Topological phases of matter
and
fractional quantum Hall effect

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Overview:

1) Symmetry-breaking paradigm for phases of matter

2) Topological phases and properties

3) Basic notions: ground state degeneracy, quasiparticles, fusion, statistics

4) Fractional quantum Hall effect

5) Conformal field theory constructions

6) Statistics calculation

7) Topological Quantum Computation

8) Conclusion

For a short introduction, see also N. Read, Physics Today, July 2012, p. 38

What is an (equilibrium) phase of matter?

Given some matter, we would like to

--- characterize the state of the matter independent of microscopic details of
the Hamiltonian, or of thermodynamic parameters . . . or even of constituents

--- be able to decide when phases count as the same, and when they differ
. . . difference should be sharp

So phases remain invariant under continuous change of Hamiltonian/parameters, until some boundary is crossed, when a distinct phase is entered

E.g. liquid versus gas --- surely distinct phases as there is a transition (boiling)?

But can continuously connect them without a transition, at high pressure & temp
--- same phase: “fluid”

Symmetry paradigm (Landau)

Suppose there is a symmetry of the Hamiltonian. Note: always assume we have microscopic degrees of freedom local in space, and Hamiltonians involve only short-range hopping, interactions, whatever

There may be parameter regions (phases) in which the symmetry is spontaneously broken in the infinite-size (“thermodynamic”) limit, or broken in different ways.

“Thm”: These cannot be continuously connected without encountering a boundary at which the symmetry changes.

Example: liquid—solid transition. Hamiltonian for particles in continuous space is translation and rotation invariant.

This symmetry is preserved in the liquid, broken in the solid (crystal).

There may be distinct solid phases in which trans/rot symmetry is broken in different ways

--- e.g. ice, about 20 solid phases under pressure

Widespread belief in converse **Thm(?):** if phases cannot be continuously connected without crossing a boundary, there **must** be a difference involving symmetry breaking.

We now understand that this “symmetry paradigm” is **wrong**.

Challenges to the symmetry paradigm (70s—80s)

Kosterlitz-Thouless transition and low T phase of XY model in two dimensions
--- no symmetry breaking

Spin glasses, metal-insulator transition, . . .

But the biggest challenge to the paradigm came from:

Integer and fractional quantum Hall effect 1980, 1982

--- **Quantum** phases of matter: phases at zero temp,
dominated by quantum mechanics (no phase transition at nonzero temp)

--- no symmetry breaking; phases distinguished by quantized Hall conductivity

In response to quantum Hall systems and other examples of exotic states e.g. from models in high T_c/quantum antiferromagnets, theorists developed concept of

Topological phases

Definition: a quantum system at zero temperature is in a **topological phase** if there is an energy gap above the ground state(s) for bulk excitations in the thermodynamic (i.e. infinite-size) limit. Hereafter, simply called a “gap”.

(There could be gapless excitations on an edge.)

Folklore (?): an energy gap does not close under perturbation of Hamiltonian by sufficiently small perturbation by local (short-range) terms --- view as same phase. But at a transition between phases, the bulk gap must collapse --- 1^{st} or 2^{nd} order.

Note: the definition allows a topological phase to be trivial, for example an ordinary insulator. That is not an oversight! We will distinguish trivial/nontrivial phases separately (next slide).

Definition: topological properties
--- properties that are unchanged throughout the phase: “topological invariants”.
(Analogous to “universal” properties at critical points)

Examples:

0) Existence of bulk energy gap

1) multiplicity >1 of ground states when the Hamiltonian is constructed for the system on a space of non-trivial topology: ~~sphere~~, torus, . . . Wen, Niu 1990

2) existence of quasiparticles with non-trivial statistics Moore, NR 1991

3) robust gapless edge excitations Wen 1990

4) quantized transport properties, such as Hall conductance Laughlin 1980

In practice, all known **non-trivial** top phases possess **one or more** of last four
---they can be used to distinguish top phases from one another

Additionally, non-trivial top phases all possess some non-trivial, and top invariant, **entanglement** behavior --- seems to be the most general characterization

The most natural way to understand topological properties is by formulating an **effective field theory** description (a fixed point of RG)

--- i.e. a low-energy, long-wavelength description of response of the ground state to external probes, and of quasiparticle properties

--- the bulk part of the effective field theory action will consist of local terms, and “topological invariance” will hold, because of the (“mass”) gap in the bulk energy spectrum and short-range Hamiltonian

--- this description will be closely connected with some **topological quantum field theory** Witten 1989, Dijkgraaf lectures

In these lectures I will describe techniques for constructing non-trivial topological phases, and for finding the effective field theory in certain cases where perturbation theory cannot be used

Basic notions

Wen, Niu 1990

Topologically degenerate (=equal energy) ground states --- if they occur for some boundary conditions (or e.g. on torus):

--- the matrix elements of any local operator between any of the states $|\alpha\rangle, \ |\beta\rangle$ must give identity matrix:

$$\langle\alpha|\mathcal{O}(x)|\beta\rangle \propto \delta_{\alpha\beta}$$

Otherwise a term like

$$\lambda \int d^d x \, \mathcal{O}(x)$$

could be added to Hamiltonian, splitting the energies; we'll assume any such terms have already been added, any remaining degeneracy of ground states is topological

This means the degenerate ground states are indistinguishable by any local probes. Note: usually such statements mean "up to corrections exponentially small in system size, separation, etc"
Energy gap implies that correlations of local operators decay exponentially with separation, so there is a "correlation length" intrinsic to system.

Local = acts only on particles within some bounded distance of some point x. Local operators in this strict sense commute at well-separated positions, so can be added to Hamiltonian. Fermion operators are not strictly local!

Quasiparticles

--- I'll generally assume space is two-dimensional

--- Starting from ground state, it may be possible to "twist" it around a point and make an (intrinsic) "defect"---a state in same Hilbert space, thus some excited state (not necessarily an energy eigenstate)

--- Far from that point, the state still looks like ground state locally, so justified in calling it a quasiparticle. (But not near the location; quasiparticles can be distinguished locally from ground state, detected, and can move around due to terms in the Hamiltonian.)
Hence excitation energy will be finite (i.e. not infinite).

--- Interested in such objects that cannot be created or destroyed by a local operator: creation of an isolated such defect changes state over arbitrarily large distance. Cannot simply appear or disappear during time evolution by the Hamiltonian

--- Identify as the same "type" quasiparticles that can be obtained from each other by acting with a local operator near their location, or by transporting its position. Type of a quasiparticle can be inferred from local measurements, in principle.
Usually (or always?) number of types is **finite**.

--- Identity or "do nothing" operation on ground state is an honorary quasiparticle type, the trivial or identity type.

--- It **is** possible to create/destroy a quasiparticle and its anti-particle within some disk with an operator localized inside the disk

--- More generally, two quasiparticles of types α, β in a region can be viewed from far away as a single object of some type, say γ : **fusion**

--- **A state containing several well-separated quasiparticles may be degenerate**; the degeneracy is topological, like that for ground states.
I.e.:

--- the multiplicity (dimension of the degenerate subspace) is independent of the positions
--- which of these states the system is in cannot be discerned or flipped by a local operator --- **"non-local"** information storage

--- This is why these systems are of interest for quantum information processing: since errors/decoherence are due to **local** terms in the Hamiltonian, the state in the subspace of states for given qptcles is **topologically protected** against errors

Fusion can be described by "**fusion rules**": if ϕ_α represents quasiparticle of type α, then fusion is described formally as an associative, commutative multiplication determined by the set of formulas

$$\phi_\alpha \times \phi_\beta = \sum_\gamma N_{\alpha\beta}^{\gamma} \phi_\gamma,$$

where $N_{\alpha\beta}^{\gamma} = N_{\beta\alpha}^{\gamma}$ are non-negative integers. (I.e. we have a commutative algebra over ***Z*** generated by the elements ϕ_α .)

Let $\alpha = 0$ be the identity qptcle type $\phi_0 = \mathbf{1}$, the identity in the ring. Then for any α

$$\mathbf{1} \times \phi_\alpha = \phi_\alpha,$$

and we further assume there is a unique anti-particle type denoted $\bar{\alpha}$ such that

$$\phi_\alpha \times \phi_{\bar{\alpha}} = \mathbf{1} + \sum_{\gamma \neq 0} N_{\alpha\bar{\alpha}}^{\gamma} \phi_\gamma.$$

If for some $\alpha, \ \beta$

$$\sum_\gamma N_{\alpha\beta}^{\gamma} > 1$$

then there are degenerate states for corresponding types of quasiparticles. The multiplicities can be calculated from the $N_{\alpha\beta}^{\gamma}$s --- return to this.

Dragging quasiparticles

To define statistics, we want to drag quasiparticles around some paths adiabatically (i.e. slowly), so that at the final time their positions are the same as at the start, up to a permutation among the qptcles of each type.

This produces a Berry phase or unitary matrix times the original state.

Should do this with the qptcles **well-separated** throughout.

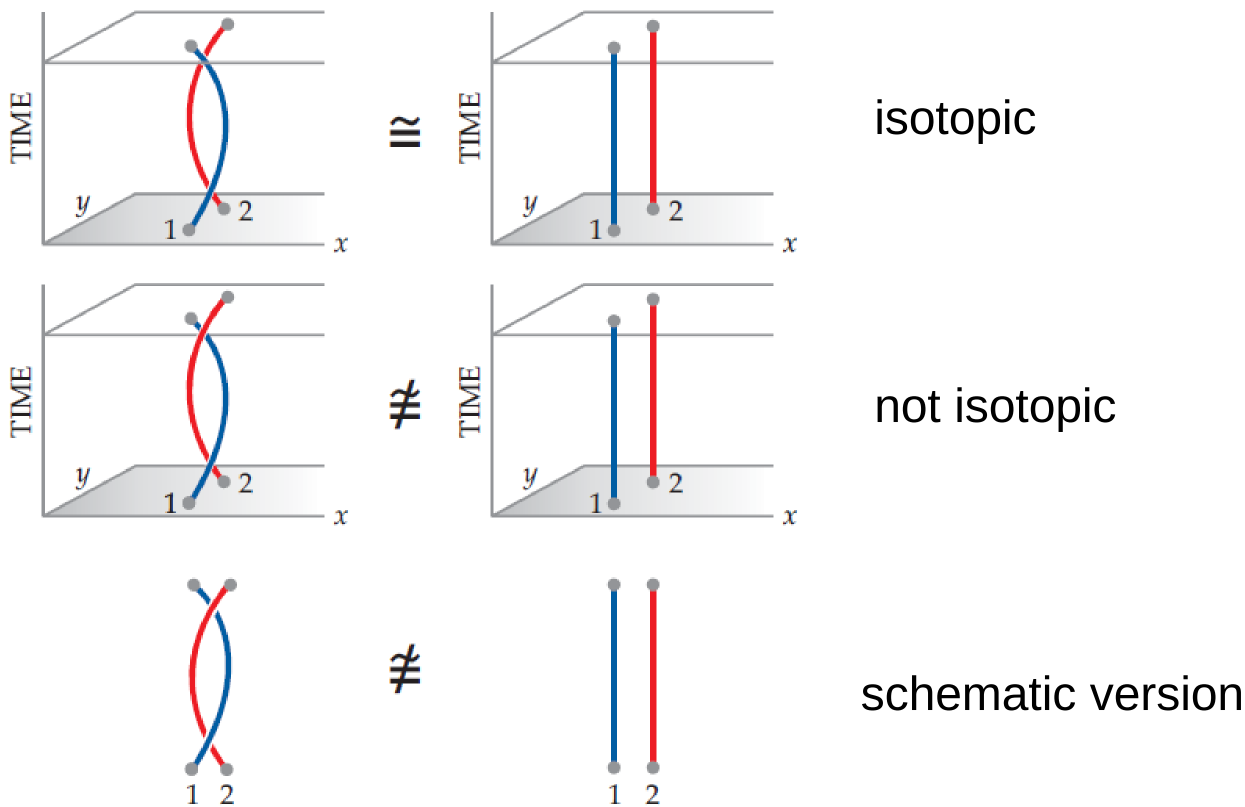
The result **can** depend on the path(s) taken by the quasiparticles. But the change under a small change of path can only be by a phase factor, even in the degenerate case, and the phase does not depend on which other qptcles are present:
Hence this part has form

$$\exp i \sum_j \oint_{C_j} dx^\mu \, \lambda_{\mu,\alpha_j}(\mathbf{x}(s), t(s))$$

where path (link) has connected components C_j and type α_j runs around C_j; $\lambda_{\mu,\alpha}(\mathbf{x}, t)$ is a real one-form function of position in spacetime, independent of the qptcles present; it is due to local coupling to background.

The point is such a change in path is a local operation, cannot detect other qptcles or change the degenerate state, hence this is only possibility

Remaining effect of exchange depends only on the “isotopy class” of the exchange, and there are distinct isotopy classes of exchanges, which cannot be deformed into each other:

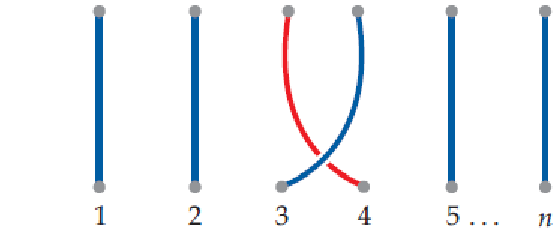


Basic case is when all quasiparticles are same type.

Exchanges of pair give operations $\tau_{i,i+1}$ which (with their inverses) generate a group:

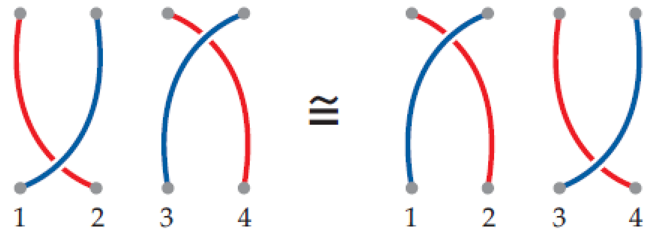
(colors only to ease visualization)

generators, e.g.



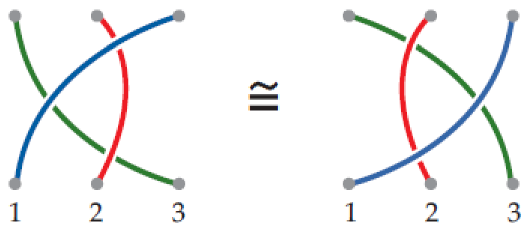
τ_{34}

obey relations, e.g.



$\tau_{34}\tau_{12} = \tau_{12}\tau_{34}$

and



$\tau_{23}\tau_{12}\tau_{23} = \tau_{12}\tau_{23}\tau_{12}$

This group is Emil Artin's **braid group** $\mathcal{B}_n$

The remaining effect of such an exchange on the degenerate states is a unitary representation matrix $\tau_{i,i+1}$, and these matrices must obey these relations

E.g. for Laughlin quasiparticles, matrices become scalars $\tau_{j,j+1} = e^{i\theta}$ --- anyons

In case with degenerate states, we arrive at notion of **nonabelian statistics**

Consistency of the whole set-up was explored in mathematical works
Doplicher, Roberts; Frohlich; … ~ 1987--90

In addition to braid group relations, there are similar relations for "mutual statistics", effect of qptcle of one type encircling one of another.
Further, fusion and braiding must be consistent: braiding one of type δ around $\alpha, \ \beta$ before fusing them must agree with fusing first, then braiding.
(Some additional structures also . . .)

The **same** structure was found in rational conformal field theory Moore,Seiberg 1989
and in quantum groups

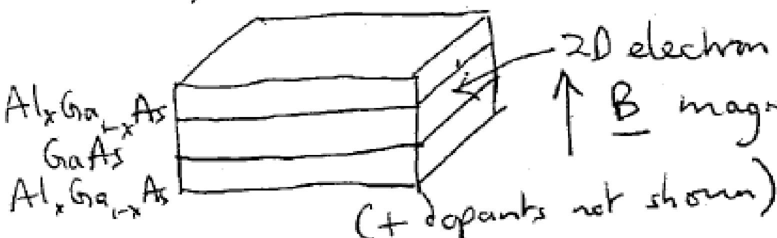
These developments culminated in definition of a mathematical structure known as a **modular tensor category** Reshitikhin, Turaev 1990

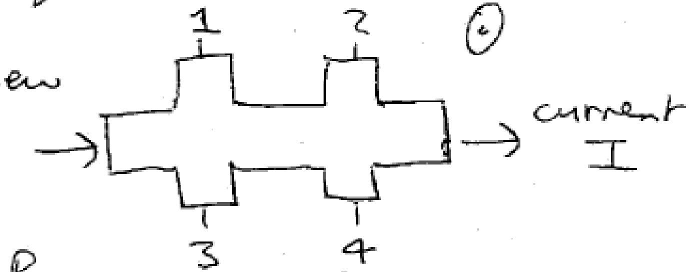
and connected with (quantum) Chern-Simons gauge theories Witten 1989

All this is also connected with topological (isotopy) invariants of knots, links, Jones 1984, . .
and 3-manifolds.

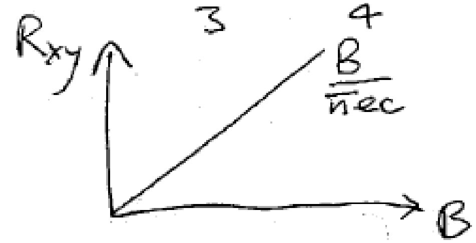
Intro QH effect

Typically is semiconductor heterostructures/quantum wells



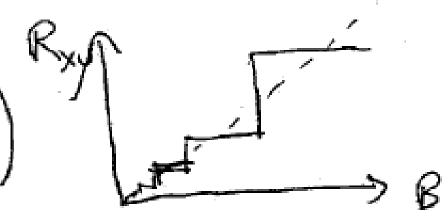


Resistance: $R_{xx} = \frac{V_{12}}{I}$, $R_{xy} = \frac{V_{13}}{I}$



<u>classical</u> Hall effect

Density $\bar{n} = \frac{N}{A} \sim 10^{11}\,cm^{-2}$ — held fixed



<u>Quantized</u> <u>Hall</u> <u>Effect</u>

Classically (for clean system) $R_{xy} = \rho_{xy} = \frac{B}{\bar{n}ec}$ $\left(= \frac{B}{\bar{n}}\frac{e}{hc}\cdot\frac{h}{e^2}\right)$ ← $\simeq 26\,k\Omega$

<u>and</u> $R_{xx} = 0$ on "plateau"

Quantized Hall effect (low T, ~ few K)

$$\boxed{\sigma_{xy} = \frac{1}{\rho_{xy}} = \frac{\nu e^2}{h}}$$

↑ $\rho_{xx} = 0$

$$\boxed{\nu = \frac{\bar{n}}{B}\frac{hc}{e}}$$

dimensionless density or <u>filling factor</u>

— Expt observation: σ_{xy} quantized with

$$\boxed{\nu = \text{integer or rational}}$$

IQHE: von Klitzing et al, 1980

FQHE: Stormer, Tsui, Gossard 1982

Landau levels

Single charged ptcle in a magnetic field $(\nabla \times \underline{A} = \underline{B})$

$$H_1 = \frac{1}{2m_e}\left(-i\hbar\nabla - \frac{e}{c}\underline{A}\right)^2$$

Energy eigenvalues

$$E_n = (n+\tfrac{1}{2})\hbar\omega_c \,, \quad n = 0,1,2,\ldots$$

Larmor/cyclotron freq: $\omega_c = \dfrac{eB}{m_e c} > 0$

In "symmetric gauge", $\underline{A} = \frac{1}{2}\underline{r}\times\underline{B}$, one set of eigenfunctions for $\underline{n=0}$ (lowest Landau level or LLL) is

$$u_m(z) = \frac{z^m e^{-\frac{1}{4}|z|^2}}{\sqrt{2\pi 2^m m!}}$$

(we set $l_B^2 = \dfrac{\hbar c}{eB} = 1$)

$z = x+iy$
$m = 0,1,2,\ldots$

$u_m(z)$ peaked at $|z| = \sqrt{2m}$ $\Rightarrow$ no. of states in LLL per unit area $= \dfrac{1}{2\pi}$ //

I.e. one state per LL per area covered by one flux quantum $\Phi_0 = \dfrac{hc}{e}$

(Same for higher LLs, of course $u_{m,n>0}$ differs.)

Many non-interacting fermions (electrons), ground state $\Rightarrow$ occupy lowest energy levels. If occupy $n = 0, 1, \ldots, \nu-1$ (ie ν levels), the density is

$$\bar{n} = \frac{\nu}{2\pi}$$

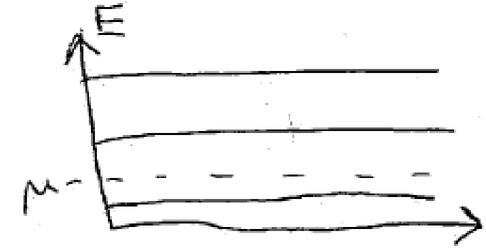
and is uniform. Here $\nu = 2\pi\bar{n} = \frac{\bar{n}}{B}\frac{h}{e}$

I.e. ν is fractional no. of levels occupied.

For this trans. inv. Ham, can show

$$\sigma_{xy} = \frac{\nu e^2}{h}, \quad \sigma_{xx} = \rho_{xx} = 0$$

Values $\nu = 0, 1, 2, \ldots$ are <u>special</u> because creating/destroying an electron costs energy (ie change in $H - \mu N$) $\geqslant 0$. — <u>energy gap in spectrum</u>



Really as fn of $\bar{n}$, μ jumps at ν=integer from $(\nu-\frac{1}{2})\hbar\omega_c$ to $(\nu+\frac{1}{2})$

So $\frac{d\mu}{d\bar{n}}$ has δ fn spikes at these values — <u>incompressibility</u>.

$$\kappa = \frac{d\bar{n}}{d\mu}$$

($\kappa = \infty$ for $\nu \neq$ integer.)

Trial wavefunctions in FQHE

General wavefunction for many particles in the LLL has the form

$$\Psi(z_1, \ldots, z_N) = f(z_1, \ldots, z_N) e^{-\frac{1}{4}\sum_j |z_j|^2}$$

where f is holomorphic in each z_j, and symmetric (bosons), antisymmetric (fermions). Use such functions, assuming interactions are weak.

Laughlin proposed a trial wavefunction to explain observed 1/3 FQHE state:

$$\Psi_{\text{Laughlin}}(\{z_j\}) = \prod_{i<j} (z_i - z_j)^Q e^{-\frac{1}{4}\sum_j |z_j|^2}$$

Laughlin 1983

where Q is a positive integer; Q is odd for fermions, even for bosons.

Highest power of any z_j is

$$m_{\max} = N_\phi = Q(N-1)$$

so if density particle density is uniform inside radius $\sqrt{2m_{\max}}$, then the filling factor as $N \to \infty$ is

$$\nu = 2\pi\overline{n} = \lim_{N\to\infty} \frac{N}{N_\phi} = \frac{1}{Q}.$$

To study density etc, Laughlin used plasma mapping:

$$|\Psi_{\text{Laughlin}}|^2 = \exp Q \left[\sum_{i<j} \ln |z_i - z_j|^2 - \frac{1}{2Q} \sum_j |z_j|^2 \right]$$

is the Boltzmann weight for a 2D plasma of particles of charge 1, with uniform background charge density $-1/(2\pi Q)$, and temperature 1/Q.

This plasma is in a screening phase if $Q < 70$. In that case, the density of particles must locally cancel the background density, so it is uniform inside the edge of the drop.

For Q=1, this can also be seen directly, as the wavefunction is a Slater determinant (the Vandermonde determinant, times the Gaussian), representing fermions filling the LLL out to $m = m_{\text{max}}$.

There is a “special” interaction Hamiltonian for which the Laughlin state is the exact ground state Haldane 1983

Fractionally-charged quasiparticles in the Laughlin state

A LLL state with one quasihole:

$$\Psi_{\text{1qhole}} = \prod_j (z_j - w)\Psi_{\text{Laughlin}}$$

--- particles avoid w, hence there is a “hole” in the density

--- in the plasma, there is an “impurity” fixed at w, with charge 1/Q. Hence by screening, the deficiency in particle number there is **exactly** 1/Q (located inside of about a screening length away).

--- more quasiholes similarly

--- can also construct states with “quasielectrons”, with fractional charge added, but no unique nice way to write a function.

These excitations are quasiparticles in our sense. Away from w, state resembles ground state. Note Q quasiholes is equivalent to removing a particle --- i.e. to a “real” hole, created by a local operator. Hence there are just Q types of quasiparticles.

Nonzero (expectation of) repulsive interaction energy implies that these quasiparticles cost energy to create, so system has gap, we have a top phase, and get FQHE.

Conformal field theory construction

Moore, NR 1991

Obtain trial QH wavefunctions from a 2D CFT, a single scalar U(1) theory times another one.

Thus write

$$\Psi(z_1, \ldots, z_N) = \langle 0|\mathcal{O} \prod_{i=1}^{N} a(z_i)|0\rangle$$

Chiral correlator or ``conformal block''

where

$$a(z) \;=\; e^{i\varphi(z)/\sqrt{\nu}}\psi(z),$$

$$\mathcal{O} = e^{-i\frac{\sqrt{\nu}}{2\pi}\int d^2z\,\varphi(z)}$$ represents background density

and ψ has Abelian fusion rules (it is a simple current)---e.g. identity operator

Example: $\psi = I$, drop second CFT. Expand exponentials and use

$$\langle\varphi(z)\varphi(0)\rangle = -\ln z$$

with Wick's Theorem, we obtain the Laughlin wavefunction

$$\prod_{i=1}^{N}(z_i - z_j)^{1/\nu} e^{-\frac{1}{4}\sum_j |z_j|^2}$$

(up to a singular gauge transformation) where we should put $\nu = 1/Q$ to be in LLL.
$\nu = 1$ is free chiral Dirac fermion, bosonized.

To get quasihole states, introduce additional primary fields in correlator:

$$\tau(w) = e^{iq_{\rm qh}\varphi(w)/\sqrt{\nu}}\sigma(w)$$

where $q_{\rm qh}$ is a constant (minus the charge of the qhole), and $\sigma(w)$ is another primary field in the ψ CFT. (Really "chiral vertex operators".)

Thus in Laughlin example, $q_{\rm qh} = 1/Q$ for the basic quasihole.

"Simple current" condition on ψ: we require that ψ have Abelian fusion rules, and so do its operator products, that is if $\psi = \psi_1$, then in an operator product expansion (ope)

$$\psi_1(z)\psi_1(0) \sim \frac{1}{z^{2h_1-h_2}}\psi_2(0) + \ldots$$

and so on, so generally

$$\psi_k(z)\psi_l(0) \sim \frac{c_{k,l}}{z^{h_k+h_l-h_{k+l}}}\psi_{k+l}(0) + \ldots$$

or as fusion rules in the CFT

$$\psi_k \times \psi_l = \psi_{k+l}$$

and further that $\psi_1 \times \sigma = \sigma^*$ (a single term), and so on. This ensures that the wavefunctions obtained, with suitable choices of ν and $q_{\rm qh}$, are single-valued, in fact polynomial, functions of z_j, times the Gaussian.

Note σ can still have non-Abelian fusion rules!

In general, this construction yields the field $a(z) = e^{i\varphi(z)/\sqrt{\nu}}\psi(z)$, which generates a chiral algebra (including the U(1) current $-i\partial\varphi$ and total stress tensor $T(z)$).

Often, this chiral algebra has a finite set of primary fields, one of which is $\tau(z)$, which can have non-Abelian properties. Thus we have a CFT, possibly rational.

We will see that the above construction of wavefunctions **may** give a top phase, which can have quasiparticles with non-Abelian statistics.

(In many cases, there **is** a “special” or parent Hamiltonian for which the trial ground and quasihole states are zero-energy eigenstates, **but** in general we can’t prove there is a gap in bulk spectrum, and expect not in certain cases --- those in which CFT is not unitary or not rational. I’ll mention that later.)

Example: Moore-Read quantum Hall phase

Moore, NR 1991

A trial *N*-particle wavefunction:

$$\Psi_{\mathrm{MR}}(z_1,\ldots,z_N) = \mathcal{A}\left(\frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\cdots\right)\prod_{i<j}(z_i-z_j)^Q \cdot e^{-\frac{1}{4}\sum_i |z_i|^2}$$

where Q = 0, 1, 2, . . . is **even** for fermions, **odd** for bosons.

$$\mathcal{A}\left(\frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\cdots\right) = \mathrm{Pf}\left(\frac{1}{z_i-z_j}\right)$$

is a **Pfaffian**

(p-ip BCS state of spinless fermions.)

Filling factor (dimensionless density of particles) $\nu = 1/Q$

5/2

CFT construction of MR state: take ψ to be a free chiral Majorana field, correlator

$$\langle \psi(z)\psi(0)\rangle = \frac{1}{z}.$$

Evaluating using Wick's Theorem gives the Pfaffian wavefunction.

Majorana CFT is the Ising critical theory. It admits a "spin field" $\sigma(w)$ with ope

$$\psi(z)\sigma(0) \sim \frac{1}{z^{1/2}}\sigma(0) + \ldots$$

Then using the construction, wavefunction with two quasiholes is

$$\mathrm{Pf}\left(\frac{(z_i - w_1)(z_j - w_2) + (1 \leftrightarrow 2)}{z_i - z_j}\right)$$

times Laughlin factor. Each quasihole is a kind of vortex or half flux-quantum --- must have **even** number of qholes (with fixed bc's), exactly as in view as paired state.

More than two qholes is more complicated, though z-dependence can be found.
Multiplicity of qhole states (with fixed bc's), of

$$2^{n/2-1}$$

for *n* qholes at fixed positions, sufficiently large *N*

--- fewer than 2 per qhole: non-local storage of information

--- these arise because the relevant conformal blocks form a space of distinct functions of z's. $\sigma(w)$ is a chiral vertex operator, not ordinary operator

Fusion rules

For *Q*=1, three quasiparticle types: $\phi_\alpha = \mathbf{1}$, ψ, or σ ,
charges 0, 0, ½ (mod 1)

“Fusion rules” (same as in Ising RCFT):

$$\begin{aligned} \psi \times \psi &= \mathbf{1}, \\ \psi \times \sigma &= \sigma, \\ \sigma \times \sigma &= \mathbf{1} + \psi. \end{aligned}$$

Can understand in terms of Majorana zero modes on vortices in p-ip paired state!
NR, Green 2000

Then repeated multiplication of σ s gives, e.g. *n* = 4:

$$\sigma \times (\sigma \times (\sigma \times \sigma))) = 2.\mathbf{1} + 2\psi$$

For *N* even, we must finish up with $\mathbf{1}$ to satisfy bc’s, so (similar for general *n*)

$$\text{multiplicity} = 2 = 2^{n/2-1}$$

For general Q, have 3Q types, but same multiplicities for n qholes containing σ

Parafermion states

NR, Rezayi 1999

Extend to case $\mathbf{Z}_k$ parafermions $(k = 1, 2, \ldots)$: CFT as above with

$$\psi_l,\ l = 0, \ldots, k-1 \qquad \text{i.e.} \quad \psi_0 = \psi_k = I$$

with conformal weights and central charge

$$h_l = l(k-l)/k,\ c = 2(k-1)/(k+2)$$

Zamolodchikov, Fateev 1985

In MR construction, put $\psi = \psi_1$. Then for single-valued ground state

$$\nu^{-1} = M + 2/k \qquad (M = 0, 1, 2, \ldots)$$

or

$$\nu = \frac{k}{Mk+2}$$

Special cases: Laughlin is $k = 1;$ MR is $k = 2.$ M is even (bosons), odd (fermions)

For $M = 0$, the CFT, and the topological phase obtained, is $\mathrm{SU}(2)_k$.
For $M = 1$, the chiral algebra is N=2 superconformal algebra, CFTs are superconformal minimal models.

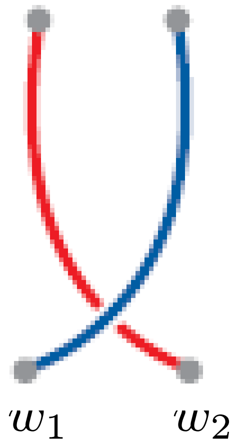
Zamolodchikov, Fateev 1986

$M = 1, k = 3$ may be relevant for observed $\nu = 12/5$!
(“Fibonacci anyons”)

Statistics of Laughlin quasiholes

Arovas, Schrieffer, Wilczek, 1984

Adiabatically exchange locations, calculate Berry phase:



Start with one quasihole:

$$\Psi_{\mathrm{Laugh},+w} = \prod_j (z_j - w)\Psi_{\mathrm{Laughlin}}$$

Berry (or adiabatic) connection (or vector potential) on parameter space:

Suppose $|\Psi(s)\rangle$ is a **normalized** non-degenerate state for each $s,\ s \in [0,1]$ and

$$|\Psi(1)\rangle = |\Psi(0)\rangle.$$

Let $$\frac{d\gamma}{ds} = i\langle\Psi(s)|\frac{d\Psi}{ds}(s)\rangle$$ (Berry connection).

Then Berry phase picked up by the state after transport from 0 to 1 is

$$\exp i \oint ds\, \frac{d\gamma}{ds}.$$

For one qhole,

$$\begin{aligned}\frac{d\Psi_{\mathrm{Laugh},+w}}{ds} &= -\frac{dw}{ds}\sum_i \frac{1}{z_i - w}\Psi_{\mathrm{Laugh},+w} \\ &= -\frac{dw}{ds}\int d^2z'\,\frac{n(z')}{z'-w}\Psi_{\mathrm{Laugh},+w}\end{aligned}$$

where $n(z) = \sum_i \delta(z_i - z)$ is the particle number density . So

$$\frac{d\gamma}{ds} = -i\int d^2z'\,\frac{1}{2}\left(\frac{1}{z'-w}\frac{dw}{ds} - \frac{1}{\bar{z}'-\bar{w}}\frac{d\bar{w}}{ds}\right)\langle n(z')\rangle_{+w}$$

and using rotational symmetry about w,

$$\begin{aligned}\gamma(1)-\gamma(0) &= -i\oint\int d^2z'\,\frac{1}{2}\left(\frac{dw}{z'-w} - \frac{d\bar{w}}{\bar{z}'-\bar{w}}\right)\langle n(z')\rangle \\ &= -2\pi\int_{\mathrm{interior}} d^2z'\,\langle n(z')\rangle\end{aligned}$$

(used Cauchy’s Theorem). Here $\langle n(z)\rangle = \bar{n}$ inside the droplet.

--- like Aharonov-Bohm phase; shows path-dependent part mentioned earlier.

Now in the case where path encloses a second quasihole, there
Is an additional Berry phase

$$\Delta\gamma = \frac{2\pi}{Q}$$

due to missing particle number 1/Q.

Similarly, if the path is exchange of two quasiholes, isotopy-invariant part is half of above, so

$$\Delta\gamma = \frac{\pi}{Q}$$

--- fractional statistics if Q>1!   (Fermi statistics if Q=1, as should be.)

--- for two quasiholes of charges $q_1/Q,\ q_2/Q\ (q_1, q_2 \in \mathbf{N})$, one gets

$$\Delta\gamma = \frac{\pi q_1 q_2}{Q}$$

so for $q_1 = q_2 = Q$, we have bosons if Q is even, fermions if Q is odd (as should be).

Different view of statistics in Laughlin case

Halperin (1984) rewrote Laughlin functions as

$$\Psi(w_1,\ldots,w_n;z_1,\ldots,z_N) = \prod_{k<l}(w_k-w_l)^{1/Q}\cdot\prod_{i,k}(z_i-w_k)\cdot\prod_{i<j}(z_i-z_j)^Q\cdot \times e^{-\frac{1}{4Q}\sum_k|w_k|^2-\frac{1}{4}\sum_i|z_i|^2}$$

He argued these are normalized independently of the w's.

That is because the 2D Coulomb interaction between the impurities is included.
The free energy of plasma becomes independent of the separation of the qholes.

We can also see that if we exchange (by analytic continuation or “monodromy”) two qholes along a path not enclosing others, the phase of the function changes by

$$e^{i\pi/Q}$$

and he viewed this as fractional statistics.

This phase agrees exactly with the adiabatic calculation.

Note: if we do a Berry phase calculation using basis states that are not single-valued on going around the loop, we must make the gauge transformation back to original gauge at end, as well as calculating the exp of Berry connection: the (**gauge-invariant)** Berry phase or **holonomy** is

$$\mathcal{B} = \mathcal{M}\mathcal{P}\exp i\int_C \mathcal{A}$$

$\mathcal{M}$ is the “monodromy” (change in function) or “transition function” to get back to original gauge choice after going around loop. $\mathcal{P}$ Is path-ordering, needed when connection is a matrix (non-Abelian).

Apart from normalization, the Halperin functions agree with the original ones up to a “singular gauge transformation”. Hence in this gauge it must be that statistics comes from monodromy only, and the statistics part of the Berry connection vanishes.

Using w as parameter, and in complex components, the Berry connection is

$$
\begin{aligned}
A_{w,l}(w) &= i\left\langle \Psi(w) \middle| \frac{\partial \Psi(w)}{\partial w_l} \right\rangle \\
A_{\overline{w},l}(w) &= i\left\langle \Psi(w) \middle| \frac{\partial \Psi(w)}{\partial \overline{w_l}} \right\rangle
\end{aligned}
$$

Suppose $|\Psi(w)\rangle$ were holomorphic in w as well as normalized. Then

$$
A_{w,l}(w) = i\frac{\partial}{\partial w_l}\left\langle \Psi(w) \middle| \Psi(w) \right\rangle = 0
$$

The Halperin functions fail to be holomorphic only because of interaction with background. Thus the part of the connection that contributes to statistics does indeed vanish!

The Halperin form is exactly what is obtained from the CFT construction.

Thus we were led to the **conjecture**: Moore, NR 1991

Apart from the background field contribution, the statistics of the quasiparticles In the MR construction is exactly given by the monodromy of the functions:

holonomy = monodromy

i.e. the statistics part of Berry connection vanishes. That means the qholes In MR and RR states have non-Abelian statistics which can be read off from CFT.

To show this is true, it will suffice if the qhole functions can be shown to be normalized independent of w.

The norm-square of one of our functions is

$$\int \prod_i d^2 z_i |\langle \mathcal{O} \prod_{i=1}^{N} a(z_i) \cdot \prod_k \tau(w_k) \rangle|^2$$

which is an integral of a correlator in a **non-chiral** CFT.

Let's write it “grand-canonically” as

$$\left\langle \bar{\mathcal{O}} \mathcal{O} e^{\lambda \int d^2 z \, \bar{a}(\bar{z}) a(z)} \prod_k \bar{\tau}(\bar{w}_k) \tau(w_k) \right\rangle$$

then the term of order λ^N. Then we realize this is a correlator of $\bar{\tau}\tau$ s in a non-chiral CFT perturbed by

$$\lambda \int d^2 z \, \bar{a} a - i \frac{\sqrt{\nu}}{2\pi} \int d^2 z \, [\varphi(z) + \bar{\varphi}(\bar{z})]$$

in the action. (Note equivalence of canonical and grand canonical ensembles.)

Now **if the perturbation causes an RG flow to a massive 2D phase**, then correlations of local operators such as $\bar{\tau}\tau$ will become constant (or zero) at large separations. Nayak, Wilczek 1996; NR 2009

Apart from dealing with matrix structure in non-Abelian case, this is essentially what we want.

This is a generalization of the screening in the plasma for the Laughlin state.

In this case, we obtain essentially all of the properties of a MTC describing a topological phase. (Quasiparticle spin?)

Does the hypothesis of "generalized screening" actually hold in our wavefunctions?

--- numerical evidence suggests it does in e.g. simplest cases Bonderson, Gurarie, Nayak

--- when CFT is non-unitary, the results are inconsistent with the consequences of unitarity in the 2+1 TQFT --- we require a "unitary" MTC Turaev 1991
Similarly when not rational

In such cases the screening should break down. NR 2009
This has been confirmed in some examples in recent numerical studies.

Quantum Computation

Desired: perform operations to make a system reach a desired quantum state starting from a simple initial state; read out using measurements.

Issues:

1) Must use only a simple set of operations, applied in a sequence; operations chosen so that any desired state can be closely approximated ("universal" set);

2) Decoherence of the quantum state due to coupling to noisy environment ---errors.

Usual "circuit" model:

---system is collection of 2-state systems or "qubits"

---"gate" operations on one or two qubits at a time---can be universal

---susceptible to errors; additional qubits can be used to provide error correction

Topological model for QC

Kitaev 1997; Freedman et al 2002

--- system is a topological phase of matter; relevant low-energy states are degenerate states of some non-Abelian quasiparticles
--- exchanges of quasiparticles provide unitary operations on these states

Advantages:
--- these states decouple from environmental noise

Disadvantages:
--- existence of non-Abelian quasiparticles & braiding properties have not yet been shown; manipulation may be difficult; other approaches are further along

Some top phases provide non-Abelian quasiparticles that can be used for "universal" QC, e.g. Read-Rezayi FQH states $k \neq 1, 2, 4$: TQC model has same computational power as circuit QC model.

Freedman, Larsen, Wang 2002

But the MR/Ising/Majorana-zero-mode example is not universal!
Would have to be supplemented by non-topological methods