Quantized transport, Berry curvature, and topological invariants in topological phases

(including Hall viscosity)

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Hall conductivity as Berry curvature

Viscosity, Hall viscosity

Calculation of Hall viscosity for fractional quantum Hall states and paired states

Relation with orbital spin density; shift

Effective field theory, quantization

Thermal Hall conductivity

Top. central charge as Berry curvature: spatially-varying changes in metric

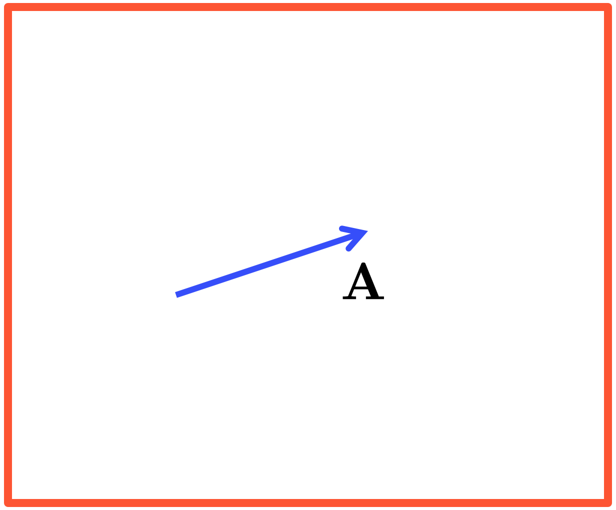
Central charge, issue of $\overline{s^2}$ invariant

Hall conductivity as Chern number or Berry curvature

Consider system of particles in a ground state with an energy **gap** for **bulk** excitations --- a **topological** phase of matter

Assume particles are charged (e.g. electrons), apply vector potential A_μ.
Time derivative is electric field; *weak* electric field is *slowly* -varying A_μ (uniform in space):

$$E_\mu = -\frac{dA_\mu}{dt}$$



periodic bcs/torus (no edges)

Square, side = L

Spatially-uniform A_μ can also be viewed as **twist in boundary condition**.

Low frequency current response can be found from **quantum adiabatic theorem** and is due to **Hall conductivity** only

Niu and Thouless (1985);
Avron and Seiler (1985)

It can be expressed as Berry "curvature" (field strength) (assume non-deg ground state):

Berry connection $\mathcal{A}_\mu = i \left\langle \varphi \middle| \frac{\partial}{\partial A_\mu} \varphi \right\rangle$, curvature $\mathcal{F}_{\mu\nu} = \frac{\partial \mathcal{A}_\nu}{\partial A_\mu} - \frac{\partial \mathcal{A}_\mu}{\partial A_\nu}$.

Or, Berry phase $\oint dA_\mu \, \mathcal{A}_\mu$ for a closed path in space of uniform A_μ

Then

$$\sigma_{xy} = \mathcal{F}_{xy}/L^2$$

which is also equivalent to the usual Kubo formula.

[However usual Kubo formula involves a sum over all excited energy eigenstates---can be difficult to evaluate in practice.]

This is a **general relation**: get response functions to external fields (Kubo); antisymmetric (i.e. Hall-like) part is a Berry curvature; response function gives transport coeff

Adiabatic response and Berry phase

Avron and Seiler (1985)

Suppose Hamiltonian $H(\lambda)$ depends on parameters $\lambda = \{\lambda_\mu\} \quad (\mu = 1, \ldots, n)$

and that $\widehat{I}_\mu(\lambda) = -\frac{\partial H}{\partial \lambda_\mu}$ is some "current" operator.

Also $H(\lambda)|\varphi(\lambda)\rangle = 0$ for each value of λ (ignore "persistent currents")
and $H(\lambda)$ is gapped.

Then as $\dot{\lambda} \to 0$, using quantum adiabatic theorem,

$$I_\mu(\lambda) = \langle \widehat{I}_\mu(\lambda) \rangle = \sum_\nu F_{\mu\nu}(\lambda) \dot{\lambda}_\nu$$

where $A_\mu = i\langle \varphi | \partial_\mu \varphi \rangle$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ are Berry or adiabatic "connection" and "curvature". Or $\oint_C A$ is a Berry phase.

For us, $\mu = (a, b)$ and $\Lambda = e^\lambda$ as matrices, then

$$\eta_{abcd} = -\frac{1}{L^d} g^{be} g^{df} F_{ae,cf}$$

Avron, Seiler, and Zograf (1995)

(Analog of "Chern number" approach to quantized Hall conductivity.)

Laughlin (1981)
Thouless et al (1982)
Niu and Thouless (1985)
Avron and Seiler (1985)

Quantization from topological invariants:

$$2\pi\overline{\sigma_{xy}} = \frac{1}{2\pi}\int_T d^2A\,\mathcal{F}_{xy} = c_1(T)$$

i.e. average of Hall conductivity over the torus T of distinct twisted boundary conditions,

$$\mathbf{A} \in [0, 2\pi/L]^2 = T,$$

is equal to the **Chern number** $c_1(T)$ of torus, which is necessarily **integer**.

But can expect/hope (without disorder?) that Berry curvature itself gives
the **quantized** Hall conductivity---averaging is not needed in those cases
(as in TKNN in fact).

Viscosity

---transport of **momentum**---total momentum must be conserved
i.e. need translation inv. Momentum density is $\mathbf{g}(\mathbf{x}) = \frac{1}{2}\sum_j \{\boldsymbol{\pi}_j, \delta(\mathbf{x}_j - \mathbf{x})\}$

---"current" of momentum is the **stress tensor** $\tau_{ab}(\mathbf{x}, t)$ (inc momentum flux), obeys

$$\partial g_b/\partial t + \partial_a \tau_{ab} = \frac{B}{m_p}\epsilon_{bc} g_c$$ (Lorentz force)

---spatial metric couples to stress, in place of vector potential---"geometric".
E.g. non-interacting particles:

$$H_0 = -\frac{1}{2m_p}\sum_j \sum_{ab} g^{ab} D_{ja} D_{jb}, \qquad \pi_{ja} = -iD_{ja} = -i(\partial_{ja} - iA_a(\mathbf{x}_j))$$

Stress tensor is functional derivative wrt spatial metric:

$$\tau_{ab} = -2\frac{\delta H}{\delta g_{ab}}$$

Response of stress to further change in metric is analog of response of current to vector potential (giving conductivity)---here, giving viscosity: Kubo formula.

Viscosity is a fourth rank tensor

Chapman and Cowling
Landau and Lifshitz, “Elasticity”

Now consider **expectation** of stress τ_{ab} in a state of matter.

In a solid, we have for expectation of stress

$$\tau_{ab} = -\lambda_{abef} u_{ef} - \eta_{abef} \partial u_{ef} / \partial t + \ldots,$$

where the local strain is $u_{ab} = \frac{1}{2} \left(\partial_b u_a + \partial_a u_b \right)$,
u_a is displacement field, λ_{abef} are elastic coefficients (moduli),
and η_{abef} is the viscosity tensor.

If rotational invariance holds, τ_{ab} and u_{ab} are both symmetric tensors.
In a (rot. inv.) fluid with local velocity $\mathbf{v}$, elastic part becomes $p\delta_{ab}$ (pressure), we replace

$$\partial u_{ab} / \partial t = \frac{1}{2} \left(\partial_b v_a + \partial_a v_b \right)$$

and also add $m_p \overline{n} v_a v_b$ (momentum flux) to stress tensor.

In Kubo point of view, replace $\partial u_{ab} / \partial t$ with $\partial g_{ab} / \partial t$ to calculate η_{abef}
---stress-stress response function.

Rate of loss of mechanical energy, or rate of entropy production, is

$$k_B T \left(\frac{\partial s}{\partial t} + \nabla \cdot \mathbf{j}_s \right) = \eta_{abef} \frac{\partial u_{ab}}{\partial t} \frac{\partial u_{ef}}{\partial t} \geq 0$$

Symmetric and antisymmetric parts:

Avron, Seiler, and Zograf (1995)

$$\eta_{abef} = \eta^{(S)}_{abef} + \eta^{(A)}_{abef},$$

$$\begin{aligned} \eta^{(S)}_{abef} &= +\eta^{(S)}_{efab}, \\ \eta^{(A)}_{abef} &= -\eta^{(A)}_{efab}. \end{aligned}$$

---at zero frequency, only symmetric part gives dissipation; if rotation invariant, it reduces to usual *bulk* and *shear* viscosities, ζ and η^{sh}

---antisymmetric part vanishes if time reversal is a symmetry; if rotation invariant, it reduces to one number η^H in two dimensions (odd under reflections), i.e. $\eta^{(A)}_{abef} = \eta^H (\delta_{be}\epsilon_{af} - \delta_{af}\epsilon_{eb})$; none in higher dimensions

Hall viscosity η^H is analog of Hall conductivity

Thus Berry curvature for gapped ground state under **adiabatic variation** of the metric should give a Hall-like viscosity response --- the Hall viscosity $\eta^{(A)}_{abef}$.

Avron, Seiler, and Zograf (1995) used this to calculate it for a non-interacting filled lowest Landau level (LLL) (see also Levay (1995)).

Recent results for Hall viscosity:
----two classes of trial states: ---paired superfluids of fermions, e.g. p+ip
---trial quantum Hall states given by conformal blocks from a CFT (Moore-Read 1991)
---e.g. Laughlin, Moore-Read, Read-Rezayi states

$$\eta^H = \tfrac{1}{2}\bar{n}\bar{s}\hbar,$$

N.R. (2009)
N.R. and Rezayi (2011)

Spin per particle appeared from 1) **minus** ang momentum of Cooper pair/2;
2) spin (conformal weight) of particle fields in conformal block trial wavefunctions.
E.g. $\bar{s} = Q/2$ for Laughlin states. Avron et al result is equivalent to $\bar{s} = 1/2$ for filled LLL.

These are also related to the "shift": $N_\phi = \nu^{-1}N - \mathcal{S}$ on sphere

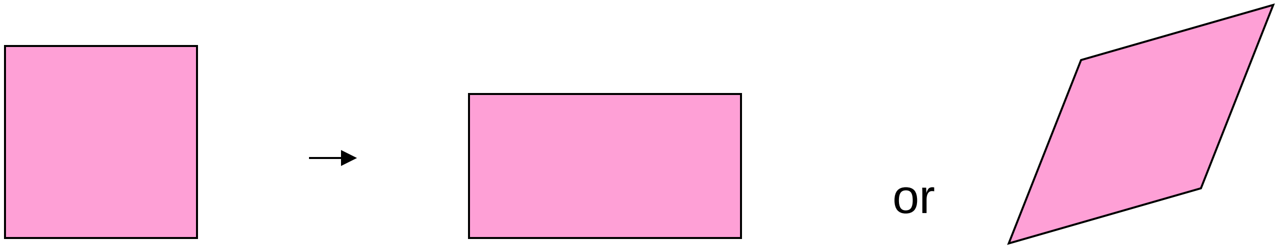
Wen and Zee (1992)

Wen-Zee idea: $\mathcal{S} = 2\bar{s}$. Find **same** mean orbital spin per particle enters η^H.
(Spin couples to curvature like charge to magnetic field.)

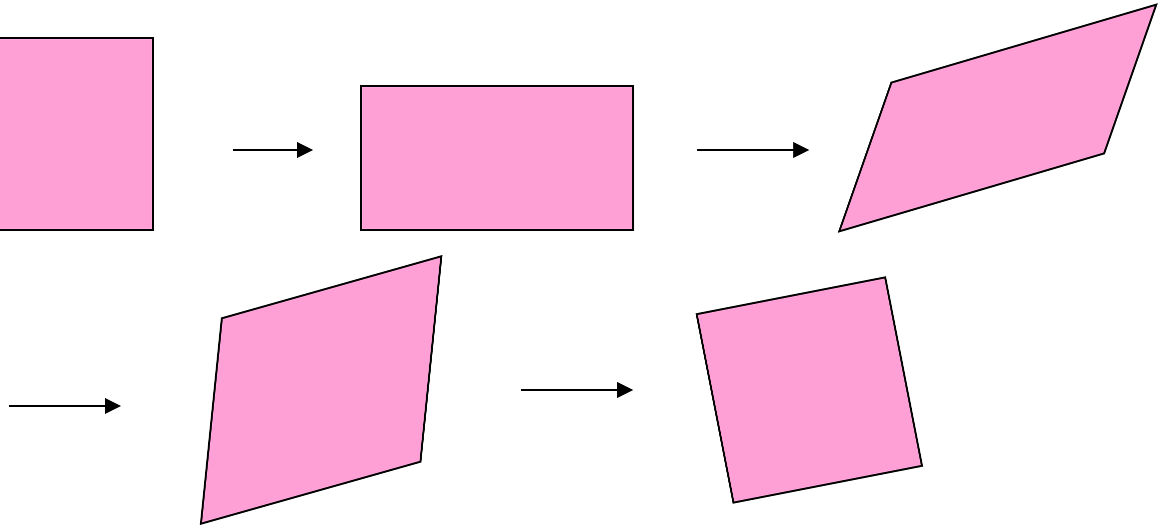
Geometry of shear

N.R. and Rezayi (2011)

Two independent area-preserving shears (equiv to change in metric) in two dimensions:



These moves don't commute: if undo in reverse order, we get a net rotation:

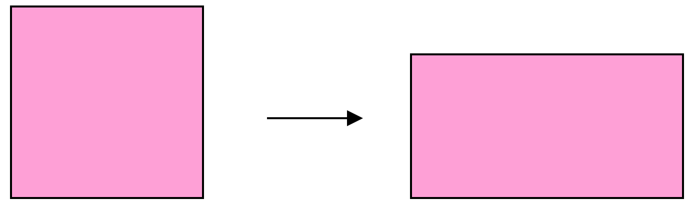


The transformations are described by $\mathrm{SL}(2,\mathbf{R})$ transformations of coordinates, i.e.

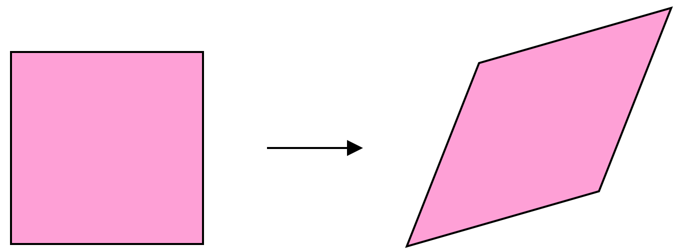
$$\mathbf{x} \to A\mathbf{x}, \qquad \det A = 1$$

Pure shears are symmetric matrices, e.g.:

$$\begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1-\varepsilon \end{pmatrix}:$$



$$\begin{pmatrix} 1 & \varepsilon' \\ \varepsilon' & 1 \end{pmatrix}:$$



and

$$\begin{pmatrix} 1 & \varepsilon' \\ \varepsilon' & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1-\varepsilon \end{pmatrix}^{-1} \begin{pmatrix} 1 & \varepsilon' \\ \varepsilon' & 1 \end{pmatrix} \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & 1-\varepsilon \end{pmatrix}$$
$$= \quad \mathbf{I} + \begin{pmatrix} 0 & -2\varepsilon\varepsilon' \\ 2\varepsilon\varepsilon' & 0 \end{pmatrix} + \mathcal{O}(\varepsilon^2, \varepsilon'^2).$$

is a small rotation. If the state has angular momentum (“spin”), then it picks up a phase related to the spin. **This is the Berry phase we calculate.**

Quantization of orbital spin $\overline{s}$?

Is $\overline{s}$ robust against pert of Ham? **Not** from “Chern number” argument
 ---return to this

Quantization as rational number:

$$N_\phi = \nu^{-1} N - \mathcal{S}.$$

If $\nu = P/Q$ and P, Q have no common factors, then $P\mathcal{S}$ is an integer.
Assuming $\mathcal{S} = 2\overline{s}$, it follows that

N.R. and Rezayi (2011)

$2P\overline{s}$ is integer,

which was not obvious initially. Note shift is robust **if** rot inv holds.

($P = 1$ for BCS paired states, in which $\nu = \infty$; $2\overline{s} = -\ell$ is an integer.)

Effective or “induced” field theory approach

Geometric definitions: background fields--- “frames” or “vielbeins” (Cartan):
vector fields $e^{\mu}_{\alpha};\ \mu = 0, 1, 2;\ \alpha = 0, 1, 2$ at each point in 2+1 dim spacetime
---define “time” and “space” directions

Dual vector fields e^{α}_{μ} . Spatial metric (used earlier) is

$$g_{\mu\nu} = \eta_{ab} e^{a}_{\mu} e^{b}_{\nu},\ a, b = 1, 2;\ \eta_{ab} = \delta_{ab}$$

Also $e^{\alpha=0}_{\mu=0} = 1 + \psi$, where ψ is Luttinger’s “thermal” potential (1965)

Arbitrariness of spatial rotation on internal indices $a,\ b$
---also have “spin connection”

$$\omega_{\mu} = \frac{1}{2} \epsilon_{a}^{\ b} \omega_{\mu\ \ b}^{\ a}$$

Wen and Zee (1992)

vector potential for internal spatial rotation

---allow background spacetime to have **curvature** and **torsion**

Bradlyn and NR (2014)

Integrate out matter and obtain effective or “induced” action.
For a gapped system, the induced action is **local** in the background fields in the bulk, and **invariant** under coordinate transformations, internal rotations, and U(1) gauge transformations

---corresponds to symmetries of translations in time and space, rotations, and U(1) gauge (**no** Lorentz or Galilean invariance), which . . .

---correspond in turn to conservation of energy, momentum, angular momentum, and particle number.

---induced action is a “**generating function**” that describes response of system to background fields, **even in trivial flat spacetime**

Formalism can be used to study general thermoelectric and stress responses

Bradlyn and NR (2014)

For **bulk** induced action, distinguish “locally-invariant” and “Chern-Simons-like” terms.
CS terms:

$$S_{\text{eff}} = \frac{\nu}{4\pi}\int d^3x\,\hat{\epsilon}^{\mu\nu\lambda}\left(A_\mu\partial_\nu A_\lambda + 2\overline{s}\omega_\mu\partial_\nu A_\lambda + \overline{s^2}\omega_\mu\partial_\nu\omega_\lambda\right)$$
$$+\frac{c}{96\pi}\int d^3x\,\hat{\epsilon}^{\mu\nu\lambda}\left(\Gamma^\rho_{\mu\sigma}\partial_\nu\Gamma^\sigma_{\lambda\rho} + \frac{2}{3}\Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\theta}\Gamma^\theta_{\lambda\rho}\right)$$

Wen and Zee (1992)
NR and Green (2000)

Each term is gauge invariant **only** up to boundary term for *at least one* type of gauge transformation

It follows that coefficients must be **robust** against perturbations of underlying Ham provided a phase boundary is not crossed

Final term is “gravitational Chern-Simons” term, actually equal to second Wen-Zee term in the bulk (up to boundary term):

$$S_{\text{GCS}} = -\frac{c}{48\pi}\int d^3x\,\hat{\epsilon}^{\mu\nu\lambda}\omega_\mu\partial_\nu\omega_\lambda$$

See also discussions in:
Gromov and Abanov
Can, Laskin, and Wiegmann
Cho, You, and Fradkin
---all 2014--2015

The Chern-Simons induced action determines **Berry curvature** for various adiabatic processes

E.g. Hall viscosity

---assume no “reduced torsion”, then can express

Bradlyn and NR (2014)

$$\omega_\lambda \equiv \frac{1}{2}\epsilon_a{}^b \omega_\lambda{}^a{}_b = \frac{1}{2}\epsilon^{ab} e_a^\mu e_b^\nu \partial_\nu g_{\mu\lambda} + \frac{1}{2}\epsilon^{ab} e_a^\mu \partial_\lambda e_\mu^b$$

For e_μ^α constant in space, space-space components varying in time (hence same for space metric)

Second derivative gives viscosity response: Hall viscosity
---implies both $\mathcal{S} = 2\bar{s}$ (Wen-Zee) and $\eta^H = \frac{1}{2}\bar{n}\bar{s}\hbar$,

N.R. and Goldberger (2009), unpublished
Hoyos and Son (2012)

also **robustness** against perturbations

Thermal Hall conductivity

From an edge state argument, the thermal Hall conductivity of gapped phase in 2+1 should be

$$\kappa^H = \frac{\pi T}{6} c$$

where $c = c_R - c_L$ is difference of central charges in edge (related to specific heat capacities) ---“topological” central charge

Kane and Fisher (1997)
NR and Green (2000)

Top. central charge is **same** as coeff of grav Chern-Simons term because of anomaly inflow effect for energy and momentum---Callan and Harvey (1985)

No thermal Hall cond from bulk in transport
(pure edge effect, unlike Hall conductivity)

M. Stone (2012)
Bradlyn and NR (2014)

but is there some Berry curvature
or bulk response function that gives the top central charge?

Must use **spatially-varying** metric variation

Bradlyn and NR (2015)

Berry curvature under condition that $A_i + \overline{s}\omega_i$ is held fixed

Results for conformal-block trial states

---we find Berry curvature with coeff c the same as in underlying CFT
(from gravitational anomaly in the underlying chiral CFT)

---in conjunction with known central charge from edge, implies that

$$\overline{s^2} = \overline{s}^2 = s^2$$

For a number of states considered, slightly earlier work by others had a mistake, corrected after our paper

Gromov, Cho, Fradkin et al (2015)

Conclusion

---Hall viscosity now somewhat well understood theoretically: $\eta^H = \frac{1}{2}\bar{n}\bar{s}\hbar$

---can get useful information from computation of Berry curvature to obtain

$$c_{\text{app}} = c - 12\nu(\overline{s^2} - \overline{s}^2)$$

---as coeff of grav CS should agree with edge theory, this will give

$$\text{Var}\, s \equiv \overline{s^2} - \overline{s}^2$$

which is not a well-understood quantity.

---can also be done numerically

Collaborators: E. Rezayi
B. Bradlyn
M. Goldstein

Papers on arXiv and in PRB

See especially arXiv: 1502.04126, 1407.2911 with Bradlyn