all of spacetime, in constrast to

$$P = \frac{1}{32\pi^2} \int_X \mathrm{d}^4 x \, \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta},$$

which is an integral over the worldvolume X of the topological insulator. When we move an instanton from far outside the material to deep inside, $\widehat{P}$ is identically 1, but P increases from 0 to 1 as the instanton enters the topological insulator.

The claim that $\theta = 0$ outside the topological insulator and $\theta = \pi$ inside ought to mean that the path integral measure is positive when the instanton is far outside X and negative when it is deep inside. But **T** and **R** symmetry, which say that the path integral measure is real, do not let us interpolate from positive to negative values using a factor of constant modulus such as $\exp(\pm i\pi P)$. Instead the factor

$$|Z_\psi|(-1)^\iota$$

has all the right properties. It is real. When the instanton is far from the boundary of X, $|Z_\psi|$ is positive and ι is 0 if the instanton is far outside X and 1 if it is deep inside. The factor $|Z_\psi|(-1)^\iota$ varies smoothly from positive to negative values since (according to the APS theorem) ι jumps from 0 to 1 at precisely the point that Z_ψ vanishes (recall that this happens because the Dirac operator $\mathcal{D} = i\not{D}$ on the boundary has an eigenvalue that passes through zero).

Finally a brief explanation of the APS theorem that I used. This is an index theorem for the Dirac operator $\mathcal{D}_X$ on a four-manifold X with boundary. The $D = 4$ Dirac operator

$$\mathcal{D}_X = i \sum_{\mu=1}^{5} \gamma^\mu D_\mu$$

anticommutes with the "chirality" operator

$$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4.$$

(In contrast to last week, I use Euclidean signature $++++$ as this is more natural for index theory.) So if

$$\mathcal{D}_X \psi = \lambda \psi,$$

then

$$\mathcal{D}_X(\gamma_5 \psi) = -\lambda \gamma_5 \psi$$

If $\lambda \neq 0$, then ψ and $\gamma_5 \psi$ must be linearly independent.

The linear combinations $\psi_\pm = (1 \pm \gamma_5)\psi$ have opposite chirality ($\gamma_5 \psi_\pm = \pm\psi_\pm$) but the same eigenvalue of the "Hamiltonian" $H = \mathcal{D}_X^2$:

$$H\psi_\pm = \lambda^2 \psi_\pm.$$

So at any eigenvalue except 0, H has equally many eigenvalues of positive or negative chirality. But among the zero-modes of H, there can be a "chiral asymmetry." If n_+ and n_- are the dimensions of the space of zero-modes of H with $\gamma_5 = 1$ or -1, then the "index" $\iota(\mathcal{D}_X)$ is defined as

$$\iota = n_+ - n_-.$$

It is a topological invariant, because of arguments similar to those often given in discussions of topological states of matter: when eigenvalues of H move to or from zero energy, they have to do so in pairs consisting of two states of opposite chirality.

The usual Atiyah-Singer index theorem, in the case of $U(1)$ gauge theory on a four-manifold X without boundary, gives

$$\iota = \widehat{A}(R) - P.$$

The APS version of the index theorem governs the case that X has a boundary. One has to use a slightly unusual boundary condition because the obvious boundary condition that we discussed last week ($n \cdot \gamma\psi| = \pm\psi|$, where n is the normal to the boundary) would not let one define an index. APS found a boundary condition that does enable one to define an index, and proved that this index satisfies

$$\iota = \widehat{A}(R) - P - \frac{\eta}{2},$$

which is the formula that I used a few moments ago.

Now I want to explain this in reverse. In four dimensions, there is a 4d topological quantum field theory with $U(1)$ symmetry, with the property that its partition function on a closed four-dimensional spin manifold, coupled to a $U(1)$ gauge field A, is $(-1)^\iota$. (This is the partition function of a topological field theory because the Atiyah-Singer index theorem shows that ι is a cobordism invariant.) However, it is hard to make sense of $(-1)^\iota$ on a manifold X with boundary. That is because there is no simple boundary condition on the Dirac equation that enables one to define the index ι. With APS boundary conditions, one can define the index ι, but it is not a topological invariant. ι jumps by ± 1 when an eigenvalue of the Dirac operator $\mathcal{D}_M$ on $M = \partial X$ passes through 0.

So the theory whose partition function on a closed four-manifold is $(-1)^\iota$ does not have an elementary, gapped, symmetry-preserving boundary state. But it does have a gapless symmetry-preserving boundary state, with $(2+1)d$ massless Dirac fermions on the boundary, such that the partition function on a manifold with boundary is

$$|\det \mathcal{D}_M|(-1)^\iota.$$

But this question might puzzle you: What does a theory with partition function $(-1)^{\iota}$ have to do with a 3d topological insulator? To make this link, we will use a characterization that was explained last week: The phase transition between a trivial 3d insulator and a topological one occurs when the mass *m* of a (3+1)d charged Dirac fermion ψ passes through 0.

For simplicity let us suppose that $\iota > 0$; then generically there are ι zero modes of ψ_+ (the positive chirality part of ψ). Their charge conjugates are ι zero modes of $\overline{\psi}_-$ (the adjoint of ψ_+). At $m = 0$, the partition function for X and A such that $\iota > 0$ vanishes because of 2ι fermion zero-modes. The action contains a term

$$-m\overline{\psi}_-\psi_+ - \mathrm{h.c.}$$

so the path integral

$$Z \sim \int D\psi\, D\overline{\psi}\, \exp\left(- \cdots + m\overline{\psi}_-\psi_+ + \dots\right)$$

is proportional to m^ι, with one factor of m needed to absorb each pair of zero-modes. So if the path integral in this sector is positive for (say) $m > 0$, then its sign for $m < 0$ is $(-1)^\iota$.

This completes what I will say about a 3d topological insulator. Before getting into details, I want to give a preview concerning the other cases of real or complex fermions. (My basic examples will be a **T**-invariant 2d topological superconductor for real fermions and a 3d topological superconductor for complex fermions.) In each case there is an invariant analogous to ι and a $d+1$-dimensional TQFT whose partition function is the exponential of this invariant. In each case, the invariant in question cannot be defined as a topological invariant on a manifold with boundary.

For real fermions, the invariant in question is the mod 2 index of the Dirac operator, which I will call ζ, and for complex fermions, the relevant invariant is a $(d+1)$-dimensional η-invariant. The partition function of the $d+1$-dimensional theory is $(-1)^{\zeta}$ or $\exp(i\pi\eta)$ in the two cases.

As I have said, just like $(-1)^\iota$, the other invariants $(-1)^\zeta$ or $\exp(i\pi\eta)$ cannot be defined as topological invariants on a manifold with boundary. That is why massless fermions on the boundary (or something more sophisticated) are needed to define one of these theories on a manifold with boundary.

Although $(-1)^\iota$, $(-1)^\zeta$, and $\exp(i\pi\eta)$ are not well-defined (as topological invariants) on manifolds with boundary, when we add the standard gapless fermions that exist on the boundaries of these systems, the products

$$|\det \mathcal{D}|(-1)^\iota, \quad |\det \mathcal{D}|(-1)^\zeta, \quad \text{and} \quad |\det \mathcal{D}|\exp(i\pi\eta)$$

are all well-defined and physically sensible.

Just as in the example that we have already discussed, the fact that only the product of the fermion path integral and the partition function of a would-be bulk TQFT is well-defined means that the fermions on the boundary are not consistent by themselves – the fermion theory is anomalous. (In the case that involves the η-invariant, an explicit computation illustrating the anomaly was done by Hsieh, Cho, and Ryu arXiv:1503.01411.)

A few more comments: (1) The fact that the partition function of a (2+1)d or (3+1)d topological superconductor on a closed manifold is $(-1)^\zeta$ or $\exp(i\pi\eta)$ can be proved the same way I argued for $(-1)^\iota$ in the case of a 3d topological insulator: Starting with a trivial phase, one looks at the sign or phase that the path integral acquires when an appropriate fermion mass term passes through 0. (2) Actually, the formulas with $(-1)^\iota$ or $(-1)^\zeta$ can be viewed as special cases of the formula $\exp(i\pi\eta)$. The formula with $\exp(i\pi\eta)$ is universal and reduces to $(-1)^\iota$ or $(-1)^\zeta$ for pseudoreal or real fermions. The reason that I do not present the subject this way is that the special cases in which one can use ι or ζ instead of η are important and are much simpler than the general case. (3) The formula with $\exp(i\pi\eta)$ can be applied to better-understood cases such as the quantum Hall effect. (4) The various formulas with $(-1)^\iota$, $(-1)^\zeta$, and $\exp(i\pi\eta)$ are all elaborations on work on global anomalies going back to the 1980's.

Now I would like to discuss the case of real fermions, for example the case of a **T**-invariant (2+1)d topological superconductor (in which the boundary fermions are nonchiral real fermions in dimension $1+1$). The first step is to define the mod 2 index of the Dirac operator.

For physicsts, I think the simplest explanation is this. Suppose that we have a fermion theory in D dimensions. All I care about is that there is some sort of fermion action

$$I = \int \mathrm{d}^D x \, (\psi, \mathcal{D}\psi)$$

for some $\mathcal{D}$. By fermi statistics, $\mathcal{D}$ is antisymmetric. (In general, in Euclidean signature it has no reality or hermiticity properties.) The canonical form of an antisymmetric matrix is block diagonal

$$\begin{pmatrix} 0 & a & & & & & \\ -a & 0 & & & & & \\ & & 0 & b & & & \\ & & -b & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$$

with nondegenerate 2×2 blocks and some zero-modes (3 in the case shown).

From this it follows that the number of zero-modes is a topological invariant mod 2, since as an antisymmetric operator $\mathcal{D}$ is varied, zero-modes can only be removed (or added) in pairs. The number of zero-modes mod 2 is a $\mathbb{Z}_2$-valued topological invariant ζ that is called the mod 2 index of the operator $\mathcal{D}$. I want to stress that generically it is *not* the mod 2 reduction of a $\mathbb{Z}$-valued invariant such as an ordinary index of a chiral Dirac operator. (For example, there is a nontrivial mod 2 index in $D = 3$, as we discuss shortly, but there is no ordinary index in three dimensions.)

The example we will consider is a Majorana fermion in $D = d + 1 = 3$ dimensions. It certainly has a Dirac action, so the 3d Dirac operator has a mod 2 index ζ. On an orientable 3-manifold X, this mod 2 index is always 0 because of a version of Kramers doubling. However, it is in general not zero on an unorientable 3-manifold (for example on $S^1 \times K$ where K is a Klein bottle).

There is a 3d TQFT whose partition function on a closed 3-manifold is $(-1)^\zeta$, but as I have indicated, this invariant cannot be defined (as a topological invariant) on a manifold with boundary. Anticipating that the problem can be cured by coupling to massless Majorana fermions on the boundary, let us discuss the path integral for $D = 2$ massless Majorana fermions.

In two dimensions, we only need two gamma matrices, and we can pick them to be real 2×2 matrices

$$\gamma_1 = \sigma_1, \quad \gamma_2 = \sigma_3.$$

This means that the Dirac operator $\not{D} = \gamma^\mu D_\mu$ is real and antisymmetric; the hermitian Dirac operator $\mathcal{D} = i\not{D}$ is imaginary and antisymmetric. Such an operator has equal and opposite eigenvalues, since if $\mathcal{D}\psi = \lambda\psi$, then $\mathcal{D}\psi^* = -\lambda\psi^*$.

On an *orientable* two-manifold M, there is a further doubling of the spectrum because of a version of Kramers doubling. We set

$$\overline{\gamma} = \frac{1}{2}\epsilon_{\mu\nu}\gamma^{\mu}\gamma^{\nu}$$

and then the operation $\mathcal{T} : \psi \to \overline{\gamma}\psi^*$ is an antiunitary operator, obeying $\mathcal{T}^2 = -1$, and commuting with $\mathcal{D}$, so all eigenvalues of $\mathcal{D}$ have even multiplicity. Because of **T**- and **R**-invariance, it makes sense to define the 2d Majorana fermion on a possibly unorientable 2-manifold M. In this case, we cannot define $\overline{\gamma}$ and there is no Kramers doubling.

The path integral of a Majorana fermion is naturally understood as the Pfaffian (not determinant) of the antisymmetric Dirac operator $\not{D}$. The canonical form of an antisymmetric matrix U is

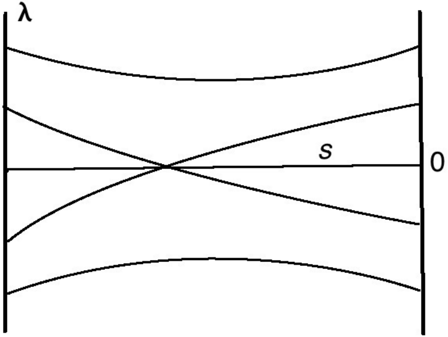
$$U = \begin{pmatrix} 0 & \lambda_1 & & & \\ -\lambda_1 & 0 & & & \\ & & 0 & \lambda_2 & \\ & & -\lambda_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

where the λ_i are uniquely determined up to sign, and the Pfaffian is

$$\mathrm{Pf}(U) = \prod_i (\pm\lambda_i)$$

where I allow for the fact that the λ_i are defined up to sign.

In any system of real (Euclidean signature) fermions, the Pfaffian $\mathrm{Pf}(\not{D})$ is naturally real, but there can be an anomaly in its sign because of an odd number of pairs of eigenvalues passing through 0 as one goes around a loop in the space of fields:



(This picture is different from the one we had for pseudoreal fermions and is possible even for an antisymmetric matrix of finite rank.)

In the specific case of the $D = 2$ Majorana fermion, Kramers doubling prevents this if we are on an orientable manifold M: the spectrum is doubled, so there are always an even number of level crossings. There is no anomaly in $\mathrm{Pf}(\not{D})$, and it is natural to define this Pfaffian to be always positive. If M is unorientable, there is no Kramers doubling, and there can be an anomaly: the 2d Majorana fermion is inconsistent on an unorientable 2-manifold.

By now you hopefully know what I am going to say. There is no way to define the 2d Majorana fermion theory on a "bare" (possibly unorientable) 2-manifold. (Even on an orientable 2-manifold, this partition function cannot be defined so as to be invariant under orientation-reversing symmetries.) But the 2d Majorana fermion can exist on a 2-manifold M that is the boundary of a 3-manifold X that supports a **T**-invariant topological superconductor. The partition function of the combined system is

$$|\mathrm{Pf}(\not{D})|(-1)^{\zeta}.$$

Here ζ is defined with APS boundary conditions and jumps by ± 1 precisely when a pair of eigenvalues of $\not{D}$ is passing through 0.

In view of the time, perhaps I should be brief with the case of a topological superconductor with worldvolume dimension $D = (3 + 1) = 4$. On an *orientable* four-manifold, the partition function of this theory is $(-1)^\iota$, where ι is the index of the Dirac operator (now coupled to only a $\mathbb{Z}_2$ gauge field). This is proved exactly as I said for the topological insulator, by considering the phase transition to the topologically nontrivial phase when a fermion (now a neutral $D = 4$ Majorana fermion) passes through 0 mass. One can determine the answer the same way on a general unorientable four-manifold X, but the answer can no longer be expressed in terms of an index or a mod 2 index. Rather the answer is $\exp(i\pi\eta)$. (This formula was guessed by Kapustin, Thorngren, Turzillo, and Wang, arXiv:1406.7329, on the basis of cobordism invariance.) In even dimensions, the APS index theorem shows that η is a topological invariant and moreover in $D = 4$, one can prove that $\exp(i\pi\eta)$ is in general an arbitrary 16^{th} root of 1. (It is $\exp(2\pi i/16)$ for $X = \mathbb{RP}^4$.) Thus there are 16 classes of superconductor in $D = 4$.

You know the story by now: $\exp(i\pi\eta)$ is no longer a topological invariant if X has a boundary, and likewise the path integral of Majorana fermions on the $D=3$ boundary of X is anomalous. Moreover, in contrast to the other examples I have given, the anomaly in their path integral is not simply a sign. It involves the η-invariant, not the more simple invariants ι or ζ, via reasoning that goes back to my old work relating global anomalies to the η-invariant ("Global Gravitational Anomalies", 1985) and an important elaboration of that result that I call the Dai-Freed theorem (hep-th/9405012). (These tools are very useful in string theory, for example see EW, hep-th/9907041 for application to the heterotic string.)

The upshot, as you should expect from what I have said so far, is that the product

$$|\mathrm{Pf}(\not{D})|\,\mathsf{exp}(i\pi\eta)$$

is well-defined and physically sensible.

There is much more to say, and in particular I would have loved to say something about the slightly delicate case $\nu = 8$, which actually arises in string theory. But this is as good a place to stop as we are going to find.