Real-space entanglement spectrum in quantum Hall trial wavefunctions

Nicholas Read

w. Jerome Dubail and Edward Rezayi

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Outline

Entanglement entropy and entanglement spectrum in topological phases, correspondence with edge

Quantum Hall (QH) states: orbital, particle, and **real-space** partitions
---non-locality and locality

Dubail, N.R., Rezayi (2012a)
See also: Sterdyniak et al (2012);
Rodriguez, Simon, Slingerland (2012)

Numerical spectra

CFT construction of trial states from conformal blocks

Edge state overlaps and entanglement:
---(generalized) screening as a conformal boundary condition

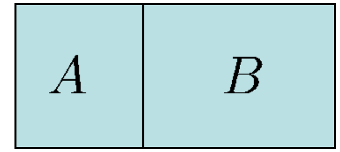
Dubail, N.R., Rezayi (2012b)

Corrections to scaling and entanglement spectra
---real space and particle partitions

Comparison with numerical data

Bipartite entanglement

Usually, partition of **coordinate** space into two parts:



Partition of Hilbert space:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Schmidt decomposition of ground state $(\langle\psi|\psi\rangle = 1)$:

$$|\psi\rangle = \frac{1}{\sqrt{\mathcal{Z}}} \sum_n e^{-\varepsilon_n/2} |\psi_{A,n}\rangle \otimes |\psi_{B,n}\rangle, \qquad \mathcal{Z} = \sum_n e^{-\varepsilon_n}$$

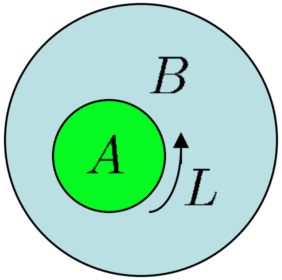
$\varepsilon_n \in \mathbf{R}$ are the "pseudoenergies" in the entanglement spectrum (ES)

Alternatively, reduced (or "marginal") density matrix

$$\rho = \frac{1}{\mathcal{Z}} \sum_n e^{-\varepsilon_n} |\psi_{A,n}\rangle \otimes \langle\psi_{A,n}|, \qquad \mathrm{tr}\,\rho = 1$$

Entanglement entropy is $S = -\mathrm{tr}\,\rho \ln \rho$

Entanglement entropy in **2D** topological phases



Kitaev-Preskill, Levin-Wen (2006): as length of cut $L \to \infty$,

$$S \sim s_0 L - \gamma + \ldots$$

γ is called the **topological entanglement entropy**:

$$\gamma = \ln \mathcal{D}, \qquad \mathcal{D} = \sqrt{\sum_\alpha d_\alpha^2} = 1/\mathcal{S}_{00}$$

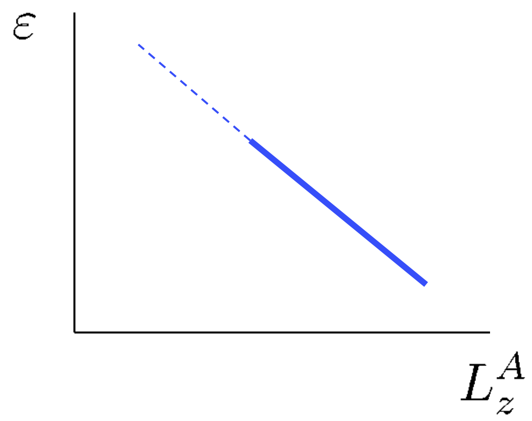
$d_\alpha \geq 1$ are quantum dimensions, $\mathcal{S}_{\alpha\beta}$ are elements of modular S matrix

Issues? Definition of partition, necessary and sufficient conditions, . . .

How to understand from entanglement spectrum?

Entanglement spectrum (ES) in 2D top phases

Kitaev-Preskill (2006):



Using a different approach, we can formulate a simpler but more heuristic derivation of the formula for γ. First we write the marginal density operator ρ for the disk as $\rho = e^{-\beta H}$.

Now we make a natural but nontrivial assumption: that H can be regarded as the Hamiltonian of a $(1+1)$-dimensional conformal field theory (CFT). This CFT ignores short-distance properties of the bulk medium, and therefore will not account correctly for the term in the entropy proportional to L, but it should reproduce correctly the universal constant term.

Li-Haldane (2008)---orbital partition in QH systems in lowest Landau level (LLL)
---focused on multiplicities in low-lying part, below "entanglement gap"
---**conjecture:** multiplicities agree with spectrum at an edge in thermo limit

Arguments for these: I. Klich, 2006 (free fermions) [KP]
A. Turner et al.; Fidkowski, 2010 (free fermions) [KP]
A. Bernevig et al., 2010--2011 (trial QH states) [LH]
Qi, Katsura, Ludwig, 2012 (generally) [KP]
Swingle, Senthil, 2012 (generally) [KP]

Entanglement in topological phases in QH systems

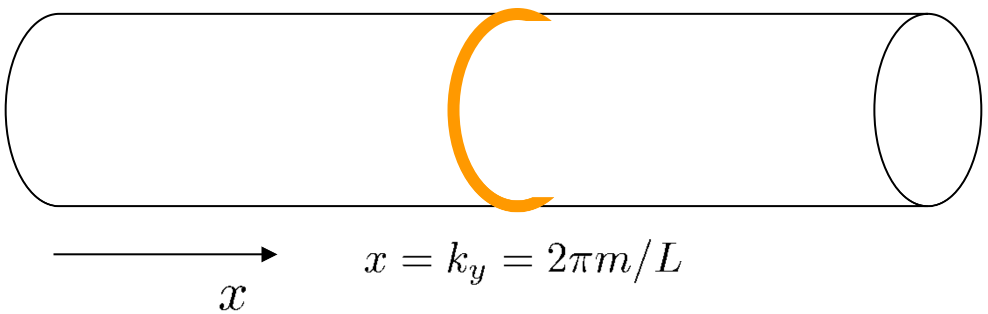
"Orbital" and "particle" partitions

Zozulya, Haque, Schoutens, Rezayi (2006, 2007)

Number of particles in A (resp B) is always $N_A \ (N_B = N - N_A)$.

Ang momentum of particles in part A is always L_z^A.

In LLL, eigenstate for translations in y direction is localized in x:



$$e^{ik_y y - (x-k_y)^2/2}$$

so restricting to $k_y < k_0$ is like $x < k_0$. **Orbital partition**.
Actually use sphere, azimuthal qu no $-N_\phi/2 \leq m \leq N_\phi/2$, $m > 0$ as part A (roughly, northern hemisphere), $m \leq 0$ as part B (southern). $(L_z^A = \sum m_i)$
(N_A, N_B not fixed)

Particle partition: divide particles into two groups of (fixed) sizes N_A, N_B, allow all quantum numbers. (Come back to this.)

Following Li-Haldane (2008) --- orbital partition

Laughlin state:
Thomale et al 2009
--Coulomb interaction

$\nu = 1/2$

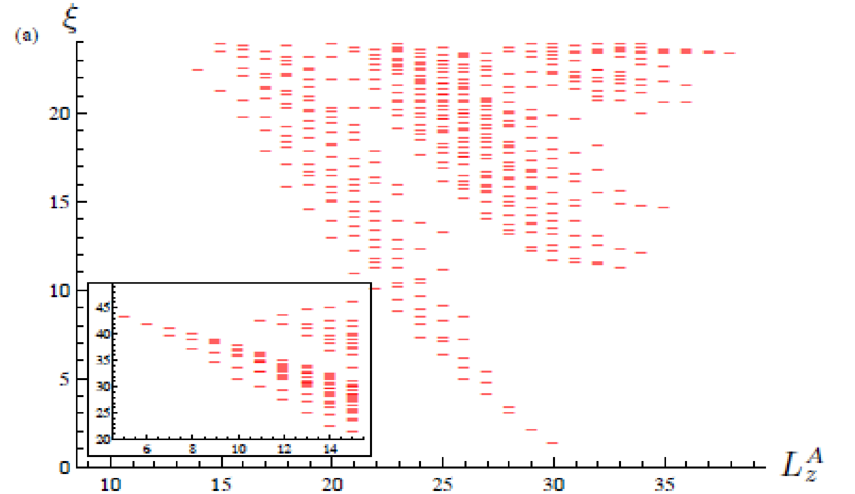
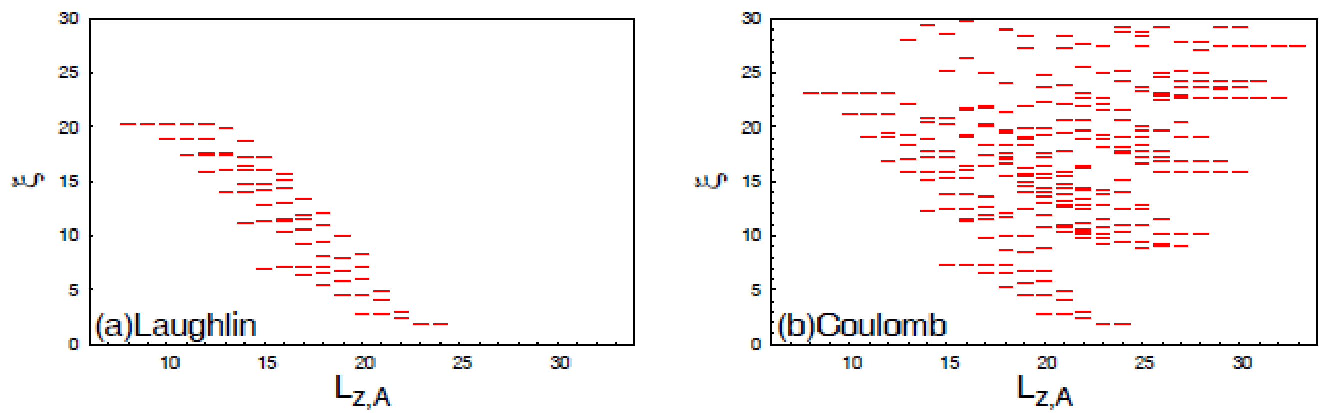


FIG. 1: Entanglement spectrum for the $N = 11$ bosons, $N_\phi = 20$ $l_A = 10$ orbitals and $N_A = 5$ bosons.

$\nu = 1/3$



Sterdyniak et al, 2011

Figure 1. Orbital entanglement spectrum for the Laughlin state, (a), and for the ground state of the Coulomb interaction, (b) for $N = 8$ fermions, $N_\Phi = 21$, $N_A = 4$ and $l_A = 11$. A small system-size has been selected for pedagogical purposes. The

Observation 1: orbital partition for filled LLL is trivial

J. Dubail (2011)

---For any given cut, there is only one state for A, and it has $N_A = N_{\text{orb}}^A$.

---**Not** the $\nu = 1$ edge; **counterexample** to Li-Haldane conjecture

Observation 2: orbital partition is not local in real space

Single-particle correlation (reduced density matrix)

$$\langle\psi|\hat{\psi}^\dagger(x',y')\hat{\psi}(x,y)|\psi\rangle = \frac{\nu}{2\pi}e^{(z+\overline{z'})^2/4-x^2/2-x'^2/2}$$

in any translationally-invariant state (Landau gauge, $L=\infty$).

When instead **restricted** to k_y<0 (i.e. in part A), one has for large y-y':

$$\sim \frac{\nu e^{-x^2/2-x'^2/2}}{2\pi^{3/2}i(y-y')}$$

---long-range along the “cut”.

Dubail, N.R., Rezayi (2012a)

However Kitaev-Preskill assumes locality of cut.

Third way: **real-space** partition

---even for LLL states, they live in full Hilbert space

Simply partition real (coordinate) space into two regions---thus to trace out part B, integrate $\psi.\overline{\psi}$ over region B

For a Schmidt decomposition of a system of N particles, we (usually) start from a decomposition of **single-particle** space into subspaces:

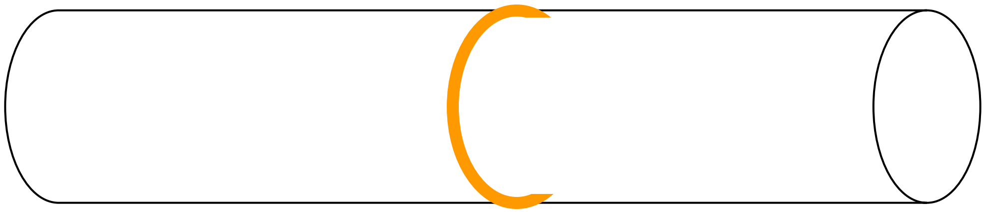
$$\mathcal{H}_1 = \mathcal{H}_{1,A} \oplus \mathcal{H}_{1,B}$$

which induces a decomposition of [(anti-)symmetrized] **N-particle** space,

$$\mathcal{H}_N = \bigoplus_{N_A, N_B : N_A + N_B = N} \mathcal{H}_{N_A,A} \otimes \mathcal{H}_{N_B,B}$$

For real-space partition here, states in $\mathcal{H}_{N_A,A}$ are **not** in LLL, even when state in $\mathcal{H}_N$ is.
Manifestly local.

Klich 2006; Rodriguez and Sierra 2008; A. Turner, Zhang, Vishwanath 2010:
Cylinder: cut at x_0, basis states are Gaussians in x, integrating over B gives an error function. (For sphere, incomplete Beta functions.)

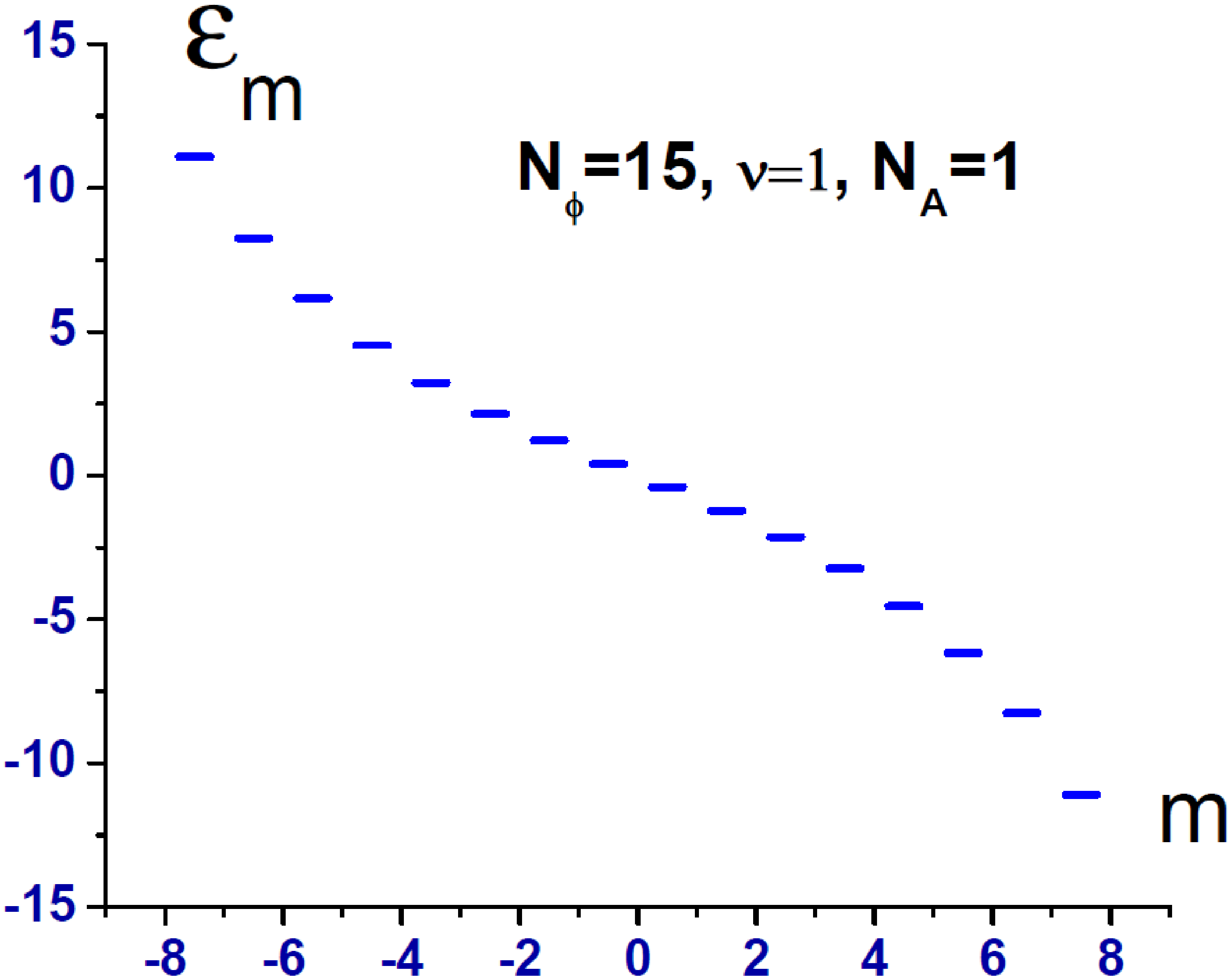


For filled LLL, pseudoenergies are sums of “single-particle” ones derived from this, namely

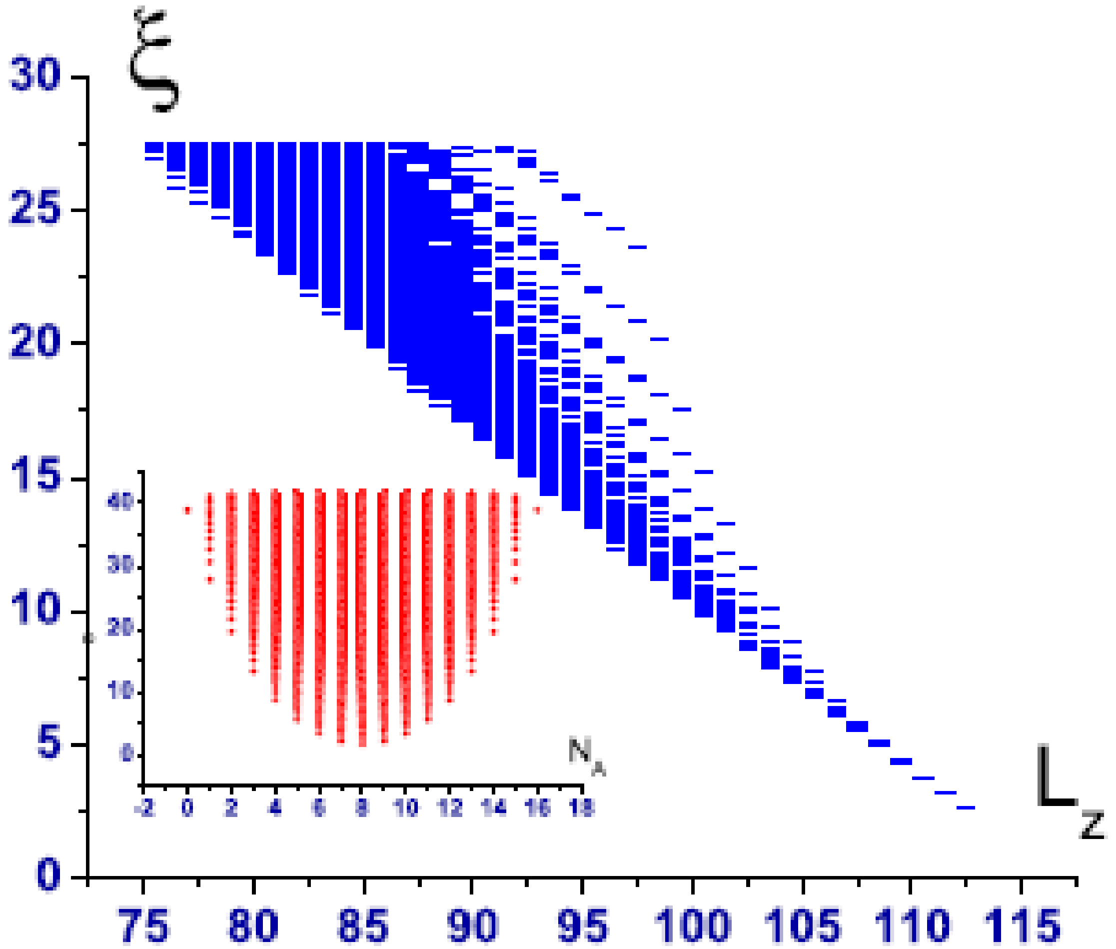
$$\xi = \sum_m \varepsilon_m n_m, \qquad \varepsilon_m = -2 \ln \frac{\alpha_m}{\beta_m} \qquad (\alpha_m^2 + \beta_m^2 = 1)$$

(plus constant). The “cut ground state” has negative pseudoenergies filled, “excited” states are particle and/or hole “pseudoexcitations”.

Orbital partition can be viewed as all $\alpha_m, \ \beta_m$ are 0 or 1.



ε_m
N_ϕ=15, $\nu=1$, N_A=1
m
15
10
5
0
-5
-10
-15
-8
-6
-4
-2
0
2
4
6
8



N=30, $\nu = 1$, N_A=15. Inset: versus N_A, for N=16.
Different values of N_A are still part of the spectrum!

Real-space entanglement spectrum for topological phases

J. Dubail, N.R., and E. Rezayi (2012a)

If the partition is defined locally in real space, then correlations within part A are restrictions of those of the full system.

In a topological phase, these are short-range correlations.

We expect that, writing $\rho = e^{-H_{\rm ent}}$, the entanglement Hamiltonian $H_{\rm ent}$ acts in the Hilbert space of some d-1--dimensional theory and is **local** (short-range) along the cut. That is, regions of space separated by more than the correlation length are approximately independent.

(No formal proof of this yet.)

In general, could contain many “massive”/gapped excitation modes penetrating further and further into bulk.

Note that power-law--decaying interactions produce power law correlations.

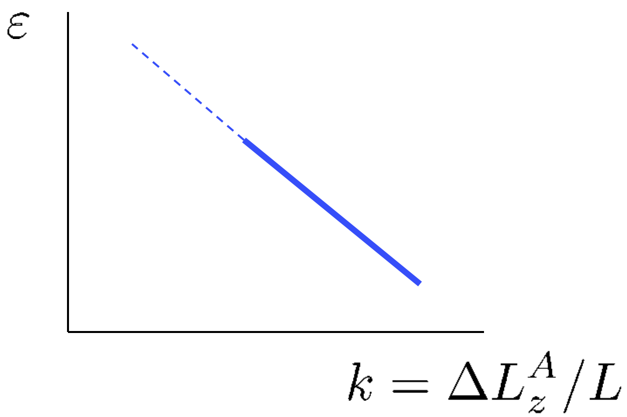
Real-space partition for fractional QH states

J. Dubail, N.R., and E. Rezayi (2012a)
See also A. Sterdyniak et al. (2012);
Rodriguez, Simon, Slingerland (2012)

Our conjecture: for **local** partitions of a ground state in a topological phase (i.e. with exponentially decaying correlations), the ES is that of a **local** (short range) Hamiltonian acting in Hilbert space of a d-1-dimensional system.

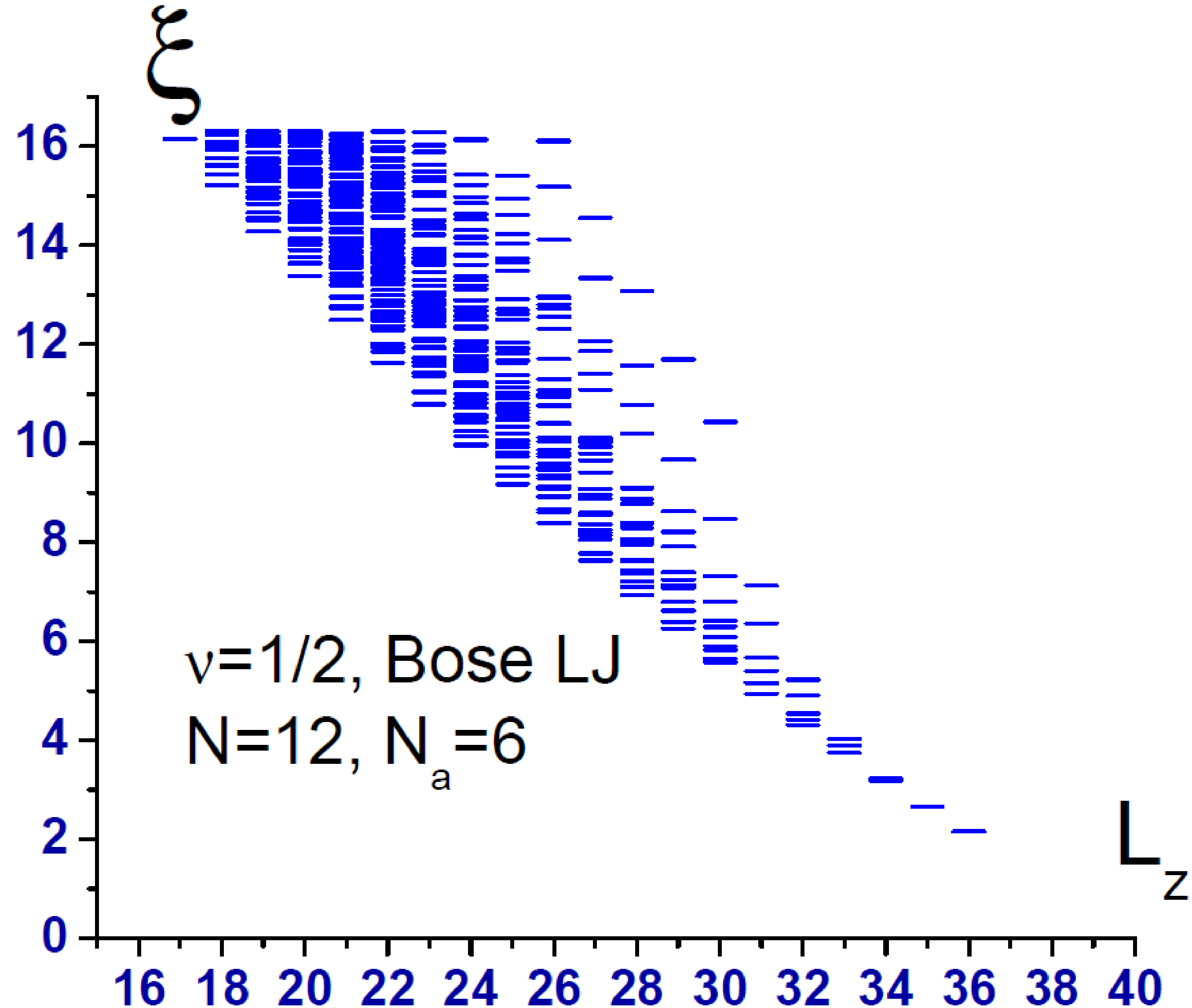
For trial QH states, we expect (following LH, KP) that the 1-dim theory is the same chiral CFT as the edge theory of the phase.

Then for a cut of length L in large system, to first order we expect ES like:



Cannot hold with $v > 0$ in all sectors for orbital partition!

with some “velocity” v


ξ
ν=1/2, Bose LJ
N=12, N_a=6
L_z
0
2
4
6
8
10
12
14
16
16
18
20
22
24
26
28
30
32
34
36
38
40

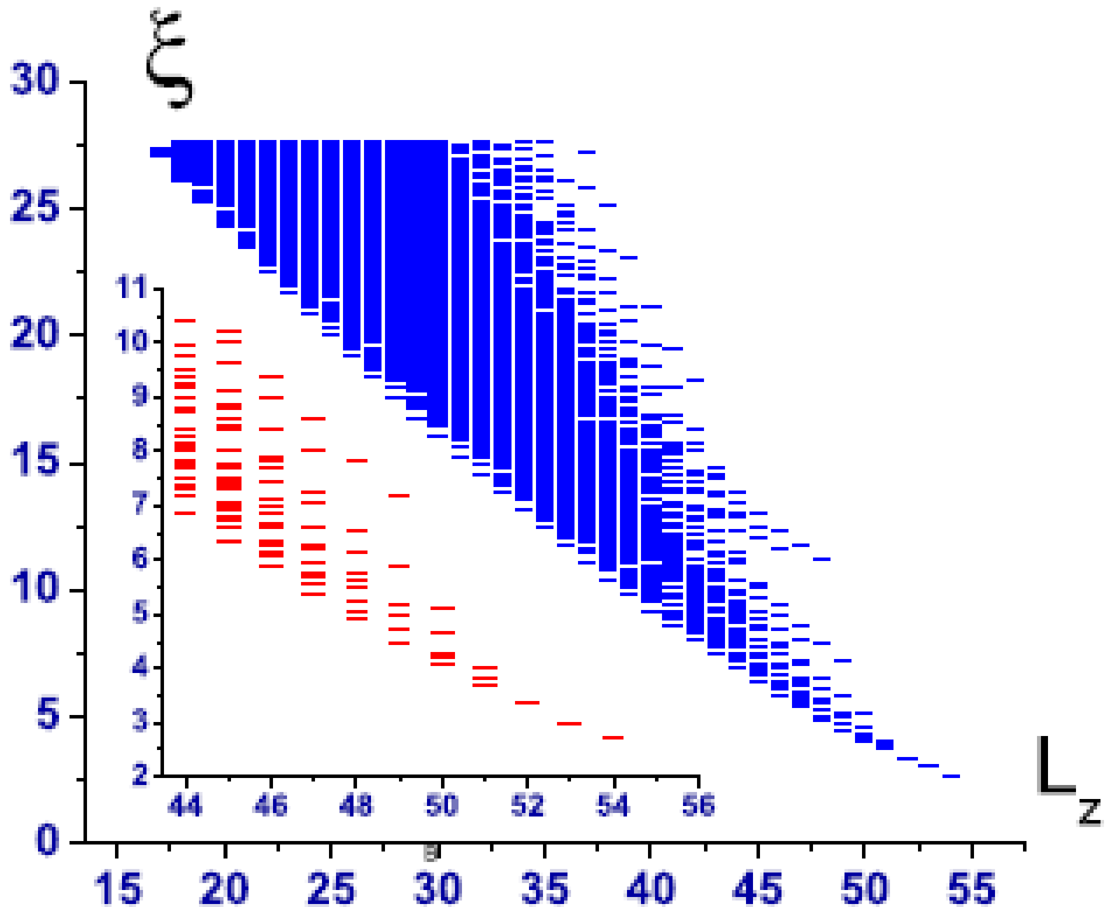
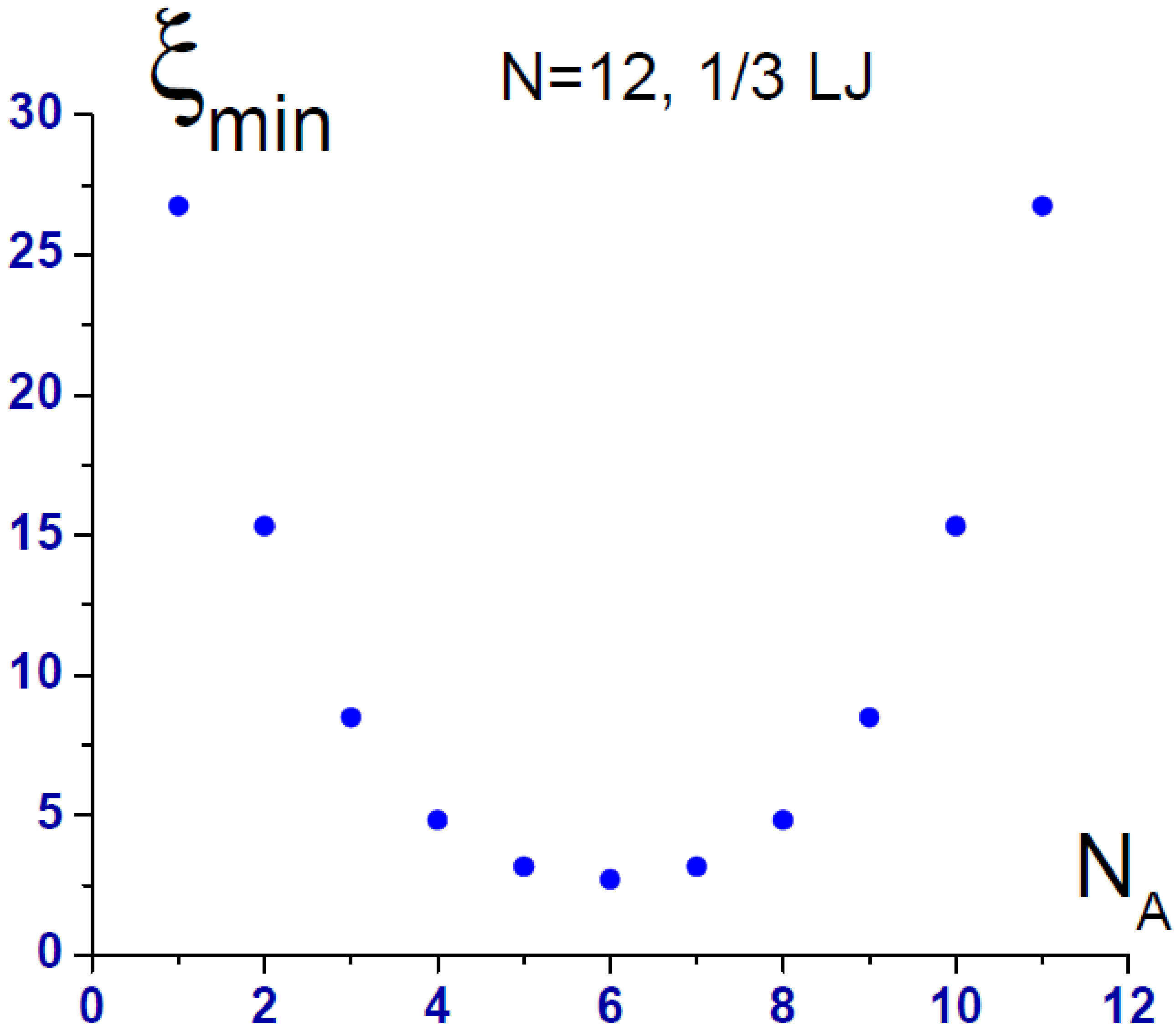


FIG. 2: (color online) The entanglement spectrum of Laughlin state for $\nu = 1/3$, for $N_A = 6$ in a 12-electron size system.



ξ_{min}
N=12, 1/3 LJ
N_A
30
25
20
15
10
5
0
0
2
4
6
8
10
12

Analysis of RSP (and PP)

Moore-Read construction: obtain trial QH wavefunctions from a 2D CFT,
a single scalar U(1) theory times another one.

Then

$$\Psi(z_1,\ldots,z_N) \;=\; \frac{1}{\sqrt{Z_N}}\,\langle N|\,\mathcal{R}\prod_{i=1}^{N} a(z_i)\,|0\rangle$$

(Radially ordered expectation of operators---leave this implicit after this)

where

$$a(z) \;=\; e^{i\varphi(z)/\sqrt{\nu}} \times \psi(z)$$

and ψ has Abelian fusion rules (it is a simple current)---e.g. identity operator

Examples: $\psi = I$, we obtain the Laughlin wavefunction

$$\prod_{i=1}^{N}(z_i - z_j)^{1/\nu}$$

For ψ a Majorana Fermi field, we obtain the MR Pfaffian state.

Moore-Read construction can be viewed as a continuous generalization
of matrix or tensor product states.

Dubail, N.R., Rezayi (2012a)

Real-space cut: use completeness relation in CFT at radius R,

$$1 = \sum_{N_A,k} R^{L_0} \left| N_A, k \right\rangle \left\langle N_A, k \right| R^{-L_0}$$

gives

$$\left| \Psi \right\rangle\rangle = \sqrt{\frac{Z_{N_{A0}} Z_{N_{B0}}}{Z_N}} \sum_{N_A=0}^{N} \sum_k \left| \Psi^A_{\langle N_A,k|} \right\rangle\rangle \otimes \left| \Psi^B_{|N_A,k\rangle} \right\rangle\rangle$$

Dubail, N.R., Rezayi (2012b)

and wavefunctions of these

$$\Psi^A_{\langle N_A,k|}(z_1,\ldots,z_{N_A}) \propto \left\langle N_A, k \right| \prod_{i=1}^{N_A} a(z_i) \left| 0 \right\rangle$$

can be identified as **edge states** of a system with N_A particles (linearly independent as $N_A \to \infty$).

This **immediately** predicts Li-Haldane-type result for multiplicities as $N_A \to \infty$ in RSP.

If these basis states are orthonormal, we have the Schmidt decomposition . . .

Examples in real space

Decomp of Laughlin state: multiplicities

Use coords $z \leftrightarrow A,\ w \leftrightarrow B$

$$\prod_{i<j}(z_i - z_j)^Q \prod_{k<l}(w_k - w_l)^Q \cdot \prod_{i,l}(w_l - z_i)^Q$$

$$= \prod_{i<j}(z_i - z_j)^Q \prod_{k<l}(w_k - w_l)^Q \prod_m w_m^{QN_A} \cdot \exp\left[Q \sum_{i,l} \ln\left(1 - \frac{z_i}{w_l}\right)\right]$$

$$= \prod_{i<j}(z_i - z_j)^Q \prod_{k<l}(w_k - w_l)^Q \prod_m w_m^{QN_A} \cdot \exp\left[-Q \sum_{i,l} \sum_{n=1}^{\infty} \frac{1}{n} \frac{z_i^n}{w_l^n}\right]$$

$$(|z_i| < R, \quad |w_l| > R)$$

$$= \prod_{k=1}^{\infty}\left[\sum_{n_m=0}^{\infty} \frac{1}{n_m!}(-j_{-m}^A j_{-m}^B)^{n_m}\right] \cdot \Psi^A(\{z_i\})\Psi^B(\{w_l\})$$

$\left(j_{-m}^A = \sqrt{\frac{Q}{m}} \sum_i z_i^m,\ j_{-m}^B = \sqrt{\frac{Q}{m}} \sum_l w_l^{-m}\right)$ gives Schmidt decomp as $N_A,\ N_B \to \infty$ --edge states!

The overlap integrals $\int \prod_i d^2 z_i \overline{\Psi'} \Psi e^{-\sum_i |z_i|^2/2}$ are difficult.

For Laughlin state, can relate them to expectations in Coulomb plasma. Assuming screening in this plasma, Wen (1992) derived leading behavior.

Dubail, N.R., Rezayi (2012b)

The normalization integrals have the form

$$\int \prod d^2 z_i \, |\langle N, k| \prod_i a(z_i)|0\rangle|^2 e^{-\sum_i |z_i|^2/2}$$

which corresponds to perturbing the left times right CFTs by adding

$$\int d^2 z \, \overline{a}(\overline{z}) a(z) e^{-|z|^2/2}$$

to action; effect of Gaussians is that the perturbation acts only inside a disk-shaped region---all particles inside the disk. (Also because of real-space partition.)

We will again assume that the **generalized screening** property holds:
all connected correlations of local operators in the perturbed CFT decay exponentially in the bulk of the disk (as they do in the plasma). I.e. perturbed non-chiral 2D CFT flows to a massive phase.

The out states $\langle N, k|$ can all be obtained from the states

$$\langle 0| \prod_{j=1}^{N} a^\dagger(w_j)$$

by doing contour integrals on all w's along concentric circles (which give modes of a's), and taking linear combinations.

That is because $a(w)$ generates the chiral algebra of our CFT, and the states of interest are in the "vacuum" sector (i.e. are descendants of the vacuum).

So therefore look instead at correlators

$$\int \prod d^2 z_i \, \overline{\langle 0| \prod_j a^\dagger(w'_j) \prod_i a(z_i)|0\rangle} \langle 0| \prod_j a^\dagger(w_j) \prod_i a(z_i)|0\rangle e^{-\sum_i |z_i|^2/2}$$

These are rather like norms of quasihole states, except it has a's instead of quasihole operators, and $w'_j \neq w_j$.

Generalized screening means that the correlations of still-chiral $a(w)$s decay exponentially $|w_j| < R$ (in charge sector, they carry magnetic (vorticity) as well as electric charge) --- unless there is a $w'_j = w_j$; then goes to constant.

Thus, screening of $a^\dagger(w)$ by pairs $\bar{a}(\bar{z})a(z)$ (think of binding as $z \to w$) turns it into $\bar{a}(\bar{w})$.
Then from large scale point of view, there is an effective **conformal boundary condition,** like

$$\boxed{a^\dagger(w) = \overline{a}(\overline{w})}$$

on boundary of region, say $\mathrm{Re}\, w > 0$ (using a conformal mapping to right-half plane). This corresponds to use of image charge techniques in plasma case (Wen).

Then unfold to single chiral CFT on whole plane, and get correlators.

Result is then, for given edge states $k,\ k'$,

$$\lim_{N\to\infty} \left\langle\left\langle\, \Psi_{\langle k|} \,\middle|\, \Psi_{\langle k'|} \,\right\rangle\right\rangle \;=\; \langle k' \,| k\rangle$$

i.e. that in the CFT (up to one overall factor). **General result** for edge excitations of trial QH states (when gen. screening holds), not previously known except for Laughlin states (Wen)

We checked this on examples in both Laughlin and MR states, using Monte Carlo.

Corrections to scaling

Dubail, N.R., Rezayi (2012b)

The preceding produces flat, degenerate ES to leading order.

Subleading terms: locality of correlations in bulk of drop leads us to expect

$$\langle\langle\, \Psi_{\langle k|} \,\big|\, \Psi_{\langle k'|} \,\rangle\rangle \;=\; \frac{\langle k'|\, e^{-S_b}\, |k\rangle}{\langle e^{-S_b} \rangle}$$

where $S_b \,=\, \sum_\alpha \lambda_\alpha \oint_{|z|=R} |dz|\, \phi_\alpha(z)$ is perturbation on the boundary

and ϕ_α are U(1)-neutral local operators in the CFT chiral algebra generated by $a(z),\ a^\dagger(z)$ and are irrelevant or marginal, i.e. $h_\alpha \geq 1$; $\lambda_\alpha \sim a^{h_\alpha - 1}$
(a is UV cutoff/particle spacing). In particular, current and stress tensor can appear:

$$S_b = \mu J_0 + v L_0^{U(1)}/R + \ldots$$

$L_0 =$ integral of stress tensor of CFT, a generator of Virasoro algebra;
it gives the linear spectrum seen earlier.

But these first terms may be trivial or absent for symmetry reasons in some cases
(e.g. edge state overlaps in plane).

For real-space partition of Laughlin states, the stress tensor is allowed, and there are two next-leading local operators.

We made a fit:

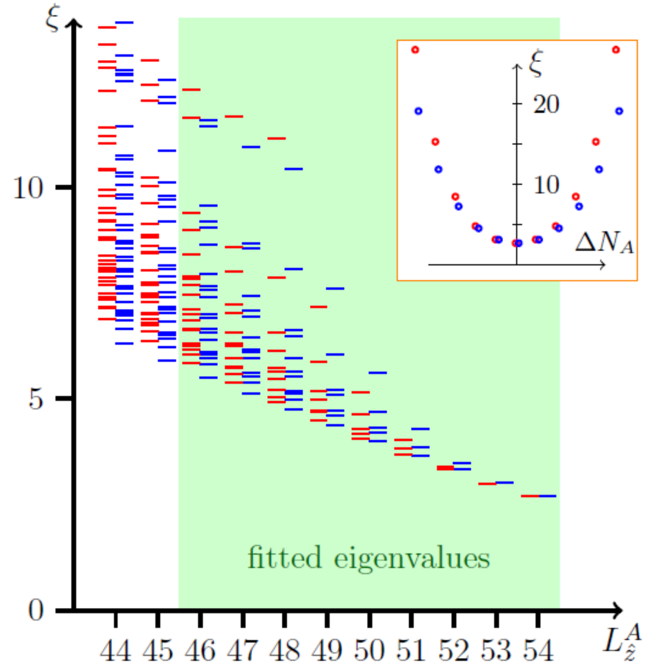


FIG. 7. (color online) Red: real-space ES of the Laughlin state at $\nu = 1/3$ on the sphere, with a cut along the equator (for $N = 12$, $N_{A0} = 6$). Blue: spectrum of the operator (4.31). Both spectra are plotted in the $\Delta N_A = 0$ sector. The inset shows the lowest eigenvalue for the other values of ΔN_A. The parameters α, β, γ are obtained from a least square fit that includes all the eigenvalues in the shaded (green) area.

More general: Moore-Read states -- expect two velocities in leading terms

Particle partition, extended

Dubail, N.R., Rezayi (2012a)

New set-up for particle partition:
Map (spinless) system into a larger Hilbert space in which each particle has a "pseudospin" of ½, call components $A, \; B$. For any state in $\mathcal{H}_N$, map it to a state that is totally-symmetric in pseudospin:

$$\psi(z_1, \ldots, z_N) \mapsto \psi(z_1, \ldots, z_N) \cdot \bigotimes_i \left(\frac{A_i + B_i}{\sqrt{2}} \right)$$

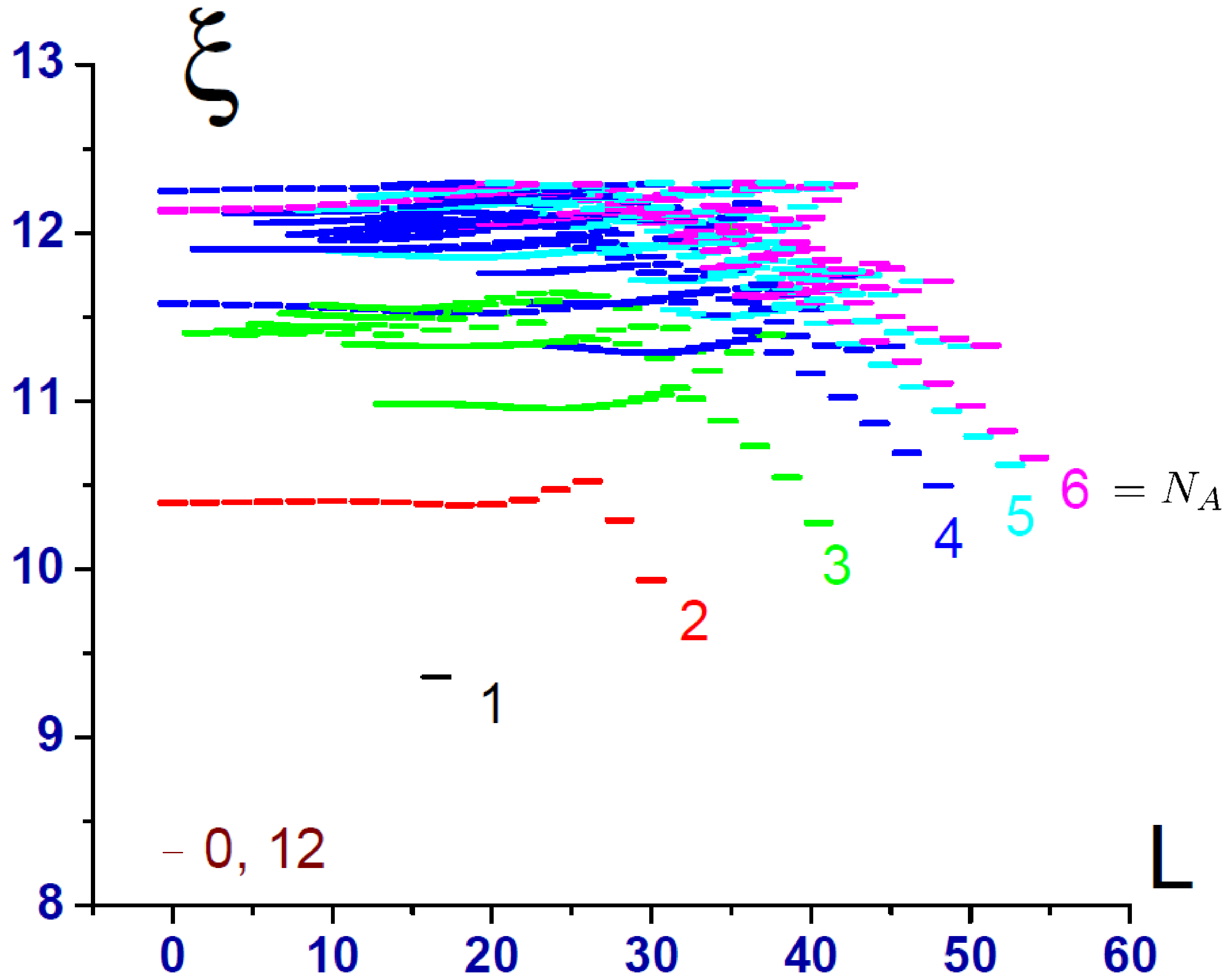
Our extended form of particle partition (EPP) simply looks at the entanglement between pseudospin A and pseudospin B (non-trivial!).

Same as setting $\alpha_m = \beta_m = 1/\sqrt{2}$ for all m.

All N_A are now included in spectrum. Trace includes sum over N_A , equals 1

Note EPP has full rotational symmetry of sphere; ES will have degeneracies

For filled LLL, **all** pseudoenergies for all N_A in extended particle partition are equal!



ν =1/3 Laughlin, 12 particles, particle partition --- L^A, not L_z^A!
Note inverted parabola versus N_A.

Analysis of particle partition: Large angular momentum on sphere means particles in part A are at north pole. Partition behaves as local in large L_z region.

Rotational invariance does not allow U(1) stress tensor to appear in pseudo-Hamiltonian.

Dubail, N.R., Rezayi (2012b)

Leading operator for Laughlin state: surface tension term. Pseudo-Hamiltonian can contain a piece proportional to length of effective edge of part A,

$$\int_0^{2\pi} d\phi \sqrt{\sin^2\theta + (d\theta/d\phi)^2} \simeq 2\pi + \frac{1}{2}\int_0^{2\pi} d\phi \left((d\delta\theta/d\phi)^2 - (\delta\theta)^2\right)$$

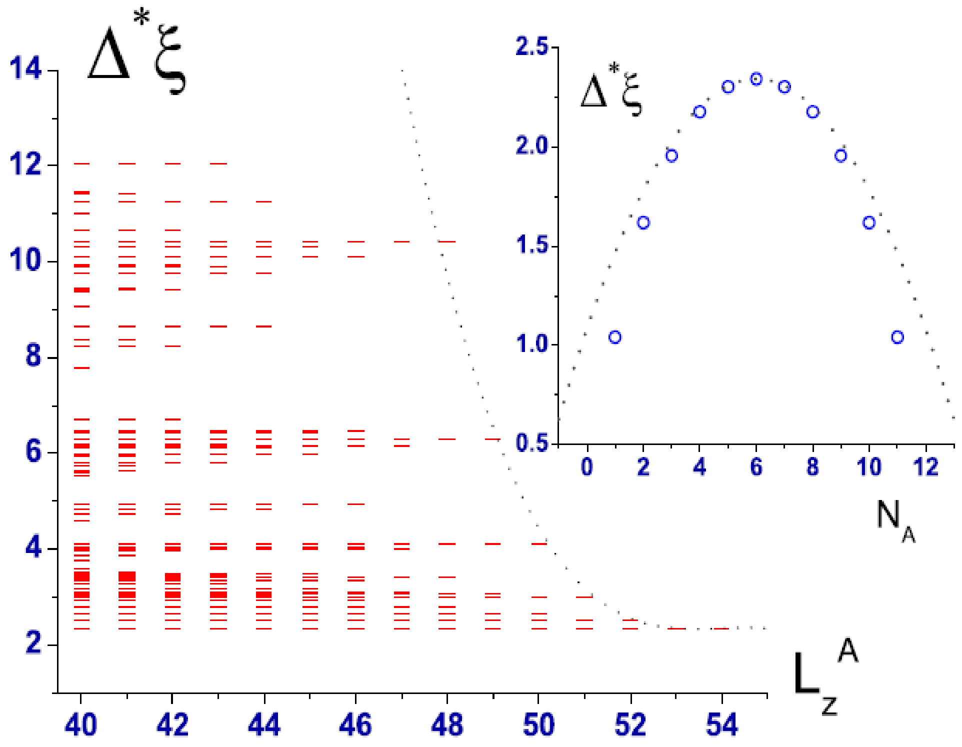
For Laughlin state, changes in shape of drop are U(1) current excitations of edge. This operator corresponds to $(i\partial^2\tilde{\varphi})^2(z)$ in CFT language; its spectrum is

$$\Delta\xi = -\frac{C}{N_{A0}^{3/2}}\left[\frac{(\Delta N_A)^2}{2\nu} - \sum_{m>0} n_m(m^2-1)m\right]$$

occupation numbers of current modes

$(a/R \sim N_{A0}^{-1/2})$

which explains the inverted parabola, as well as the banding of the levels.



One-parameter fit of dotted curve to points

L_z not L here---degeneracy due to rotation invariance

FIG. 6. Plots of PP pseudoenergies ξ versus $L_{\hat{z}}^{A}$ for $N_A = N/2$ and versus N_A for the $N = 12$, $\nu = 1/3$ Laughlin state. Here $\Delta^{*}\xi = \xi - N \ln 2$. Main figure: levels in the scaling region versus $L_{\hat{z}}^{A}$. The dotted curve is a one-parameter fit of a cubic to seven levels, explained in the text. Inset: lowest pseudoenergy for each N_A; the values for $N_A = 0$, 12, which are $\xi = N \ln 2$, are omitted. The curve is an inverted parabola, with the same parameter value as the main figure, as explained in the text.

Conclusion

---for trial states given as conformal blocks, a “proof” of locality of the pseudo-Hamiltonian for real-space and particle partitions, when “generalized screening” holds. For real-space case, this confirms Kitaev-Preskill idea for explaining top. entanglement entropy

---behavior of inner products among edge states

---fits to numerical results

---no such result is known (so far) for orbital partition, which is non-local along the cut

---expect the same for low-pseudo-energy part for general ground states in phase

---role of conformal blocks as wavefunctions is a continuous version of matrix/tensor product states

See also: Cirac, Sierra (2010--12); Zaletel, Mong (2012)

---further related but complementary results on entanglement “subspaces” and finite-size bulk-edge correspondence: Jackson, N.R., Simon (2013)