#MCF: The physics of magnetic confinement in 180 minutes

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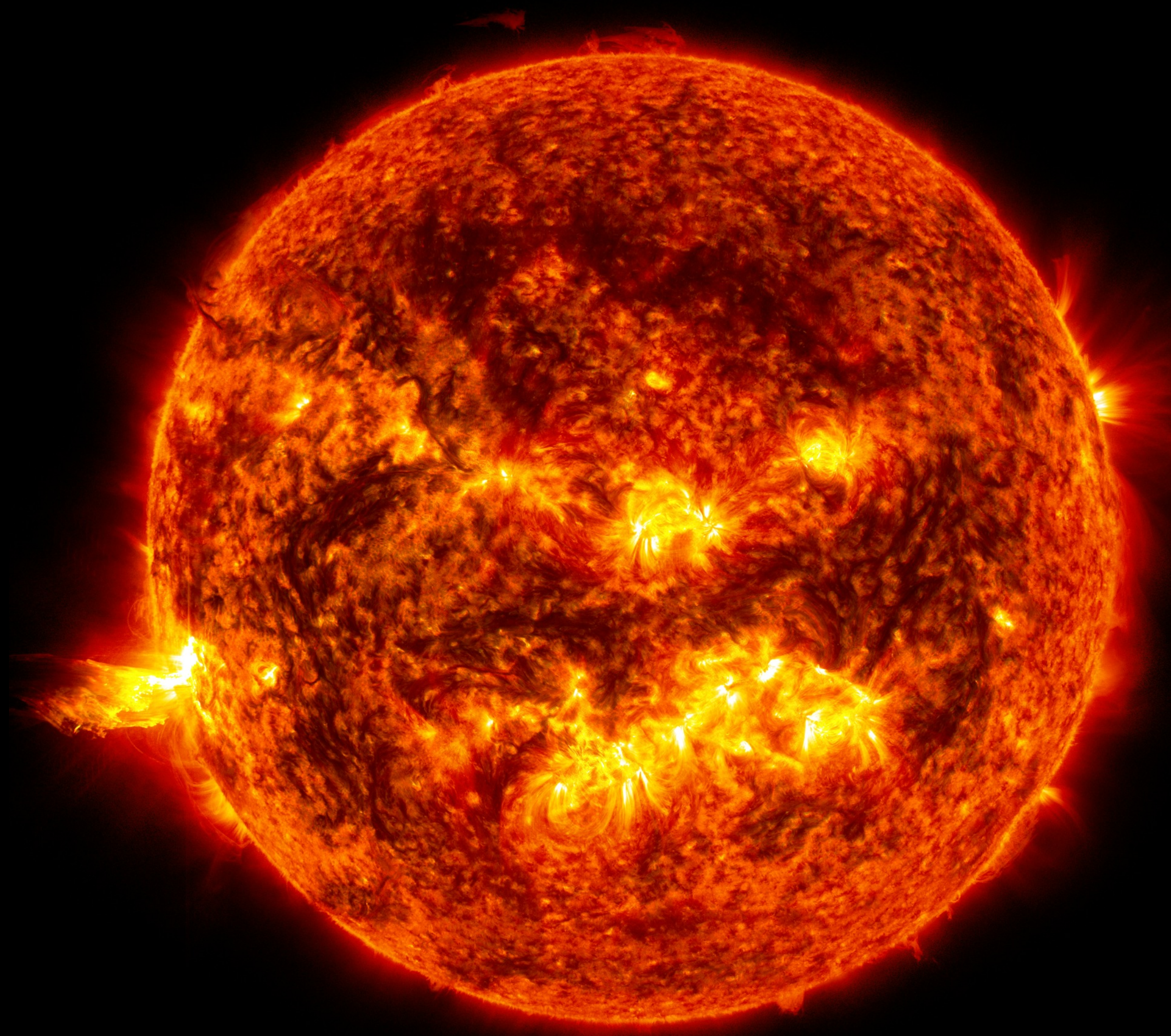
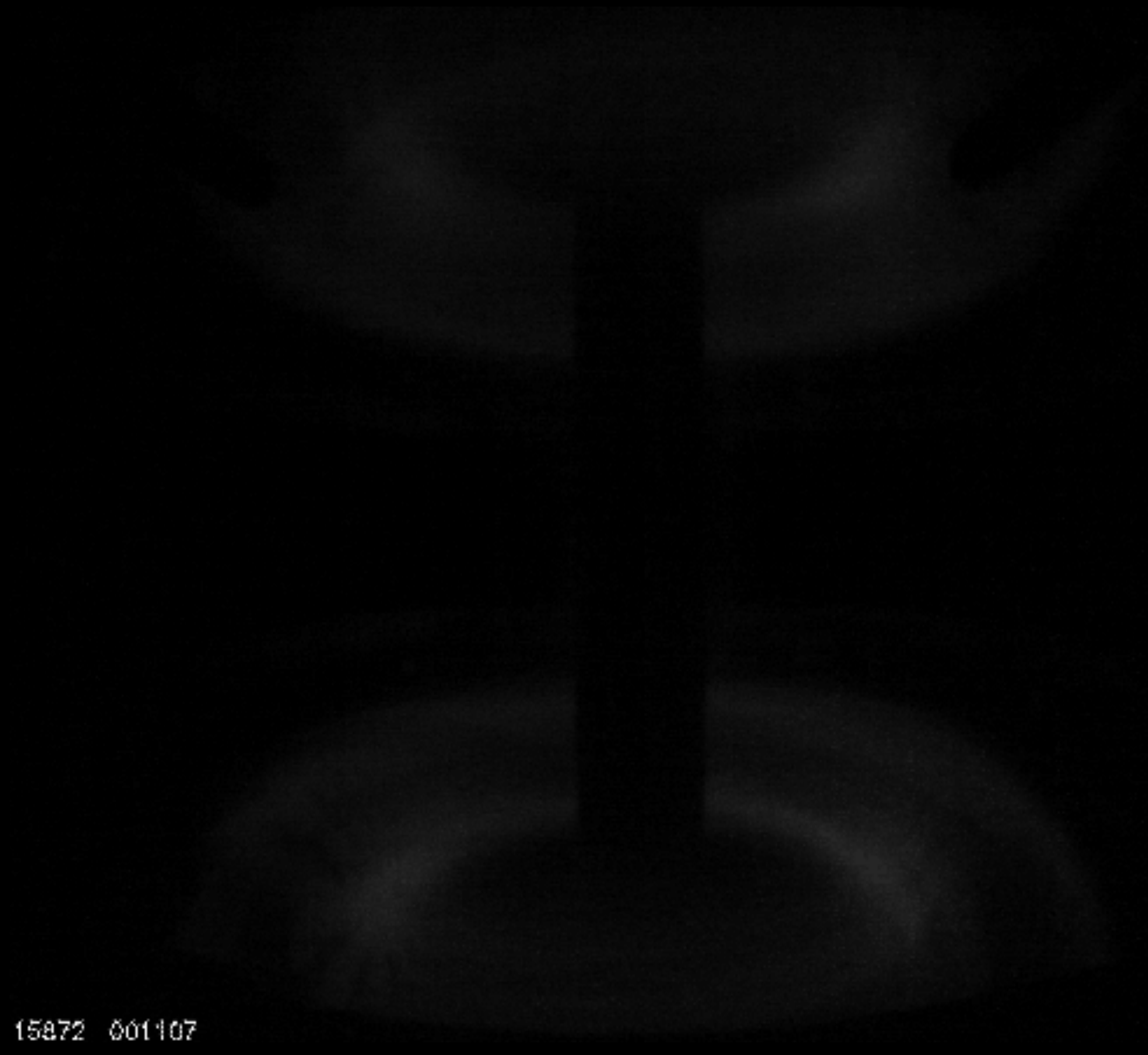
How do MCF and astro plasmas differ?



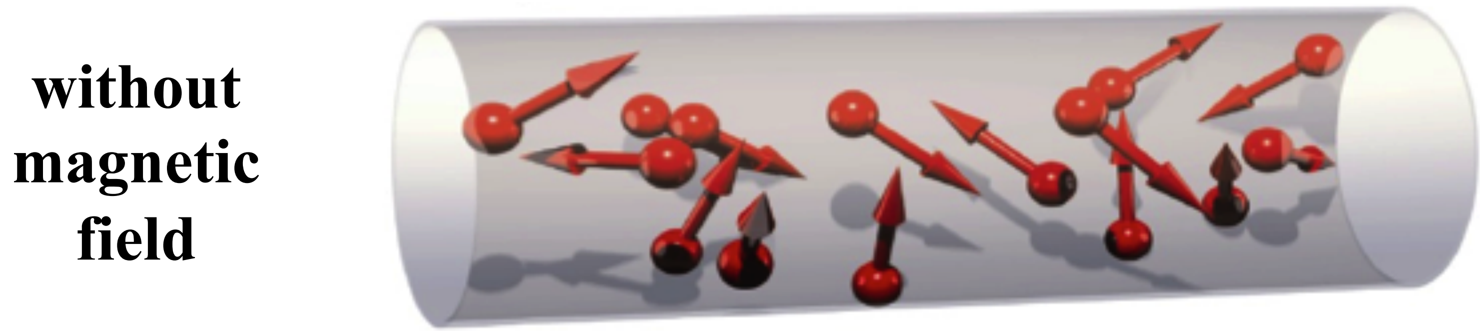
#MCF: The physics of magnetic confinement in 180 minutes

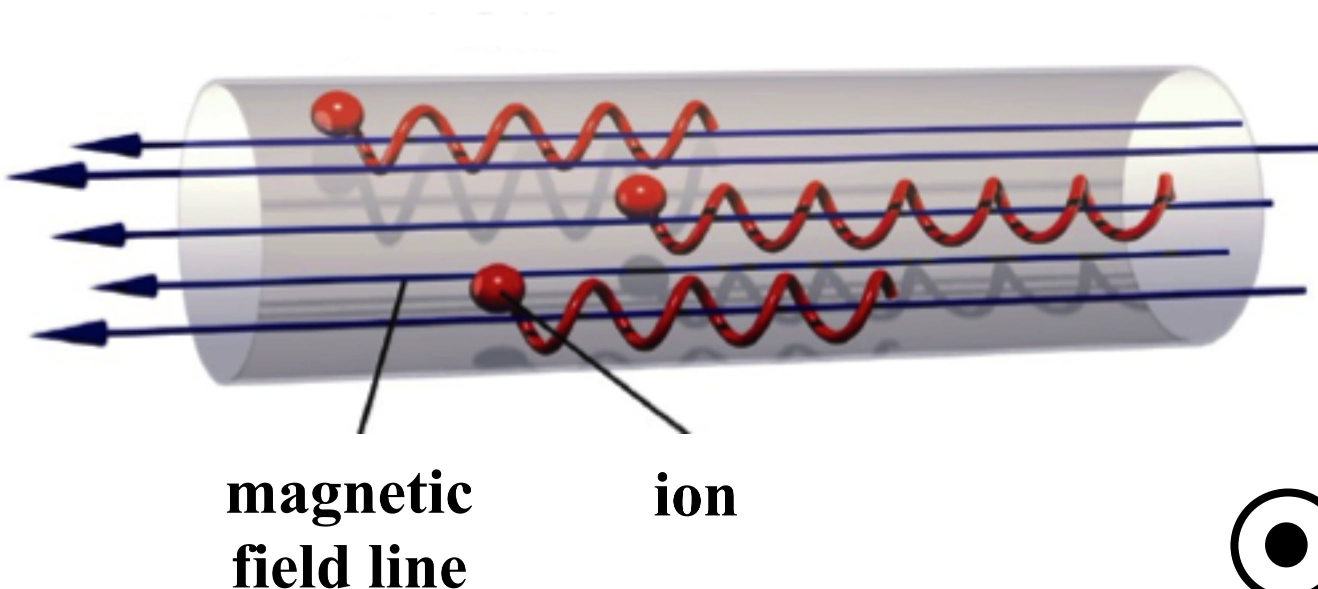


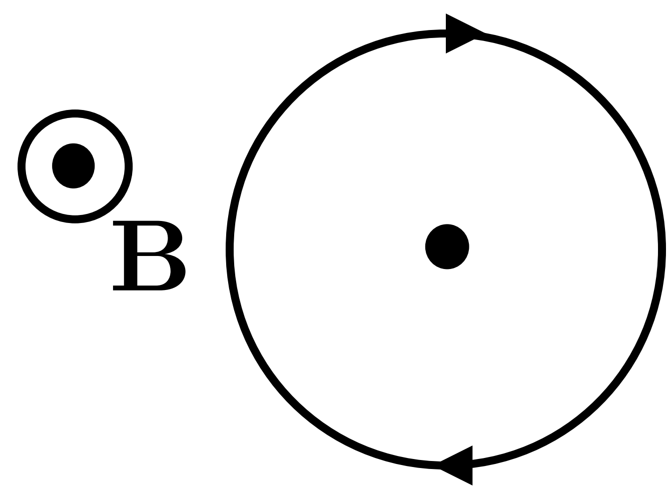
An overview of the overview

- Disclaimer
- Single particle confinement
- Magnetic topology
- Plasma equilibrium
- Macroscopic (MHD) stability
- Turbulence and transport

Basic concept of magnetic confinement

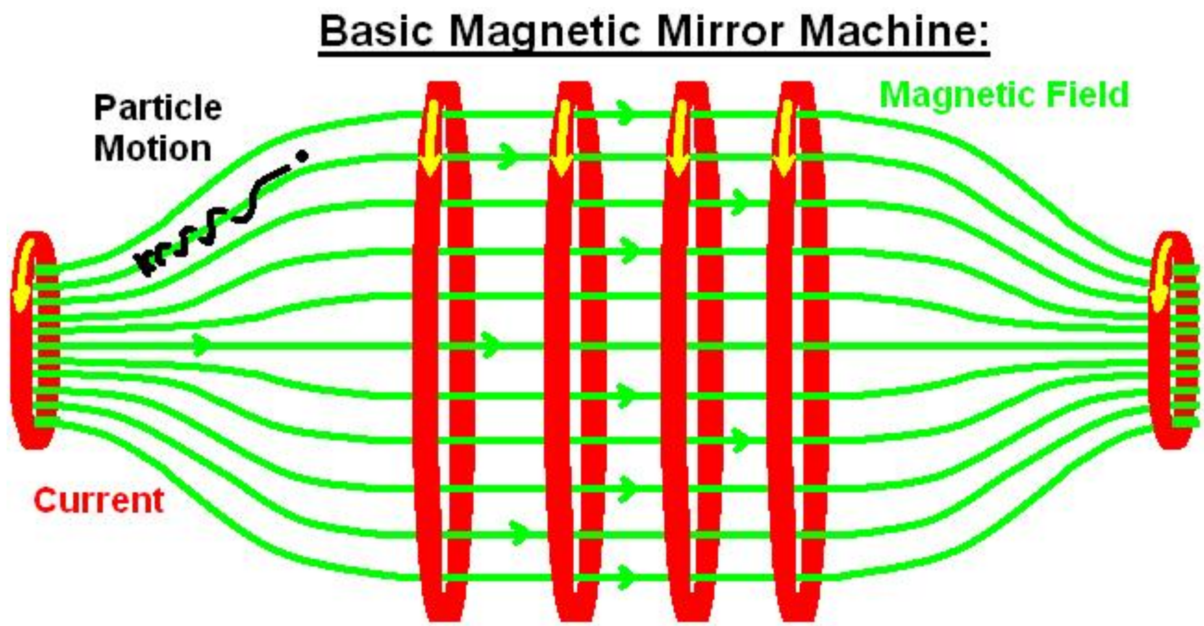


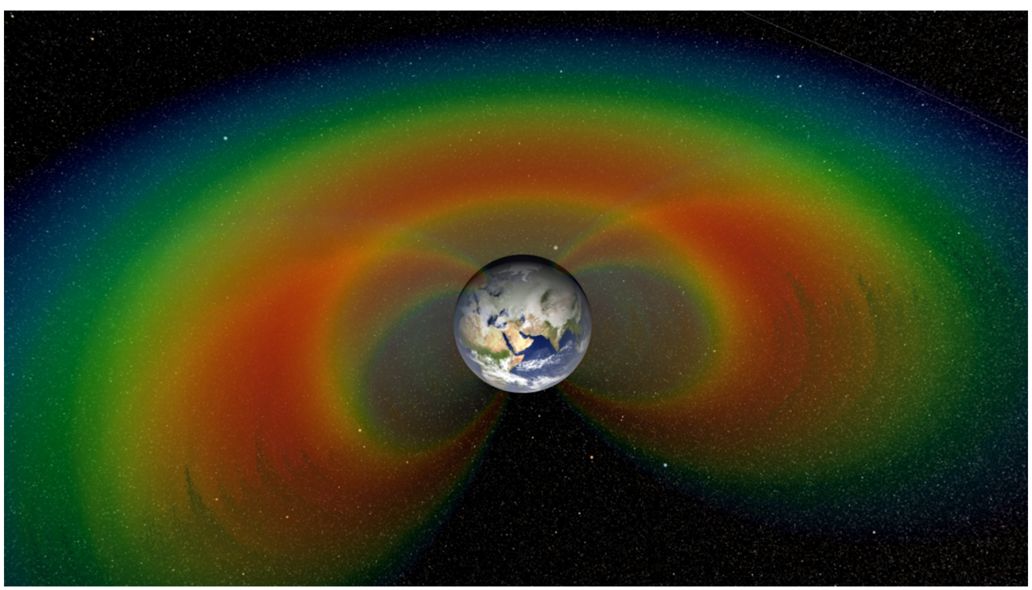




$$\rho = v_\perp/\Omega \propto v_{th}/B \sim \sqrt{\beta}$$

Solution 1: Magnetic mirror

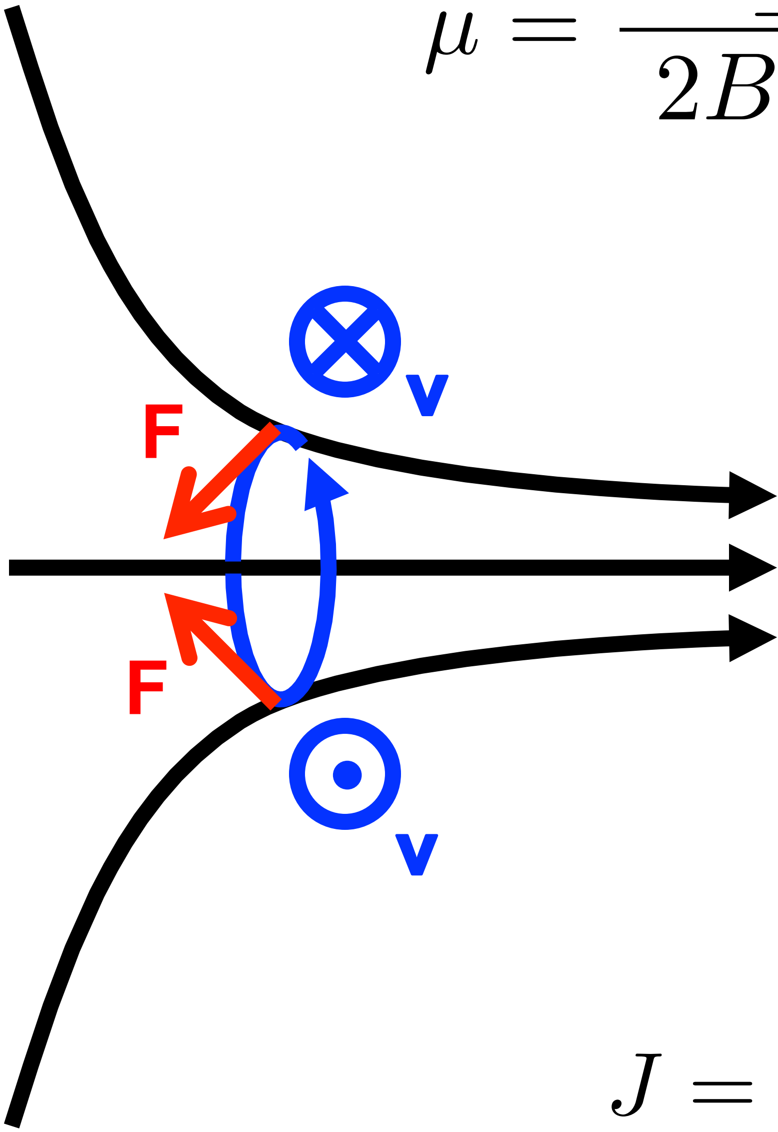




Van Allen radiation belts (NASA)

Solution 1: Magnetic mirror

$$\mu = \frac{mv_\perp^2}{2B}$$



Force on magnetic moment in inhomogeneous B-field:

$$F = -\mu \nabla B$$

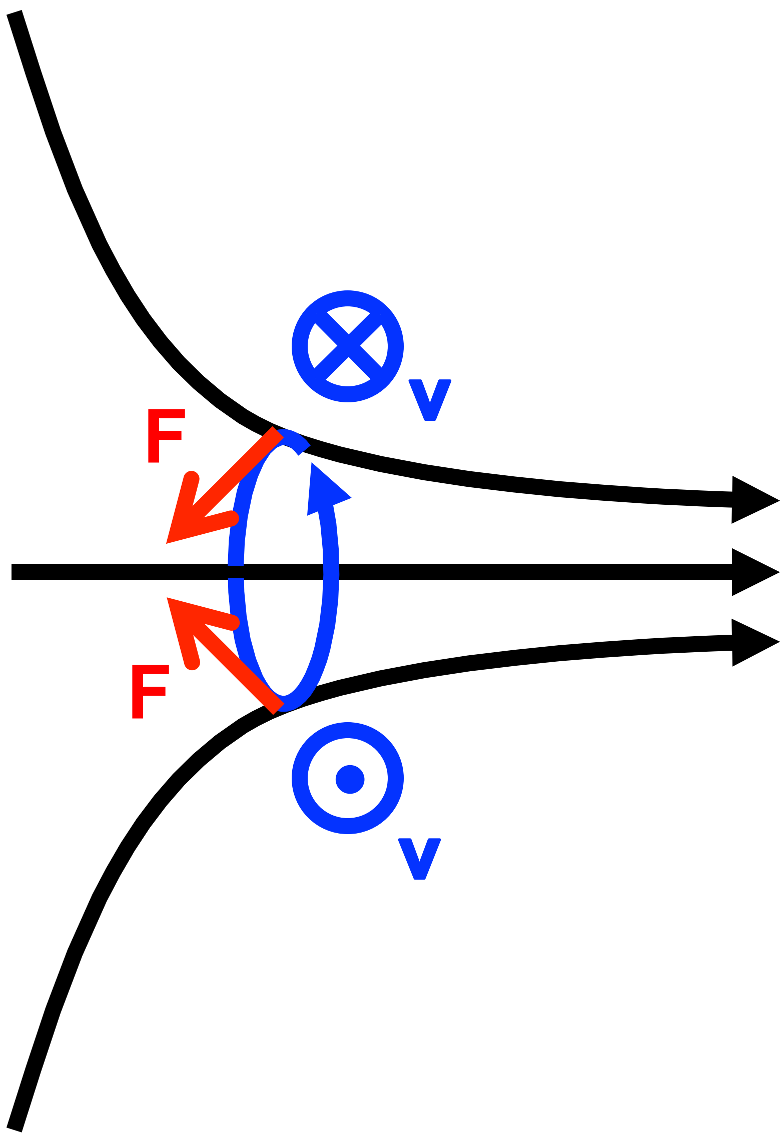
Magnetic moment is an adiabatic invariant:

$$J = \oint p dq$$

$$p = mv_x, \;\; q = x$$

$$J = \oint_0^{2\pi/\Omega} mv_\perp^2 \sin^2(\Omega t) dt \propto \mu$$

Solution 1: Magnetic mirror



Magnetic moment and energy conserved:

$$\frac{mv_{\parallel}^2}{2} = E - \mu B$$

At bounce point

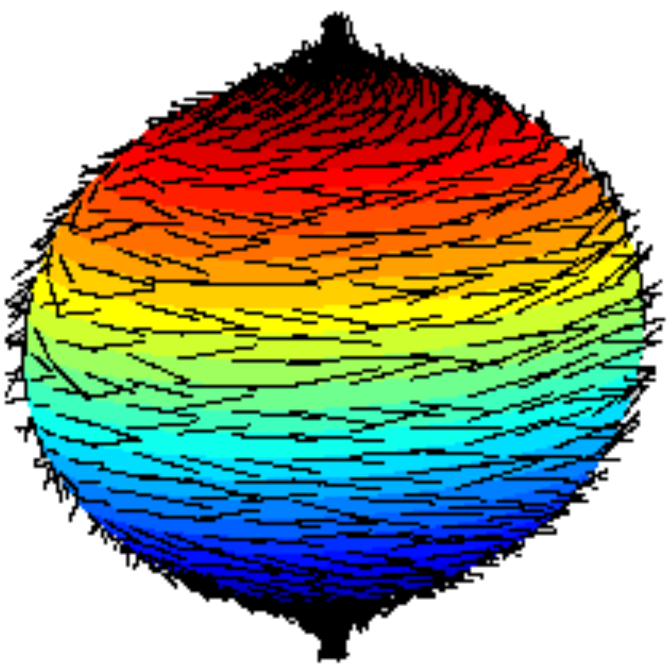
$$E = \mu B_{max}$$

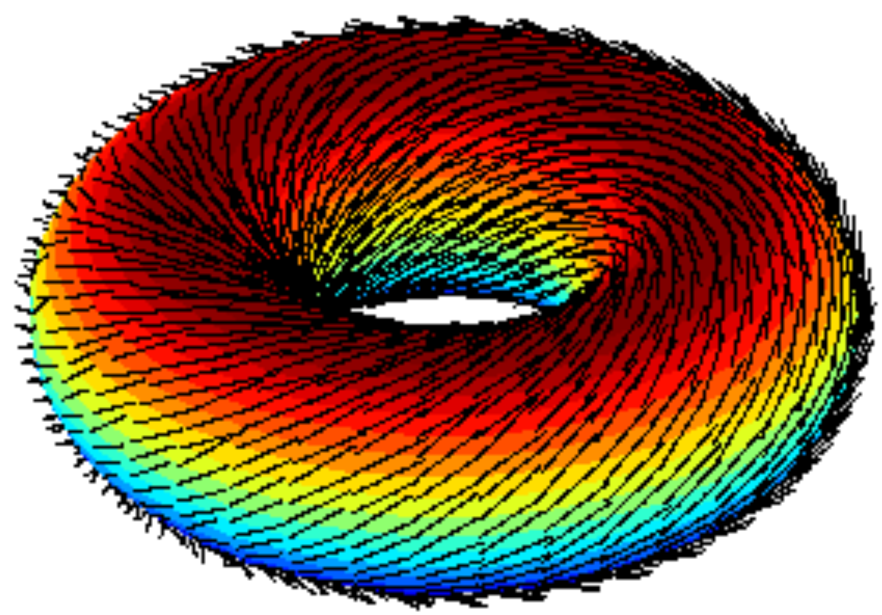
Particles bounce if

$$\left|\frac{v_{\perp}}{v}\right|_{B_{min}} > \sqrt{\frac{B_{min}}{B_{max}}}$$

Solution 2: confined field line trajectory

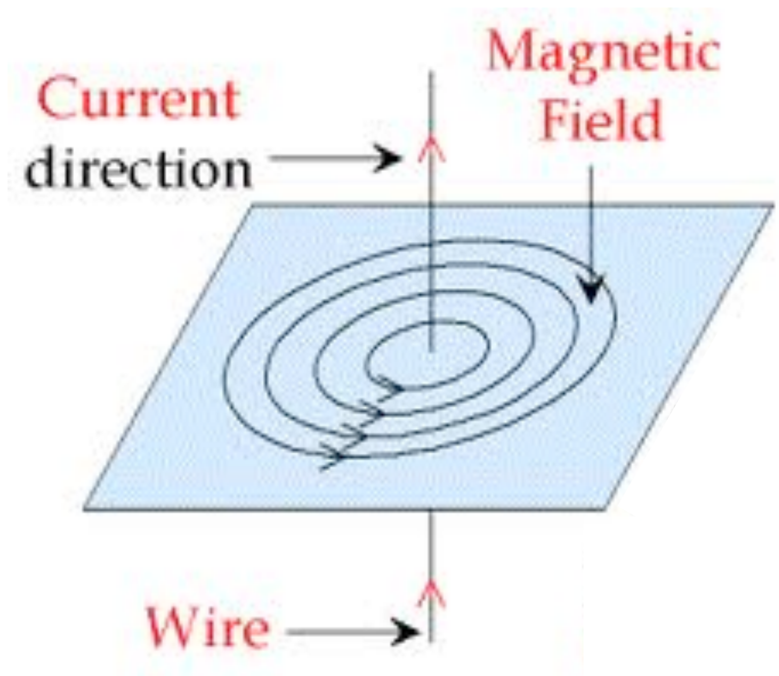
- ‘Hairy ball theorem’ → confined trajectories of vector field possible only for torii



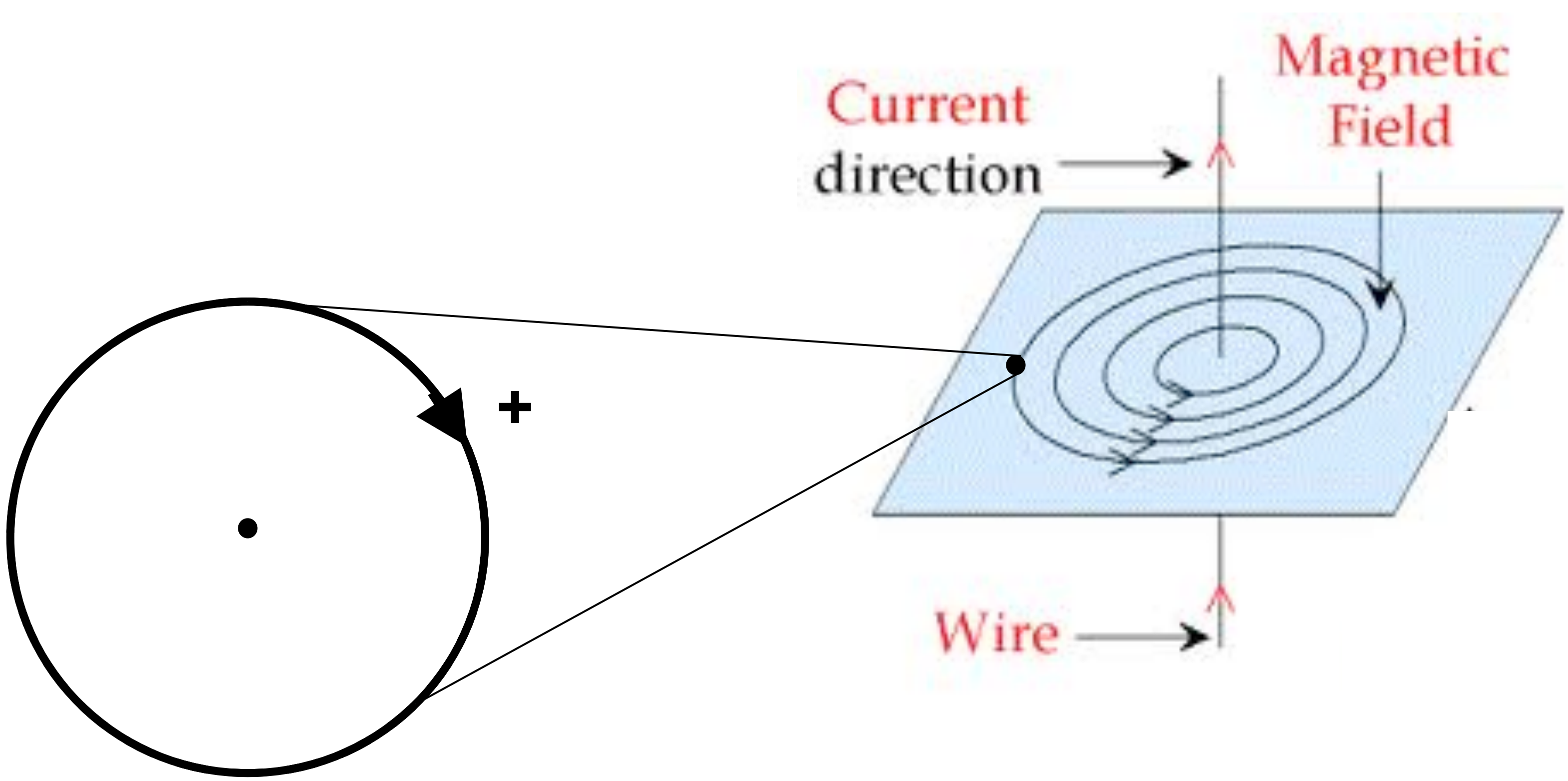


Solution 2: confined field line trajectory

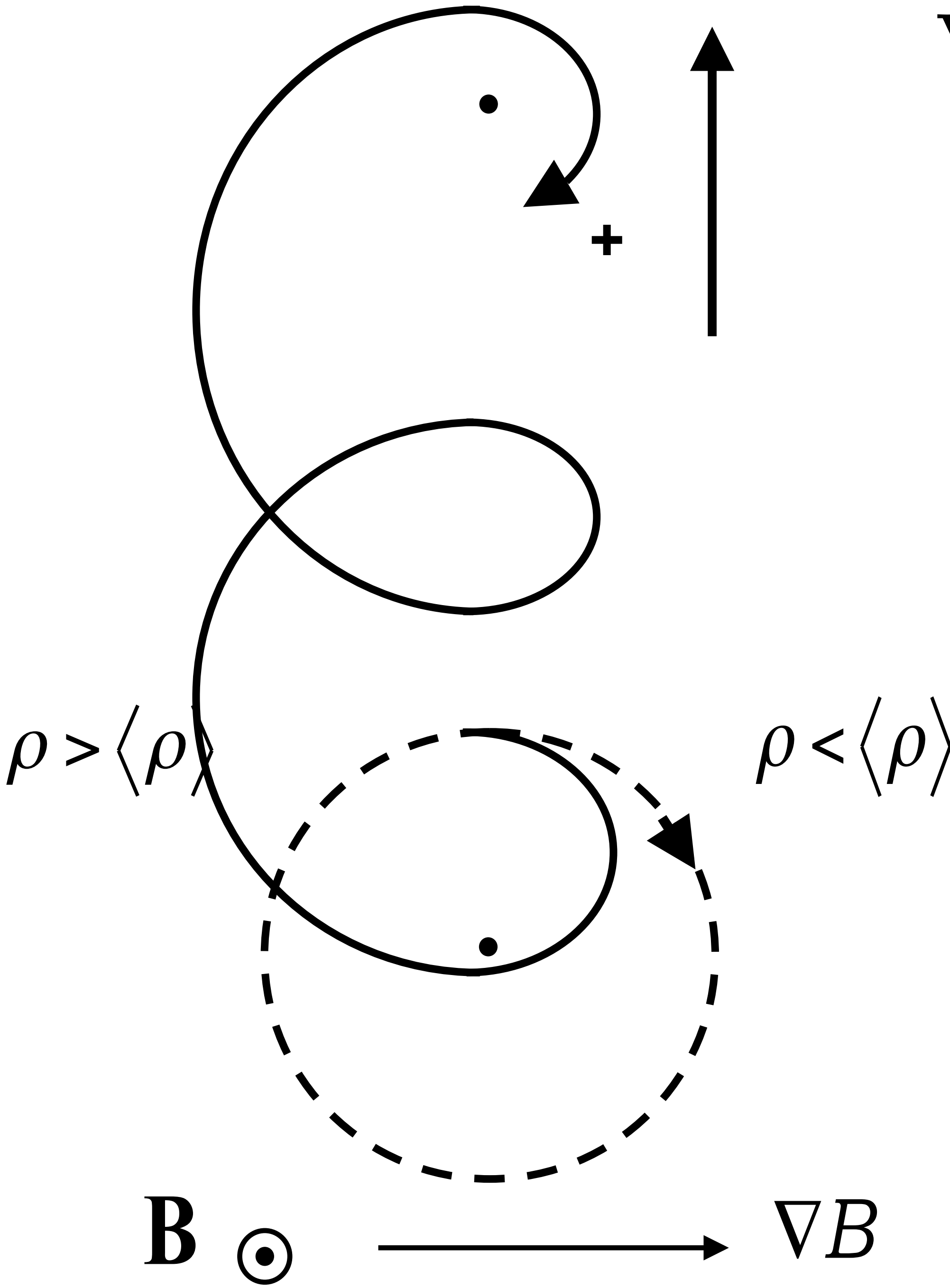
- Once we confine ourselves to torus, there are three possibilities: closed lines (1D), surfaces (2D), or toroidal annuli (3D)
- Simplest idea is circles:



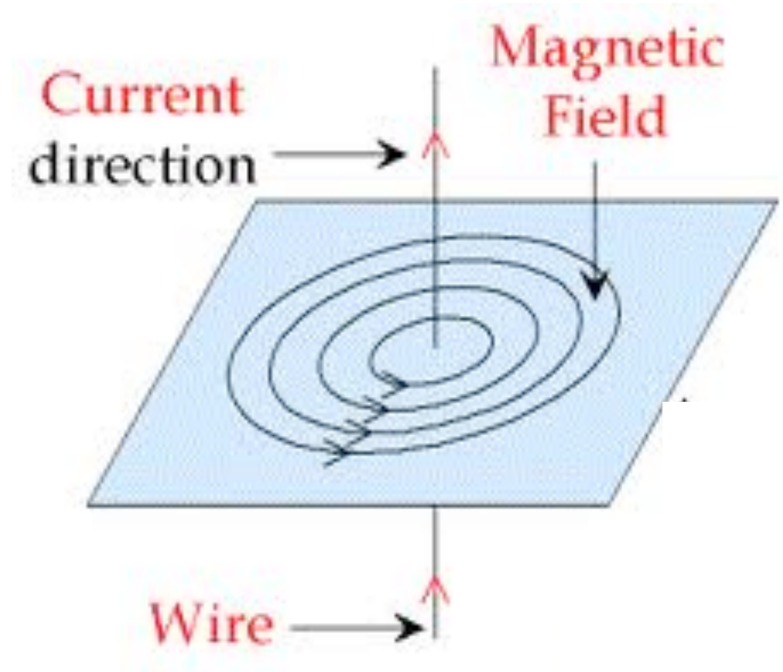
$$B = \frac{\mu_0 I}{2\pi R}$$



$\mathbf{B}\ \odot \qquad \longrightarrow\ \nabla B \qquad\qquad \rho = v_\perp/\Omega \propto 1/B$



$$\mathbf{v}_d = \frac{c\mathbf{F}_s \times \mathbf{B}}{e_s B^2} = \frac{v_\perp^2}{2}\left(\frac{\hat{b}}{\Omega_s} \times \frac{\nabla B}{B}\right)$$



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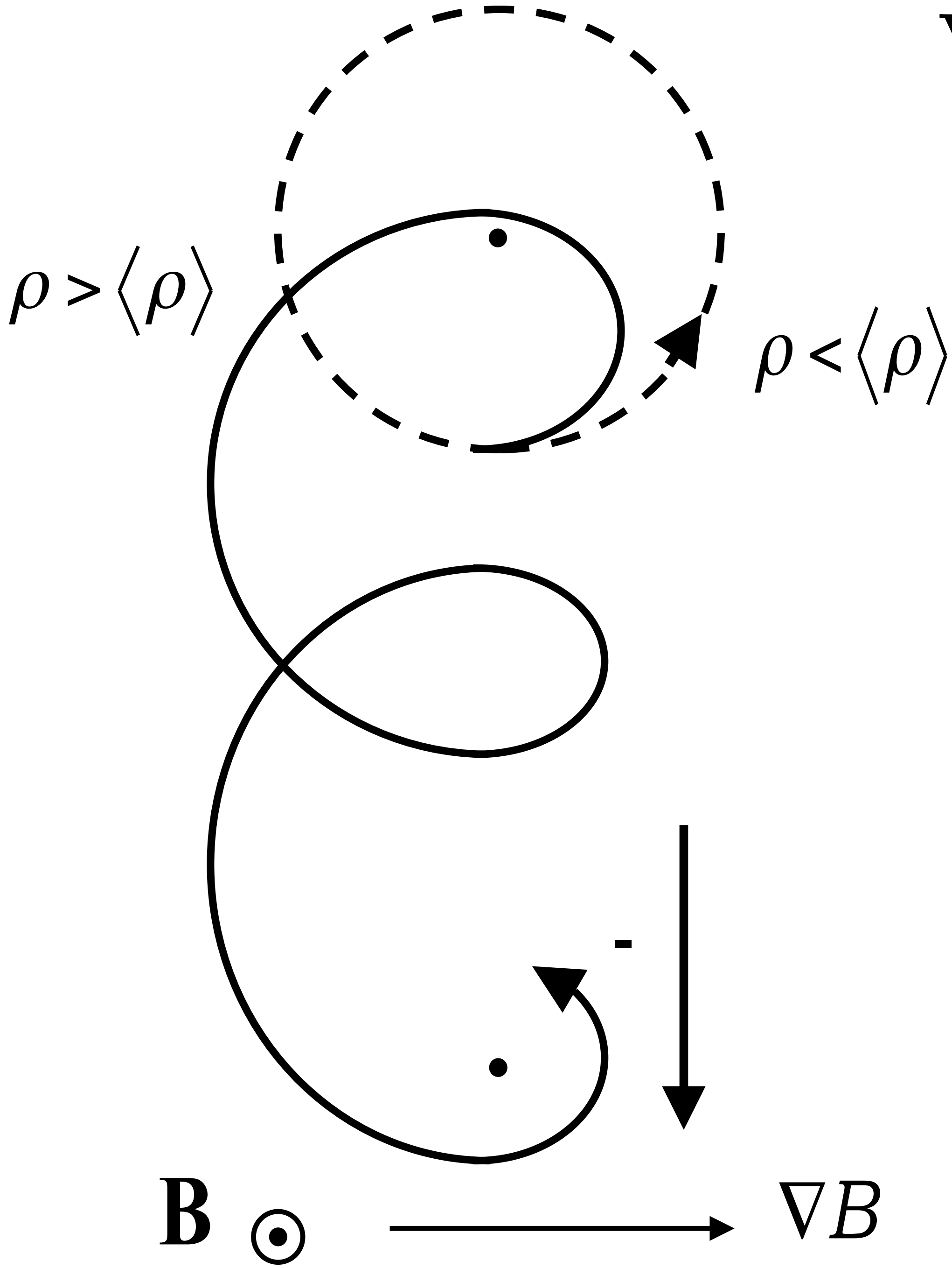

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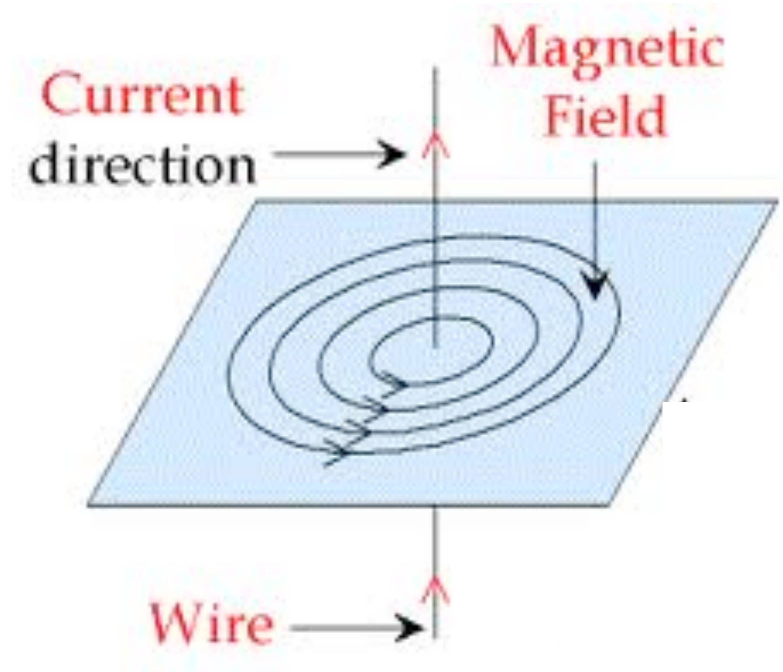

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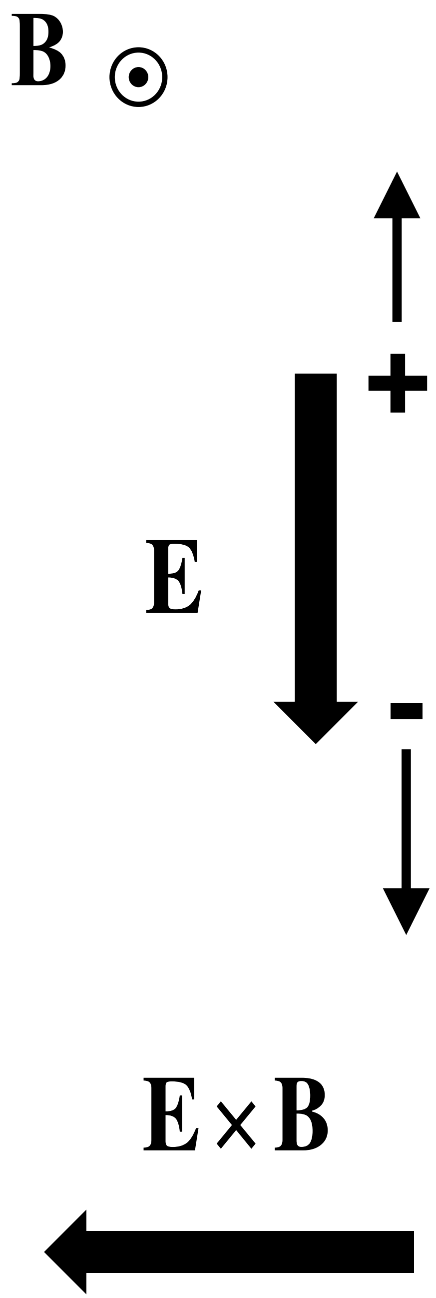
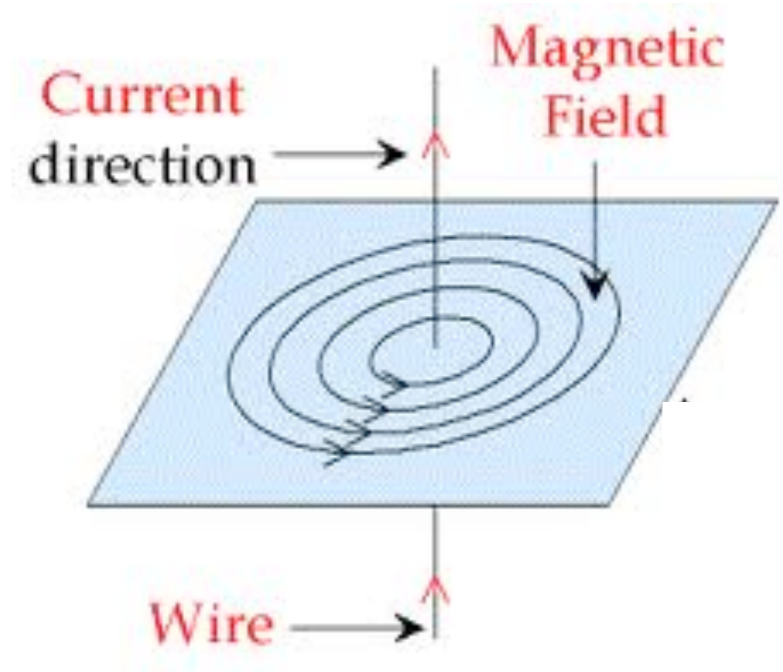
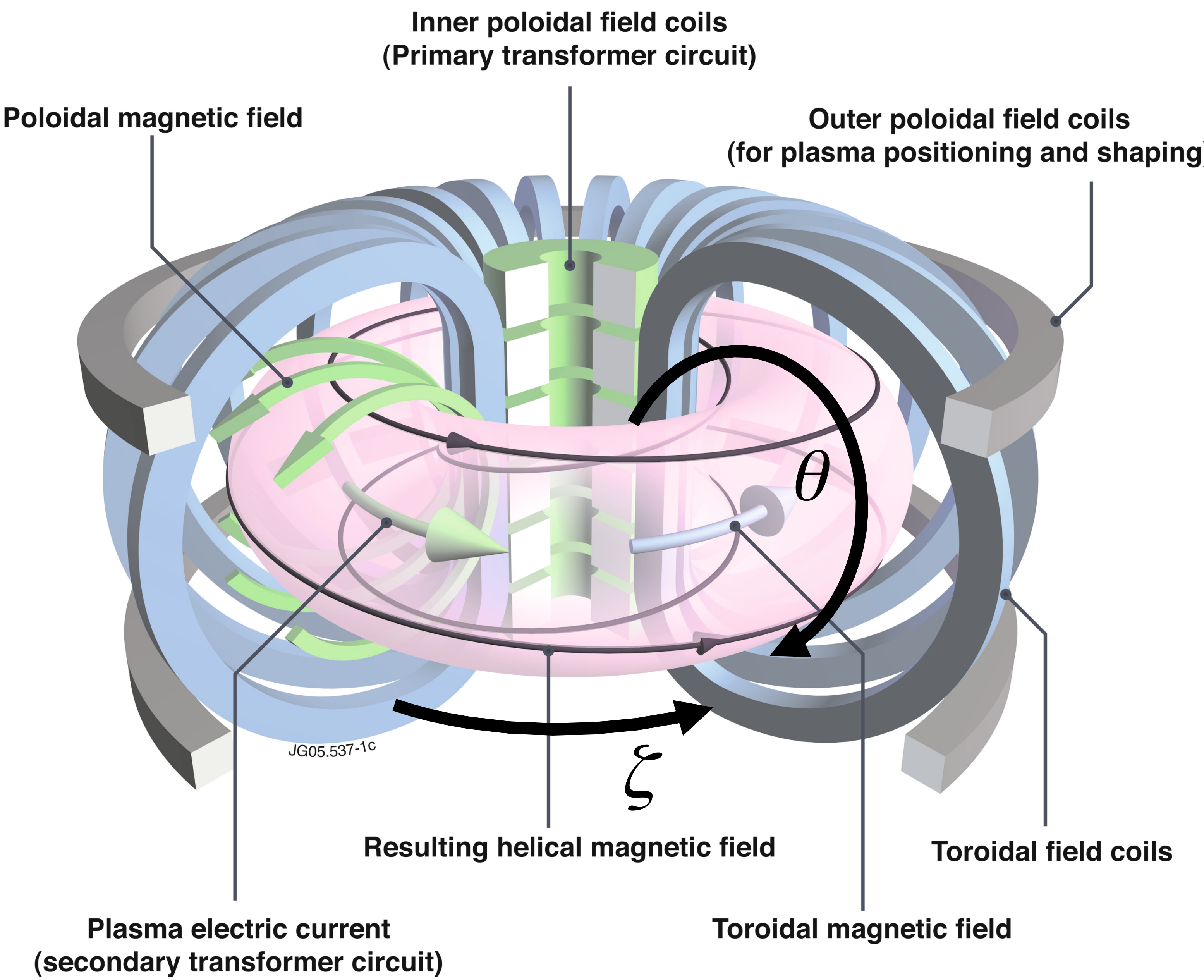

B ⊙
+
E
-
E×B

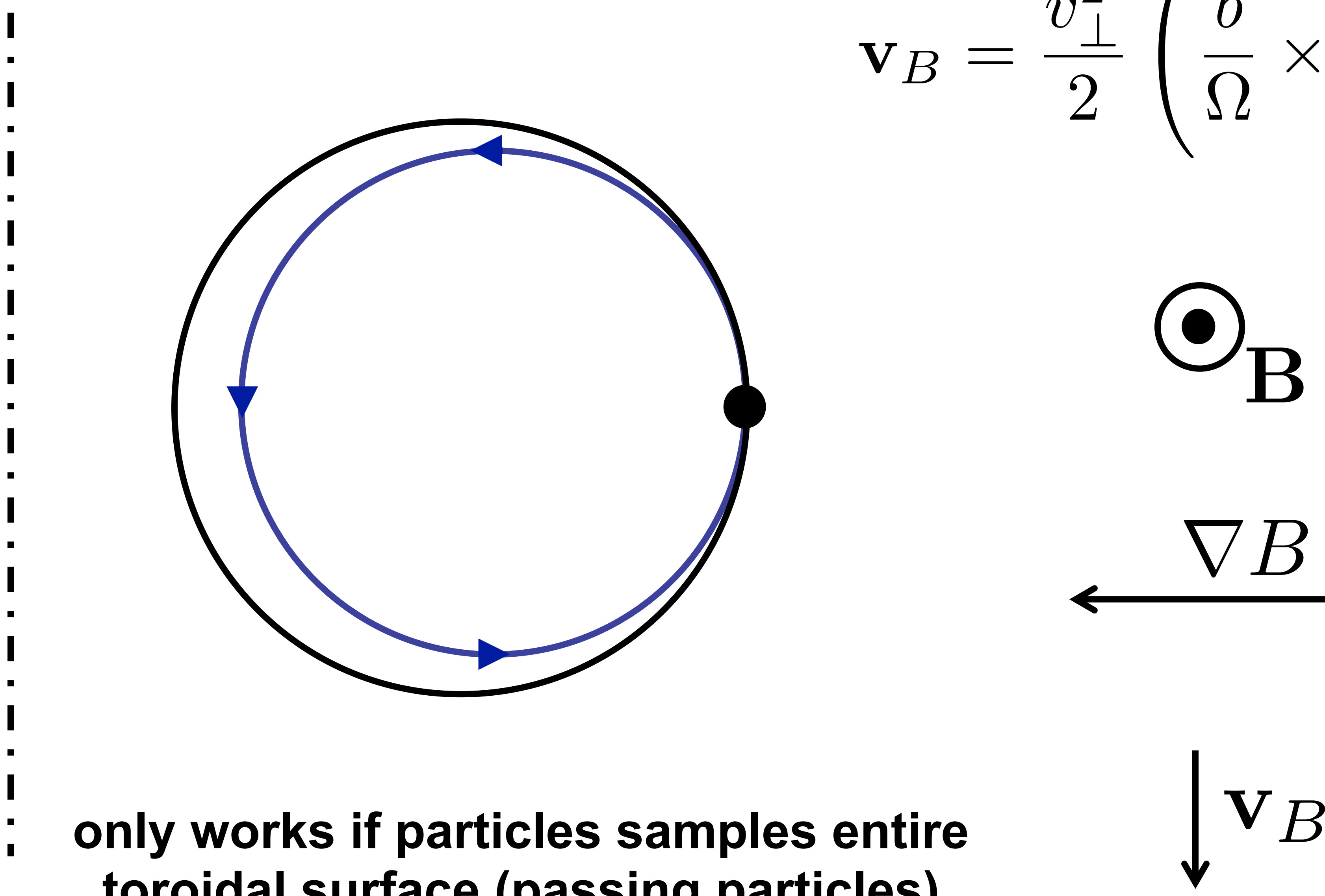

Magnetic Field
Current direction
Wire

The solution for solution 2? Add a twist



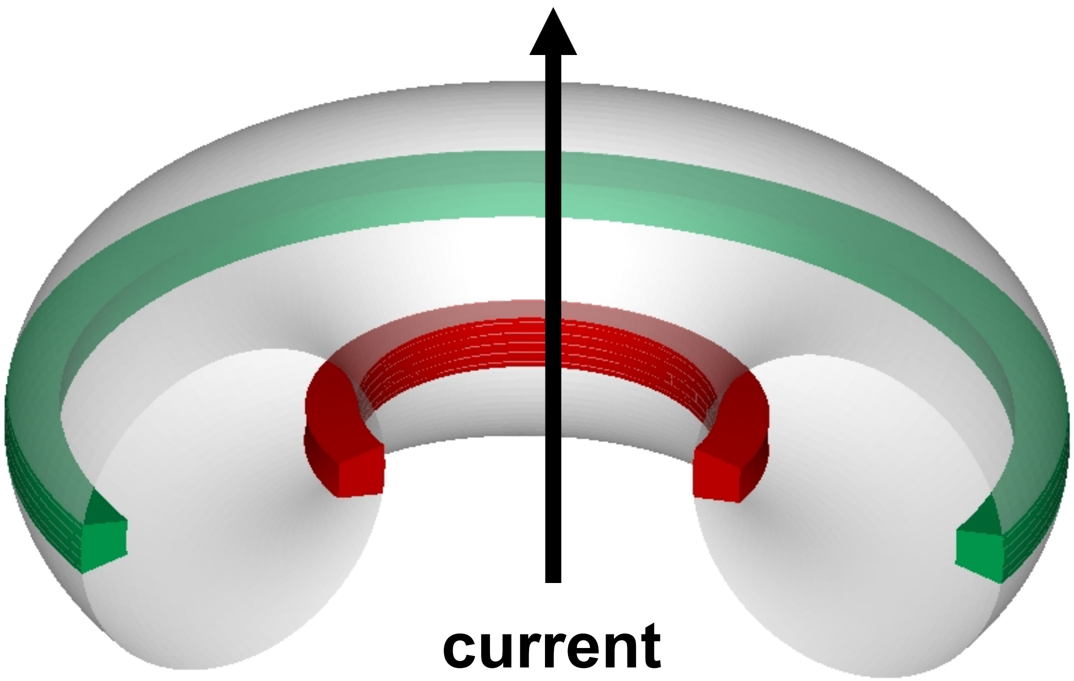
Magnetic drifts close, so no net drift

$$\mathbf{v}_B = \frac{v_\perp^2}{2}\left(\frac{\hat{b}}{\Omega} \times \frac{\nabla B}{B}\right)$$

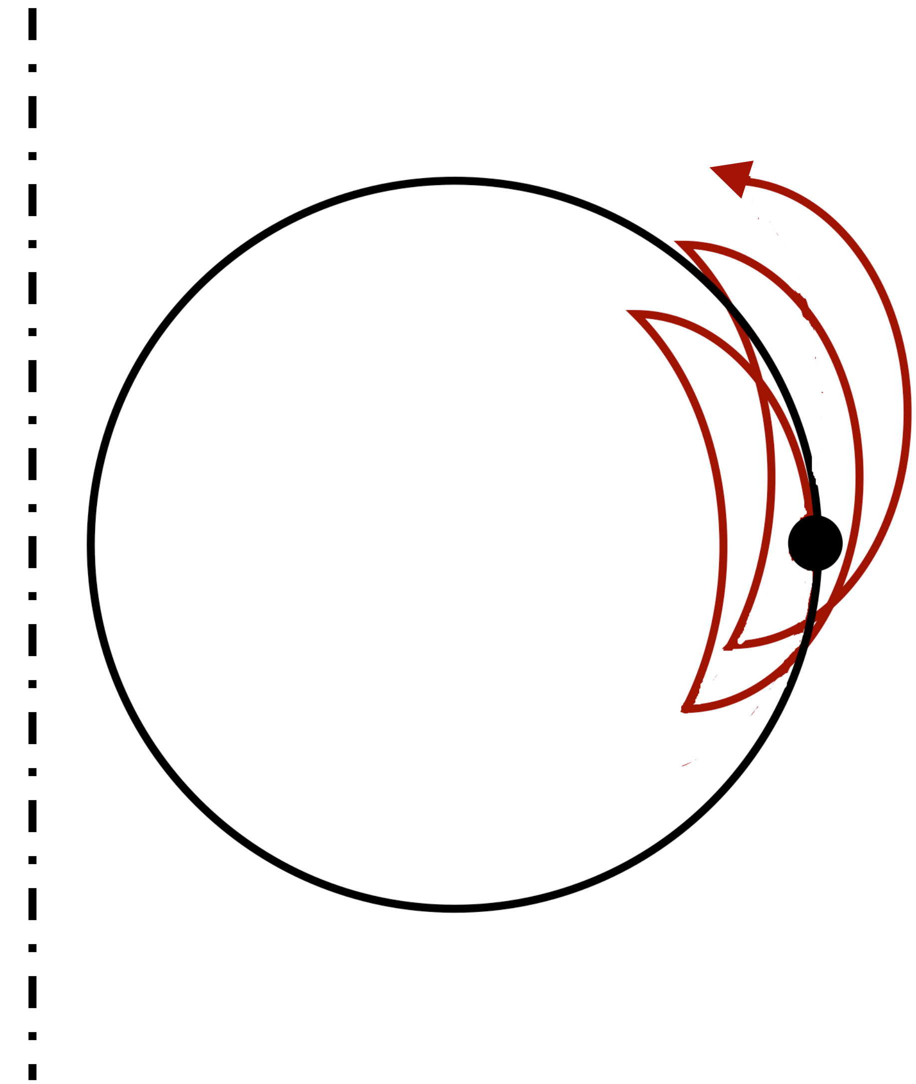


only works if particles samples entire toroidal surface (passing particles)

Complication: trapped particles

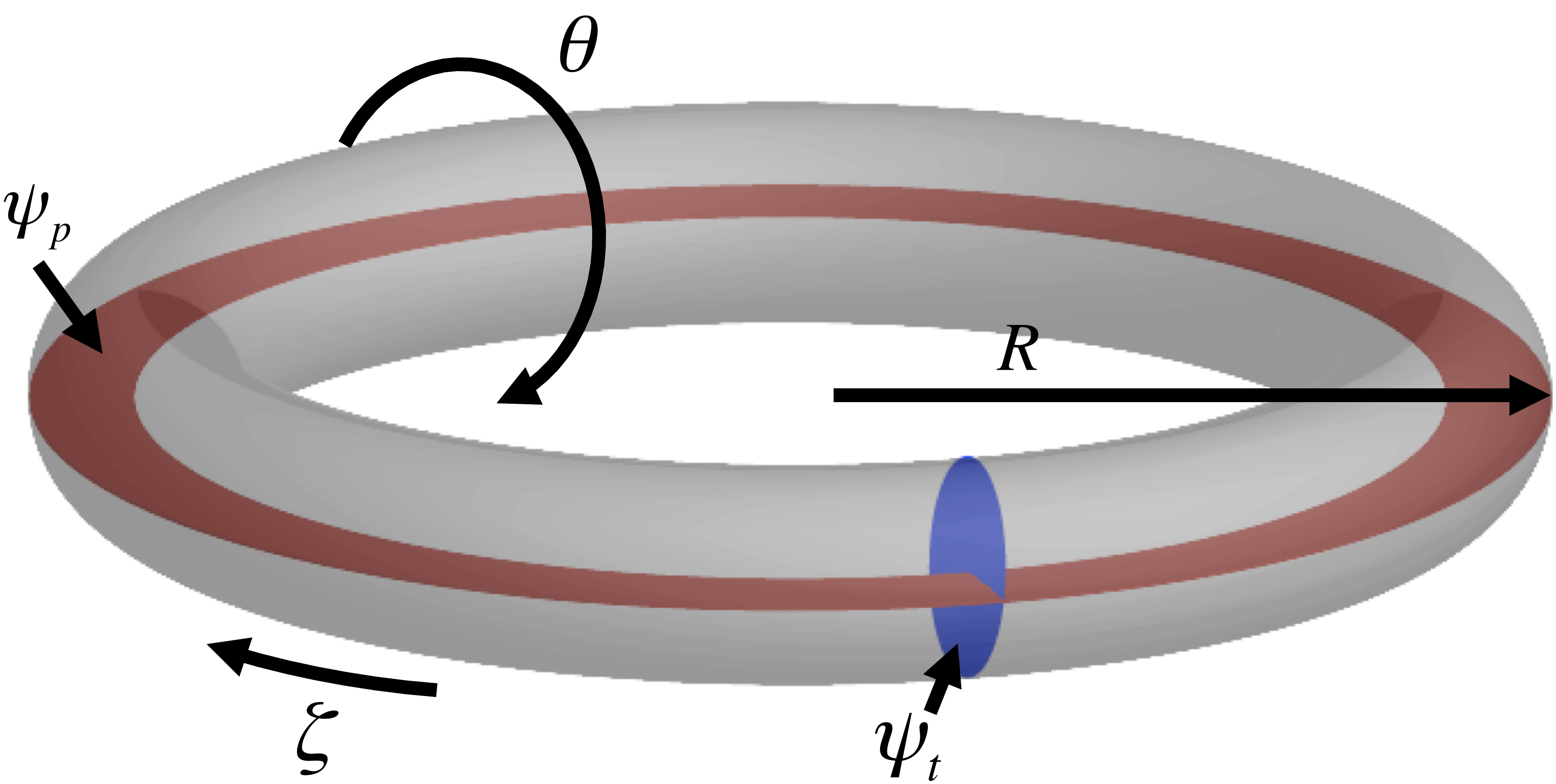


$$B_\zeta \propto 1/R \Rightarrow B = B(\theta)$$



Magnetic coordinates

$$\psi_p = \oint \mathbf{B} \cdot d\mathbf{a}^{\theta} \qquad \psi_t = \oint \mathbf{B} \cdot d\mathbf{a}^{\zeta}$$



Tamm's Theorem: no average radial drift in axisymmetric torus

If confining field axisymmetric, canonical angular momentum conserved:

$$p_\zeta = Rmv_\zeta + \frac{ZeRA_\zeta}{c} = const$$

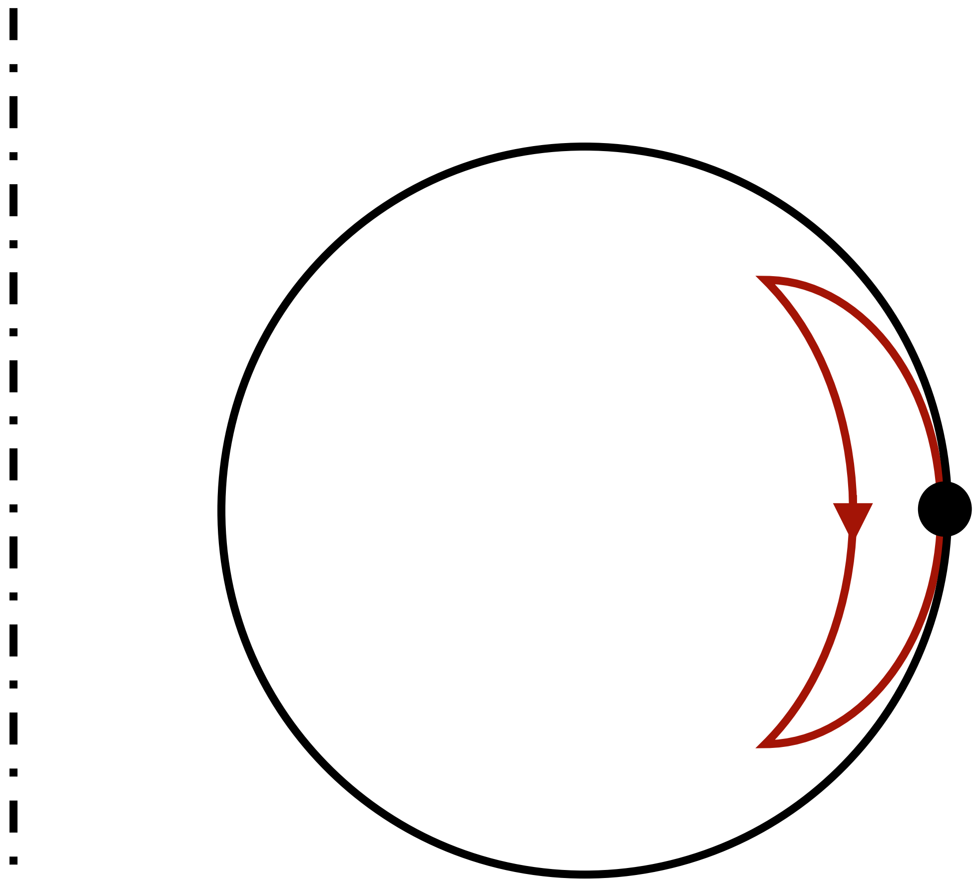
$$\psi_p = -RA_\zeta \sim R^2 B_p$$

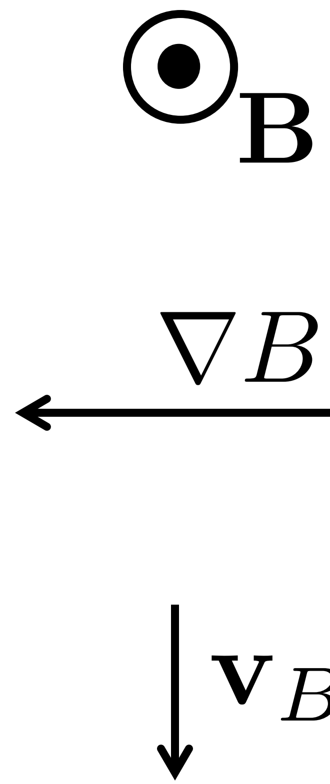
$$\psi_* = \psi_p - \frac{mcRv_\zeta}{Ze} = const$$

$$\psi_p = const \times (1 + O[\rho / R])$$

Trapped particles in axisymmetric torus

$$\mathbf{v}_B = \frac{v_\perp^2}{2}\left(\frac{\hat{b}}{\Omega} \times \frac{\nabla B}{B}\right)$$





Trapped orbits close on themselves while precessing toroidally

No average radial drift in quasisymmetric torus

Other (helical) symmetries also eliminate average radial drift. Consider

$$\psi_* = \psi_p - \underbrace{\frac{I(\psi_p)v_\|}{\Omega}}$$

Small for I ~ B/R

No average radial drift in quasisymmetric torus

Other (helical) symmetries also eliminate average radial drift. Consider

$$\psi_* = \psi_p - \frac{I(\psi_p)v_\parallel}{\Omega}$$

$$\frac{d\psi_*}{dt} = \frac{\partial \psi_*}{\partial t} + (\mathbf{v}_\parallel + \mathbf{v}_d)\cdot\nabla\psi_*$$

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$$\frac{d\psi_*}{dt} = \cancel{\frac{\partial \psi_*}{\partial t}} + (\mathbf{v}_\parallel + \mathbf{v}_d) \cdot \nabla \psi_*$$

$$= \mathbf{v}_\parallel \cdot \nabla \psi_p + \mathbf{v}_d \cdot \nabla \psi_p - I \mathbf{v}_\parallel \cdot \nabla \left(\frac{v_\parallel}{\Omega} \right) - \mathbf{v}_d \cdot \nabla \left(\frac{I v_\parallel}{\Omega} \right)$$

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$$= \underbrace{\cancel{\mathbf{v}_\parallel \cdot \nabla \psi_p}}_{= 0} + \mathbf{v}_d \cdot \nabla \psi_p - I \mathbf{v}_\parallel \cdot \nabla \left(\frac{v_\parallel}{\Omega} \right) - \cancel{\mathbf{v}_d \cdot \nabla \left(\frac{I v_\parallel}{\Omega} \right)}$$

Small in ρ / R

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$$= \frac{2v_\parallel^2 + v_\perp^2}{\Omega B^2}\left(\mathbf{B}\times\nabla\psi_p\cdot\nabla B - I\mathbf{B}\cdot\nabla B\right)$$

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$$= \frac{2v_\parallel^2 + v_\perp^2}{\Omega B^2}\left(\mathbf{B}\times\nabla\psi_p\cdot\nabla B - I\mathbf{B}\cdot\nabla B\right)$$

$$\psi_p \approx const \quad \text{if} \quad \frac{\mathbf{B}\times\nabla\psi_p\cdot\nabla B}{\mathbf{B}\cdot\nabla B} = I(\psi_p)$$

Magnetic field topology

- Choose coordinates $(\psi_t, \vartheta, \zeta)$

$$2\pi\mathbf{B} = \nabla\psi_t \times \nabla\vartheta + \nabla\zeta \times \nabla\psi_p(\psi_t, \vartheta, \zeta)$$

- Trajectory of magnetic field line given by

$$\frac{dx^i}{d\tau} = \mathbf{B}\cdot\nabla x^i = B^i$$

- Choose zeta as time-like coordinate:

$$\frac{d\psi_t}{d\zeta} = \frac{\mathbf{B}\cdot\nabla\psi_t}{\mathbf{B}\cdot\nabla\zeta} = -\frac{\partial\psi_p}{\partial\vartheta}, \quad \frac{d\vartheta}{d\zeta} = \frac{\mathbf{B}\cdot\nabla\vartheta}{\mathbf{B}\cdot\nabla\zeta} = \overset{\substack{1/q \\ \|}}{\frac{\partial\psi_p}{\partial\psi_t}}$$

Relation to Hamiltonian systems

$$\frac{d\psi_t}{d\zeta} = -\frac{\partial\psi_p}{\partial\vartheta} \qquad \frac{d\vartheta}{d\zeta} = \frac{\partial\psi_p}{\partial\psi_t}$$

- Identify $\psi_p \leftrightarrow H,\ \zeta \leftrightarrow t,\ \theta \leftrightarrow x,\ \psi_t \leftrightarrow p$

$$\frac{dx}{dt} = \frac{\partial H}{\partial p},\ \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

- 1.5 degree Hamiltonian system, allows:
  - 1D trajectories (closed lines)
  - 2D trajectories (ergodically map toroidal surfaces)
  - 3D trajectories (volume-filling)

With symmetry direction

$$\frac{d\psi_t}{d\zeta} = 0 = -\frac{\partial \psi_p}{\partial \vartheta} \qquad \frac{d\vartheta}{d\zeta} = \frac{\partial \psi_p}{\partial \psi_t}$$

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$$\frac{dx}{dt} = \frac{\partial H}{\partial p},\ \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

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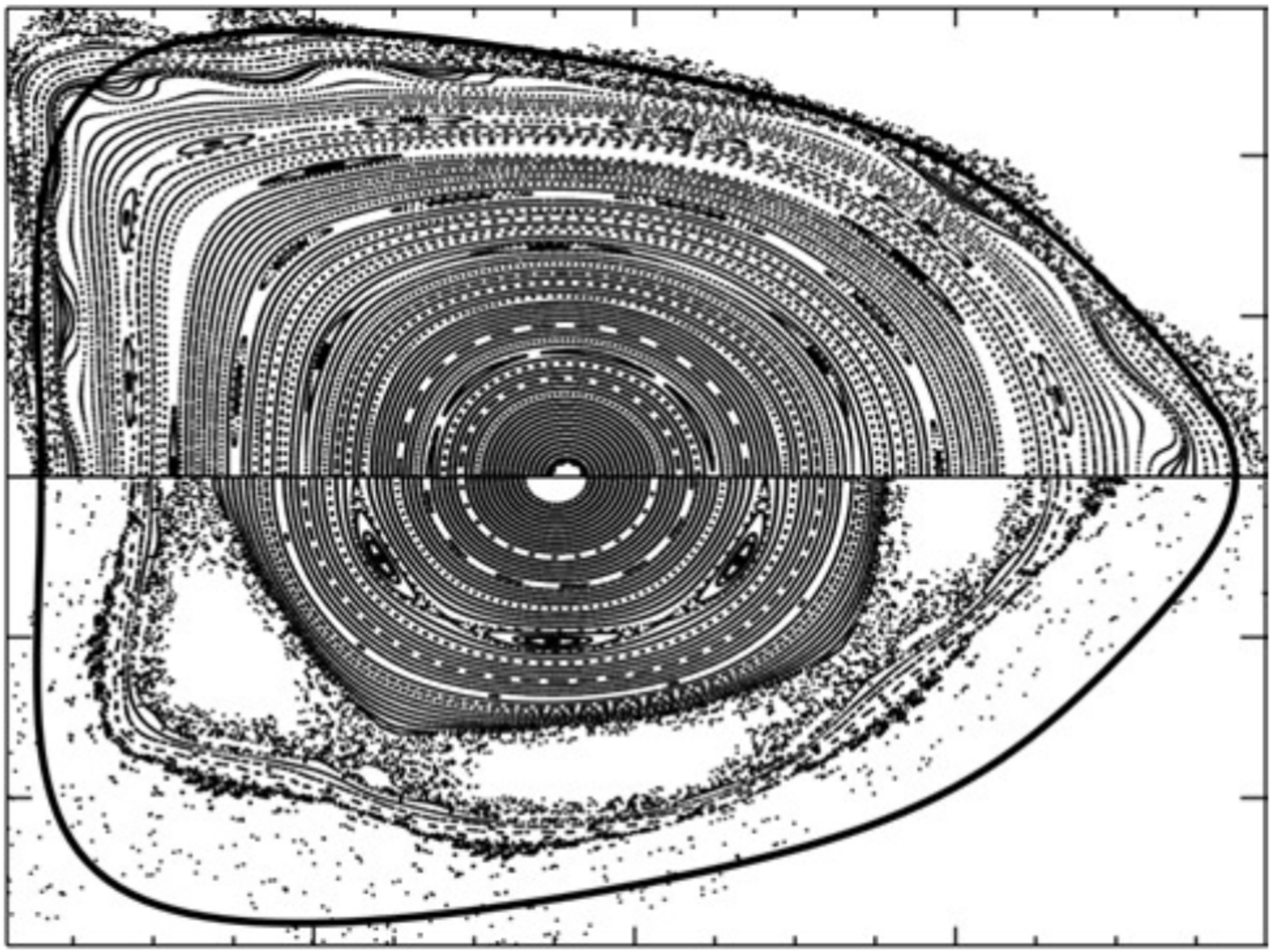
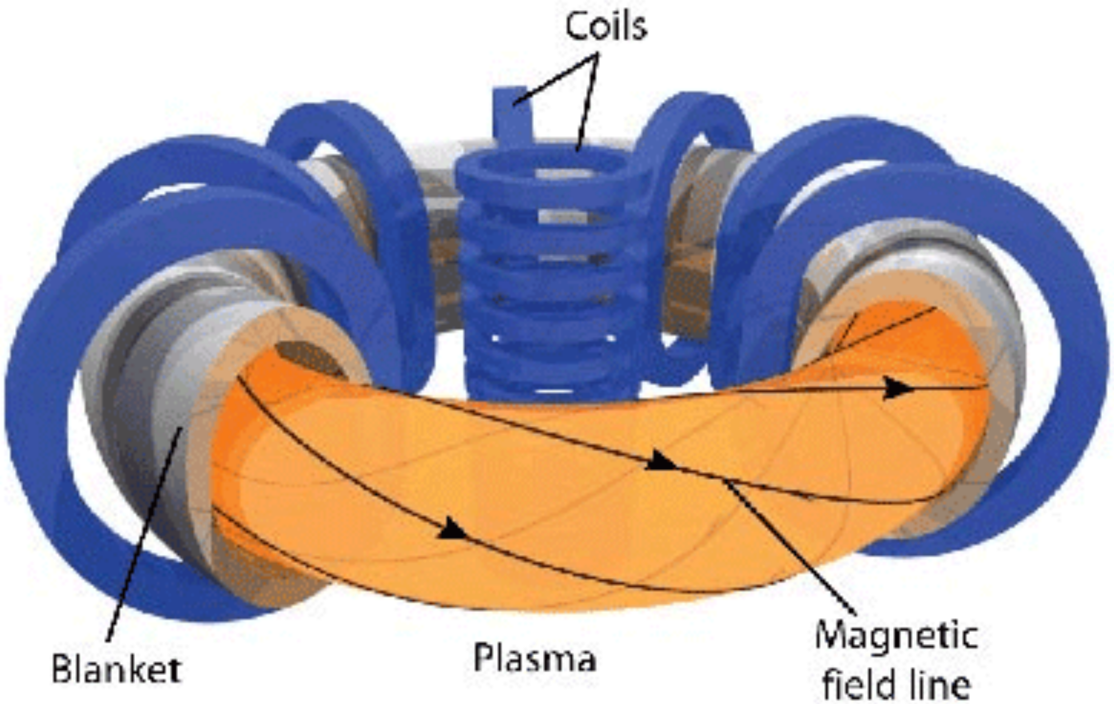
  - 1D trajectories (closed lines)
  - 2D trajectories (ergodically map toroidal surfaces)
  - ~~3D trajectories (volume-filling)~~

Field line examples



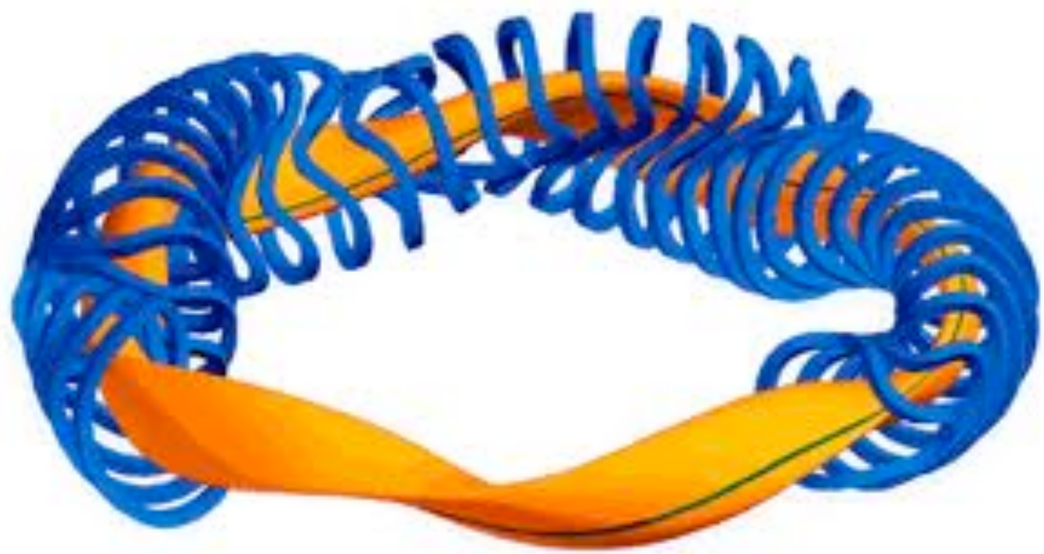
Hudson et al, PRL 2002

Tokamaks and stellarators



- Confined drift orbits
- Existence of flux surfaces
- Simpler design/ construction

- Confining poloidal field generated externally
- More flexibility in shaping



Magnetized plasma equilibrium

$$\frac{\partial(m_s n_s \mathbf{u}_s)}{\partial t} + \nabla \cdot \mathbf{P}_s - e_s n_s \left(\mathbf{E} + \frac{\mathbf{u}_s \times \mathbf{B}}{c} \right) = \mathbf{F}_s$$

Magnetized plasma equilibrium

$$\cancel{\frac{\partial(m_s n_s \mathbf{u}_s)}{\partial t}} + \nabla \cdot \mathbf{P}_s - e_s n_s \left(\mathbf{E} + \frac{\mathbf{u}_s \times \mathbf{B}}{c} \right) = \cancel{\mathbf{F}_s}$$

equilibrium

Momentum conservation (when summed over species)

Magnetized plasma equilibrium

$$\cancel{\frac{\partial(m_s n_s \mathbf{u}_s)}{\partial t}} + \nabla\cdot\mathbf{P}_s - e_s n_s\left(\mathbf{E} + \frac{\mathbf{u}_s\times\mathbf{B}}{c}\right) = \cancel{\mathbf{F}_s}$$

Quasineutrality:

$$\nabla^2\Phi = -4\pi\sum_s e_s n_s \qquad\qquad \sum_s e_s n_s \sim e n_e\left(\frac{\lambda_D}{L}\right)^2$$

Near thermodynamic equilibrium: $\nabla\cdot\mathbf{P}_s = \nabla p_s$

$$c\nabla p = \mathbf{J}\times\mathbf{B}$$

$$\Longrightarrow \quad \mathbf{B}\cdot\nabla p = \mathbf{J}\cdot\nabla p = 0$$

I.e., parallel equilibration time short compared to confinement time (parallel streaming for collisionless or sound wave propagation for collisional)

Quasineutrality and return currents

Another consequence of quasineutrality is $\nabla \cdot \mathbf{J} = 0$

Charge conservation: $$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

Quasineutrality: $$\sum_s e_s n_s \sim e n_e \left(\frac{\lambda_D}{L}\right)^2$$

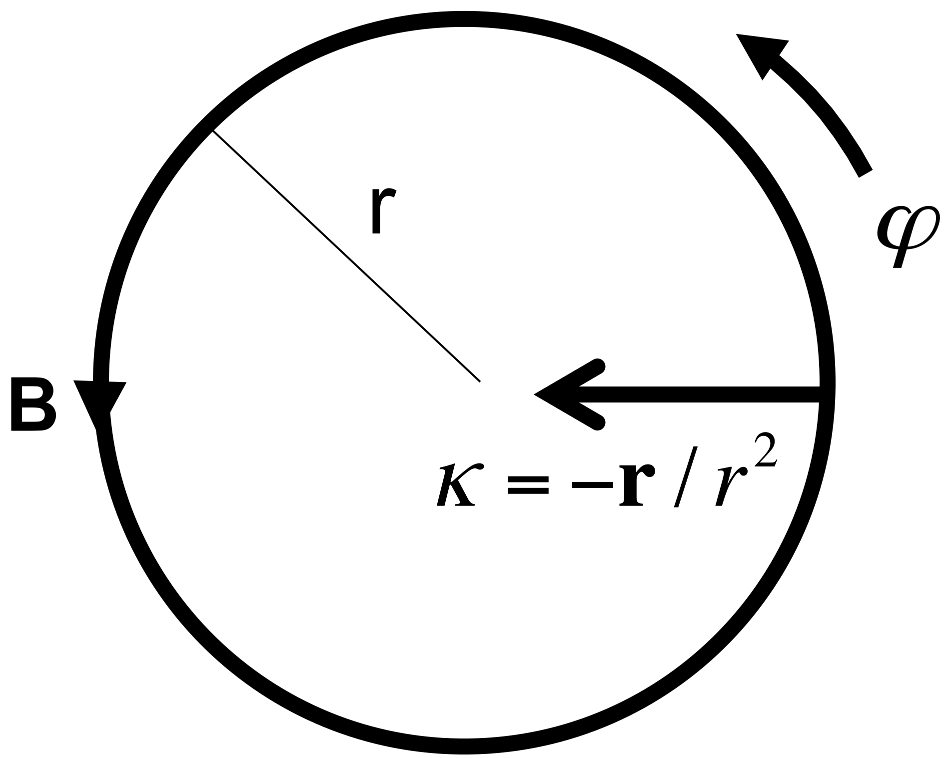
$$\Longrightarrow \quad \nabla \cdot \mathbf{J} \sim \frac{\rho}{\tau} \sim \left(\frac{\lambda_D}{L}\right)^2 \frac{e n_e}{\tau}$$

$$\sim \frac{J}{L}\left(\frac{v_{th,e}}{c}\right)^2 \frac{1}{\Omega_e \tau} \ll \frac{J}{L}$$

$$c\nabla p = \mathbf{J}_\perp \times \mathbf{B} \quad \Longrightarrow \quad \nabla \cdot J_{||} \neq 0$$

Alternative physical interpretation

$$\boldsymbol{\kappa} \equiv \mathbf{b} \cdot \nabla \mathbf{b} = \frac{4\pi}{c} \frac{\mathbf{J} \times \mathbf{B}}{B^2} + \frac{\nabla_\perp B}{B}$$



Alternative physical interpretation

$$\boldsymbol{\kappa} \equiv \mathbf{b} \cdot \nabla \mathbf{b} = \frac{4\pi}{c} \frac{\mathbf{J} \times \mathbf{B}}{B^2} + \frac{\nabla_\perp B}{B}$$

$$c \nabla p = \mathbf{J} \times \mathbf{B}$$

$$\frac{B^2}{4\pi} \boldsymbol{\kappa} = \nabla_\perp \left(p + \frac{B^2}{4\pi} \right)$$

Total pressure gradient (plasma + magnetic) balanced by field line tension in equilibrium

Low plasma beta

$$\boldsymbol{\kappa} \equiv \mathbf{b} \cdot \nabla \mathbf{b} = \frac{4\pi}{c} \frac{\mathbf{J} \times \mathbf{B}}{B^2} + \frac{\nabla_\perp B}{B}$$

$$c \nabla p = \mathbf{J} \times \mathbf{B}$$

$$\beta = \frac{4\pi p}{B^2} \ll 1 \Rightarrow \frac{B^2}{4\pi} \boldsymbol{\kappa} \approx \nabla_\perp \frac{B^2}{4\pi}$$

Magnetic pressure gradient balanced by tension in equilibrium

Axisymmetric equilibrium

Radial component of Ampere's Law:

$$\frac{4\pi}{c} J^r = \mathcal{J} \left(\frac{\partial B_\zeta}{\partial \theta} - \frac{\partial B_\theta}{\partial \zeta} \right) \qquad \mathcal{J}^{-1} = \nabla r \cdot (\nabla \theta \times \nabla \zeta)$$

$$\mathbf{J} \cdot \nabla p = 0 \Rightarrow J^r p' = 0$$

Axisymmetric equilibrium

Radial component of Ampere's Law:

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$$\Rightarrow \; B_\zeta = B_\zeta(r) \; \Rightarrow \; \mathbf{B} = \underbrace{I(r) \nabla \zeta}_{\mathbf{B}_t} + \underbrace{\nabla \zeta \times \nabla \psi_p}_{\mathbf{B}_p}$$

Axisymmetric equilibrium

Radial component of Ampere's Law:

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$$\Rightarrow \; B_\zeta = B_\zeta(r) \; \Rightarrow \; \mathbf{B} = I(r) \nabla \zeta + \nabla \zeta \times \nabla \psi_p$$

Radial force balance: $cp' = \mathcal{J}^{-1} B^\theta \left(q J^\theta - J^\zeta \right)$

$$J^\zeta = \frac{c}{4\pi} \nabla \zeta \cdot \nabla \times \mathbf{B} = \frac{c}{4\pi} \nabla \cdot \left(R^{-2} \nabla \psi_p \right)$$

$$J^\theta = -\frac{c}{4\pi} \frac{I I'}{q \psi_p' R^2}$$

Grad-Shafranov equation

$$R^2 \nabla \cdot \left(R^{-2} \nabla \psi_p\right) = -I \frac{dI}{d\psi_p} - 4\pi R^2 \frac{dp}{d\psi_p}$$

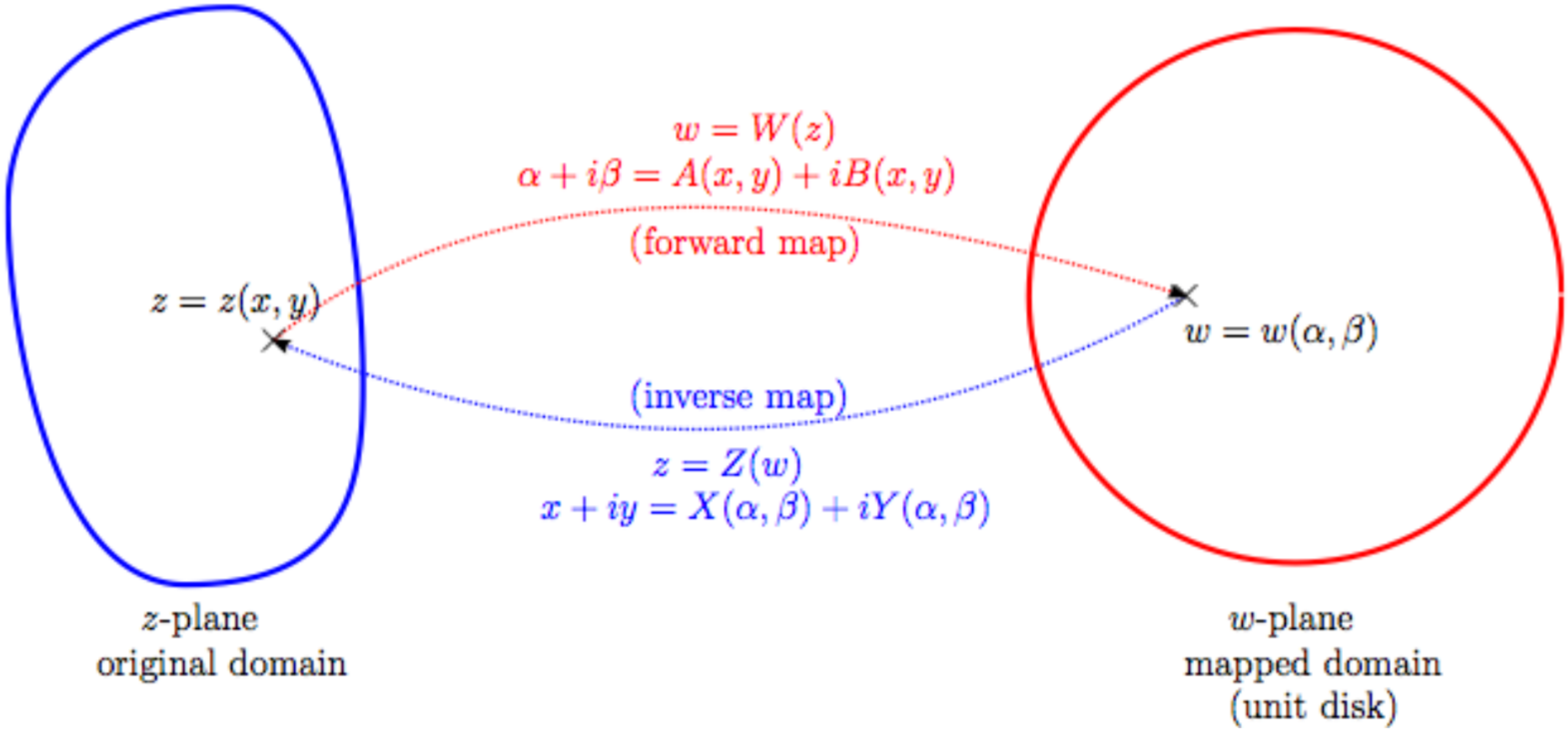
Use cylindrical (R,Z) coordinates and define $\psi_p \equiv u\sqrt{R}$:

$$\Delta u = -\frac{3}{4}\frac{u}{R^2} - 4\pi R \frac{dp}{du} - \frac{1}{2R}\frac{dI^2}{du}$$

Poisson's equation – given p(u), I(u), and boundary condition on u, can solve iteratively for u(x,y).

Let's look at an example of clever way to do this numerically.

Conformal map onto unit circle



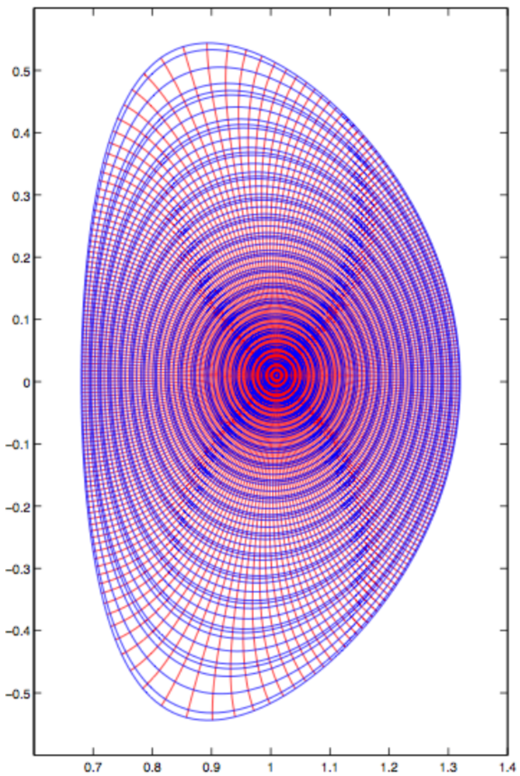
$$\Delta u(x,y) = f(x,y) \quad \rightarrow \quad \Delta v(\alpha,\beta) = f(X(\alpha,\beta), Y(\alpha,\beta)) \cdot \left|\frac{dZ}{dw}\right|^2$$

$$u|_{\partial\Omega} = g(x,y) \quad \rightarrow \quad v|_{\partial\Omega} = g(X(\alpha,\beta), Y(\alpha,\beta))$$

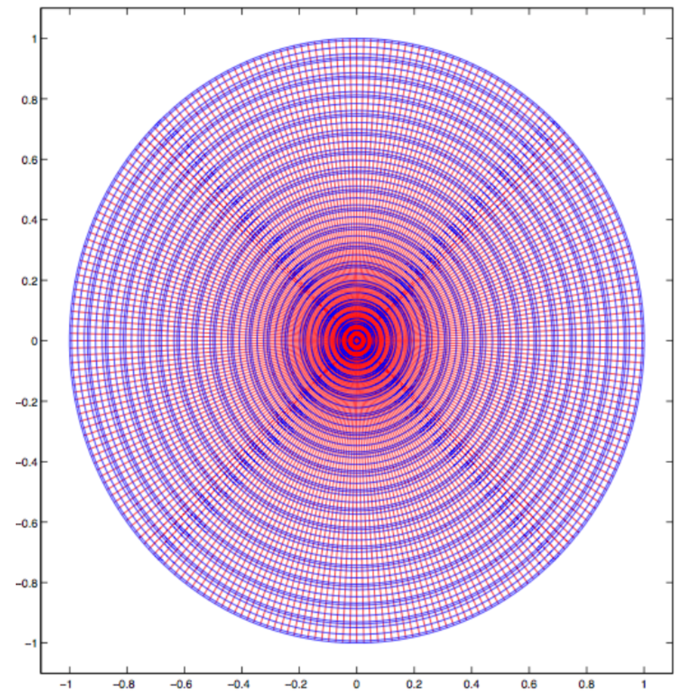
spectral: $$v(r,\theta) = \sum_n \hat{v}_n(r) e^{in\theta}, \quad h(r,\theta) = \sum_n \hat{h}_n(r) e^{in\theta}$$

$$\Rightarrow \quad r^2 \hat{v}_n''(r) + r\hat{v}_n'(r) - n^2 \hat{v}_n(r) = r^2 \hat{h}_n(r)$$

Conformal map onto unit circle



z-plane
original domain



w-plane
mapped domain
(unit circle)

Patakis et al., JCP 2013

Parallel current

$$R^2\nabla\cdot\left(R^{-2}\nabla\psi_p\right) = -I\frac{dI}{d\psi_p} - 4\pi R^2\frac{dp}{d\psi_p}$$

Calculation of $J_{||}$ and p required to close the equilibrium problem

Similar analysis that gave Grad-Shafranov gives parallel current:

$$J_{||} = -\frac{c}{4\pi}B\frac{dI}{d\psi_p} - \frac{cI}{B}\frac{dp}{d\psi_p} = \sum_s e_s\int d^3v\ v_{||}F_s$$

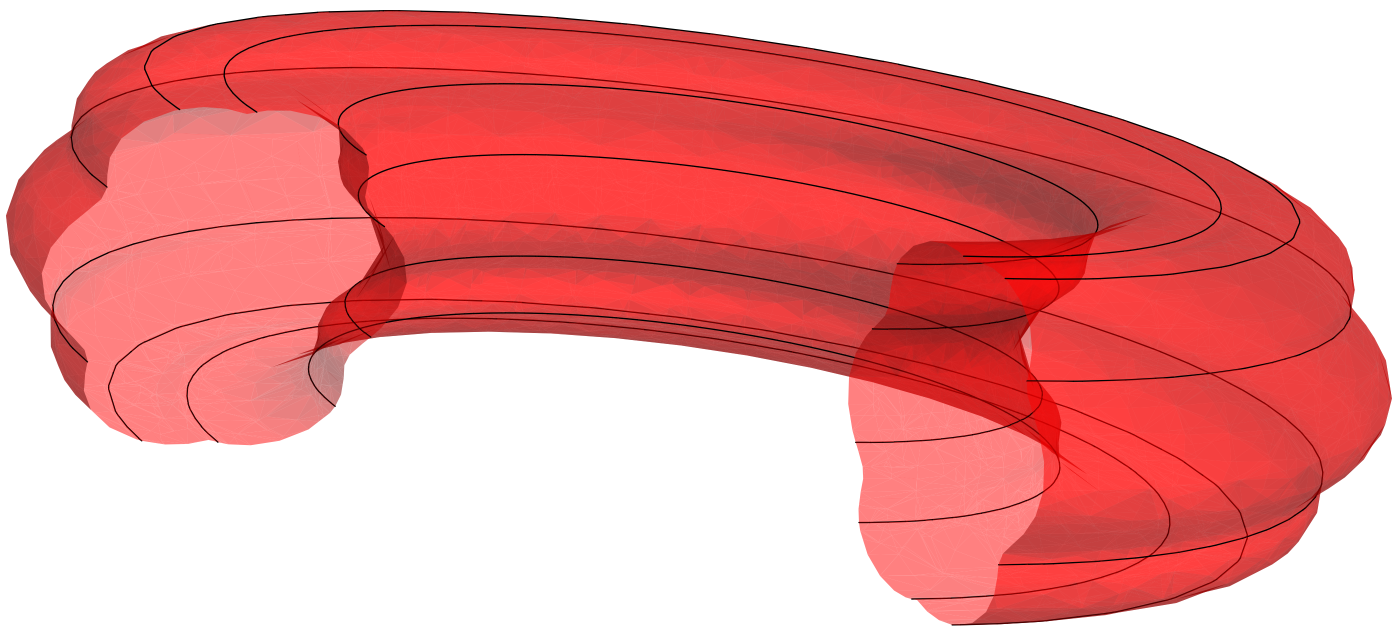
Energy moment of kinetic equation yields slow evolution of equilibrium pressure profile:

$$\frac{3}{2}\frac{\partial p_s}{\partial t} + \nabla\cdot\mathbf{Q}_s = \mathcal{S}_p$$

Calculation of F_s and $\mathbf{Q}_s$ require kinetic treatment

Stability: flute perturbations

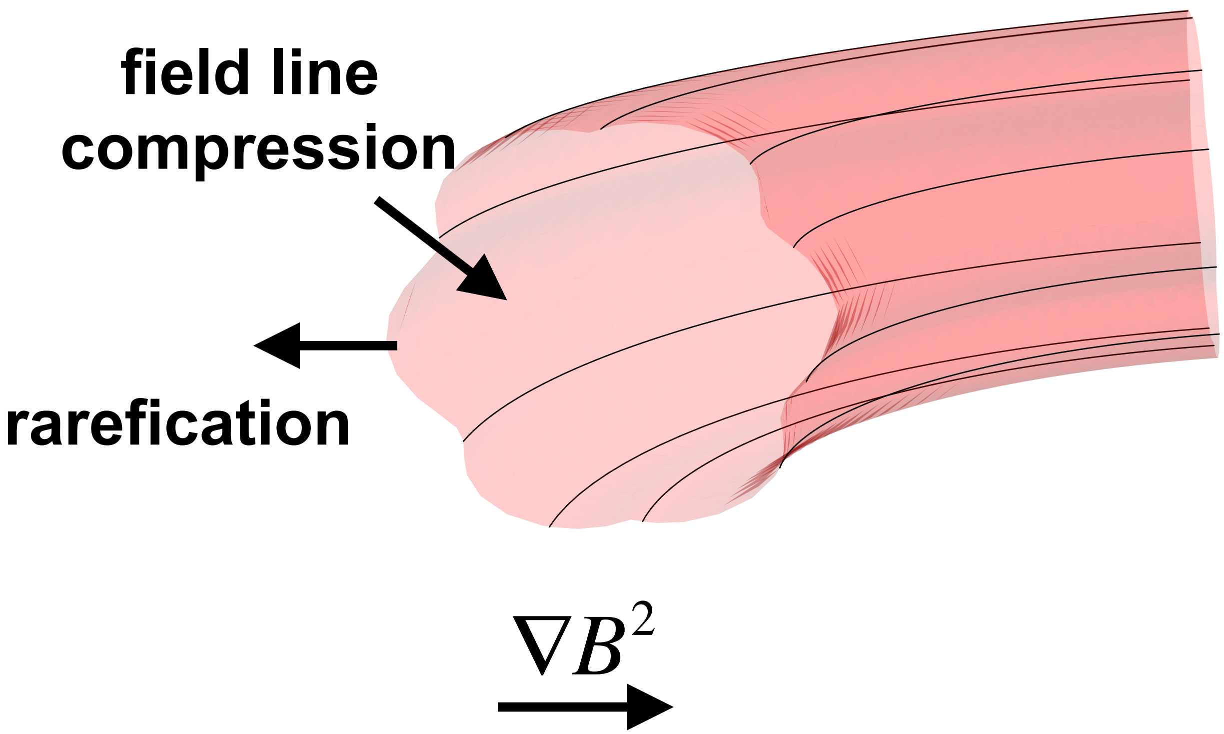
- Field-line ‘tension’ opposes bending of magnetic field lines
- Most dangerous instabilities: $k_{||} << k_{\perp}$



- NB: only possible to have perfect flute on ‘rational’ flux surfaces

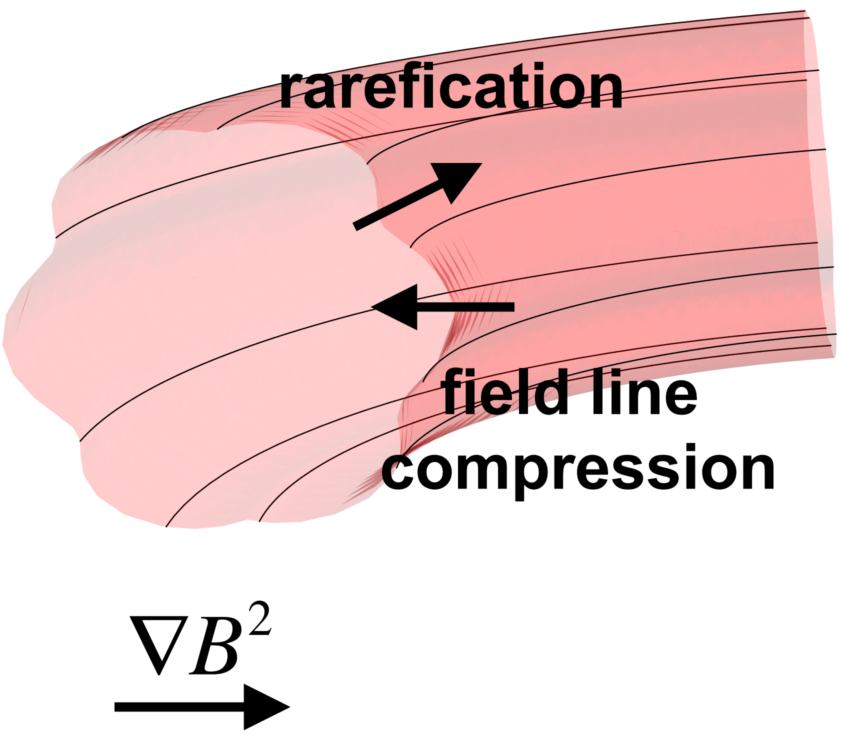
Interchange modes

- Flute perturbations unstable when field line curvature is towards plasma (‘bad’ curvature)
- Interchange plasma and magnetic flux



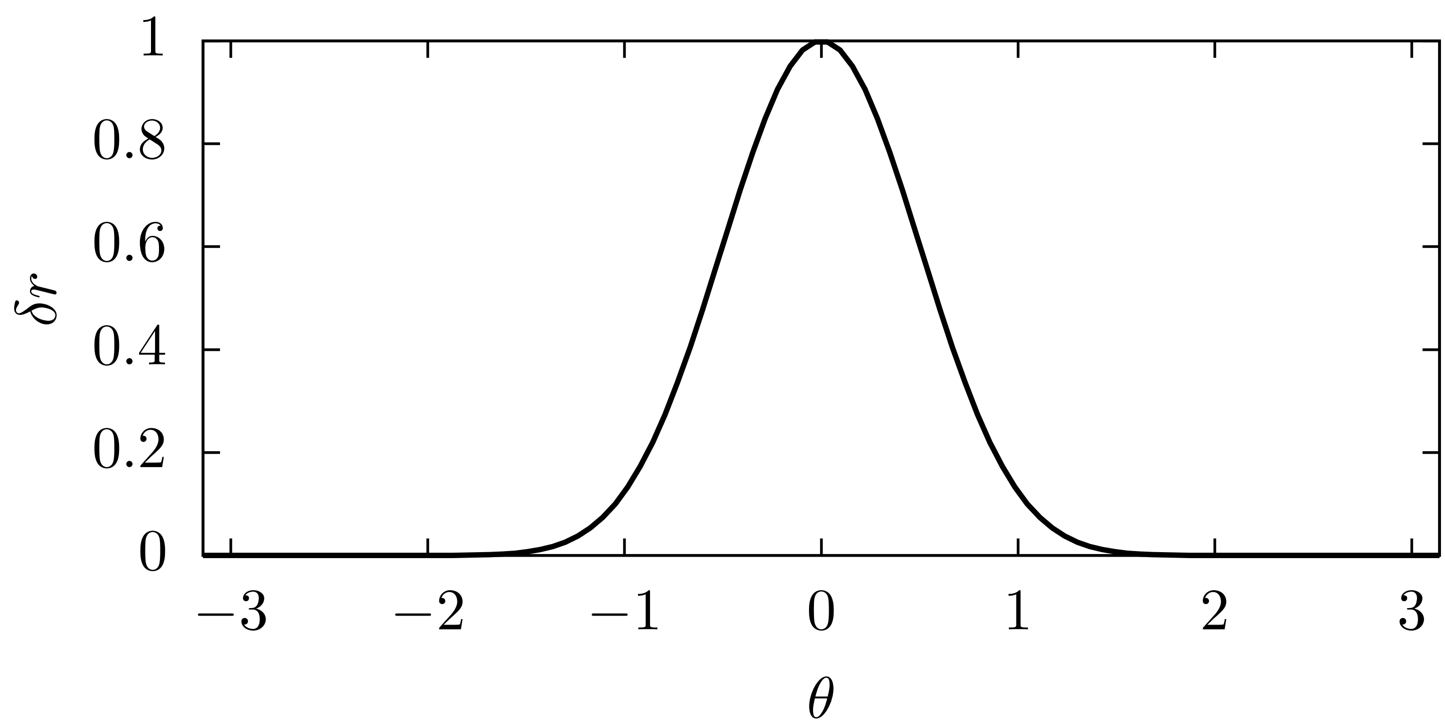
Interchange modes

- Flute perturbations stable when field line curvature is away from plasma (‘good’ curvature)



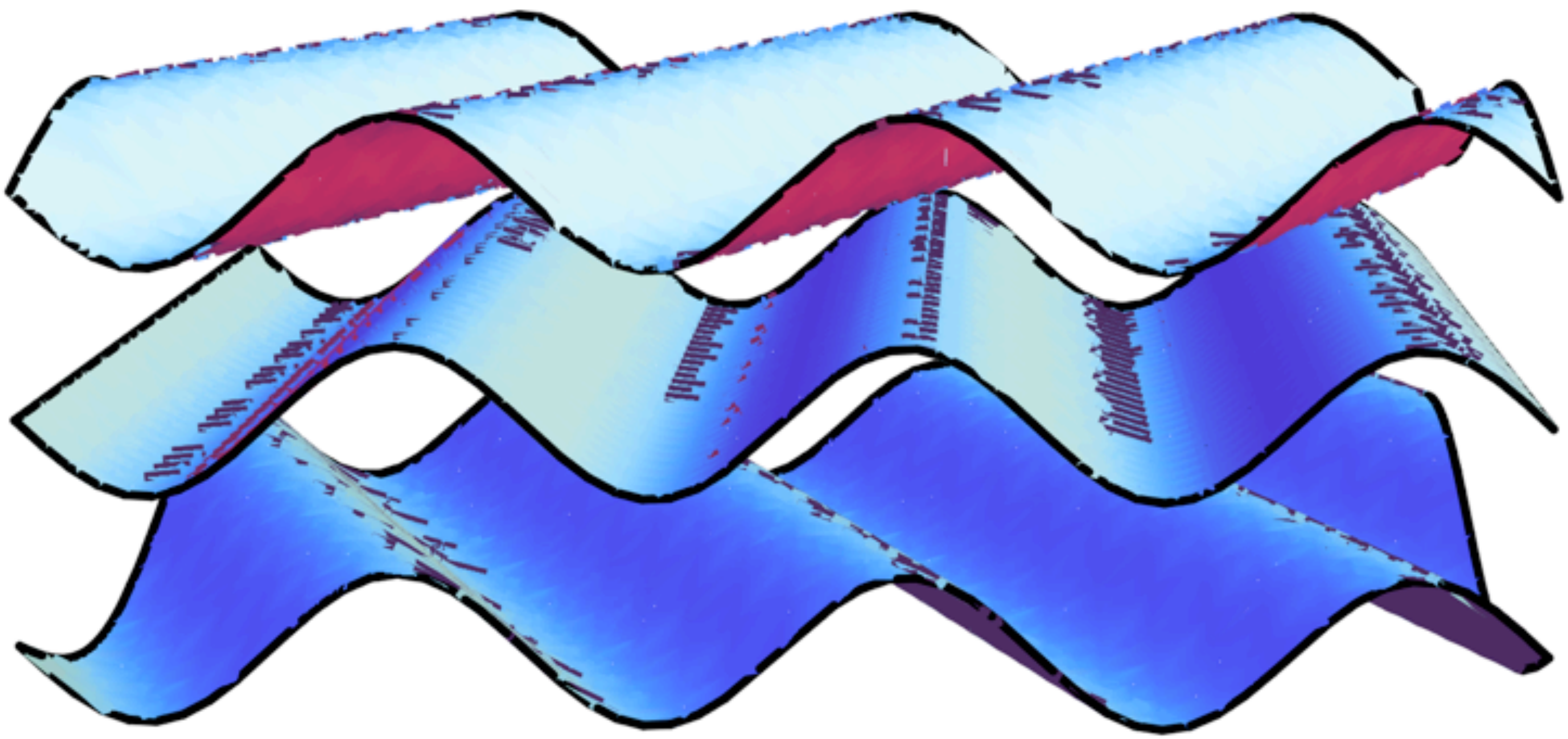
Curvature and ballooning

- Tokamak has good and bad curvature regions: instabilities ‘balloon’ in bad curvature region
- Requires field-line bending, which is stabilizing



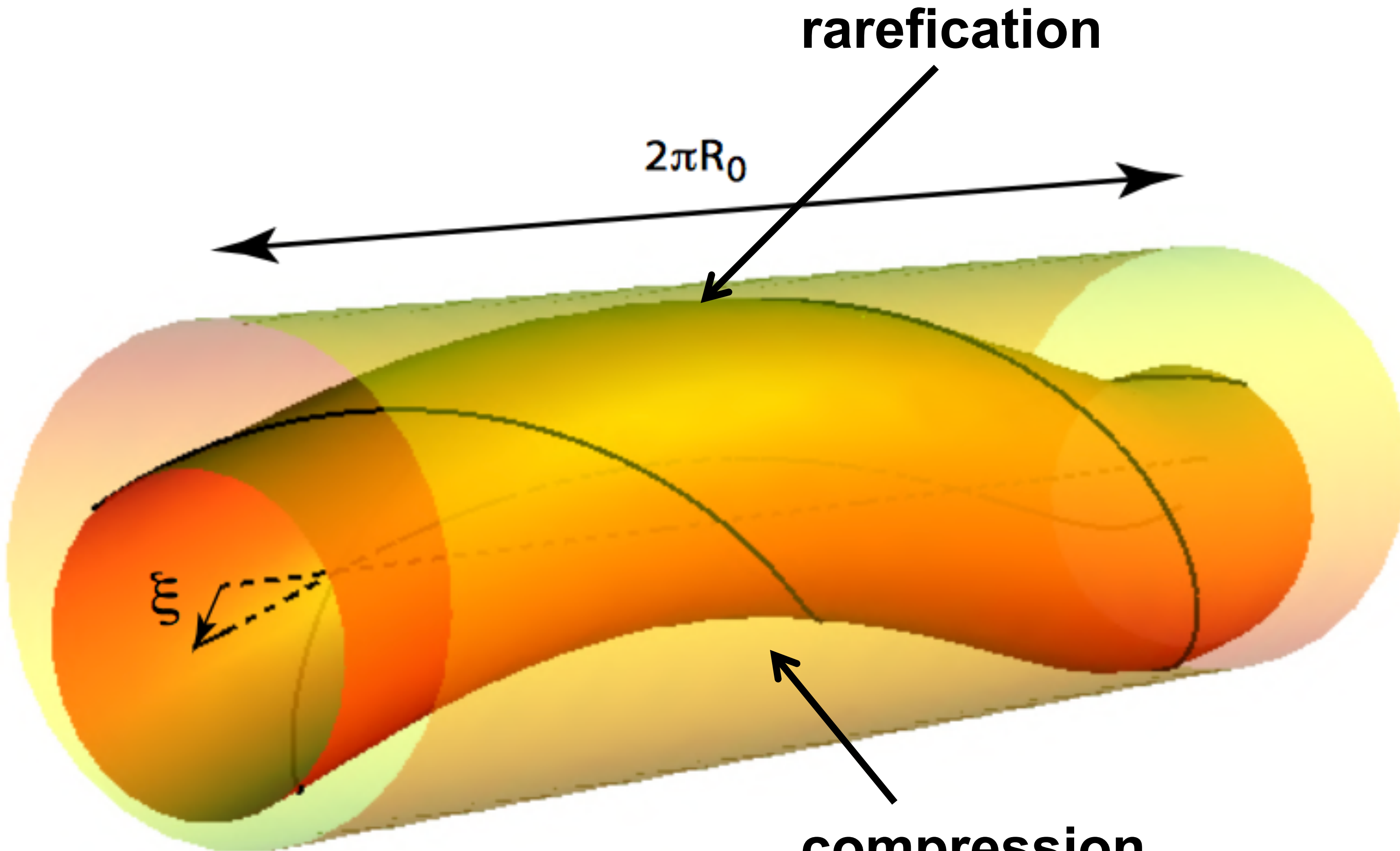
Magnetic shear stabilization

- Field-aligned perturbations at given radius not field-aligned at neighboring radius → stabilizing



Kinks

- Kink comes from helical perturbation and is (usually) current-driven



MHD linear stability

$$\rho_0 \frac{\partial \mathbf{U}_1}{\partial t} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p \qquad \mathbf{U}_1 = \dot{\boldsymbol{\xi}}$$

$$-\omega^2 \rho_0 \boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}) = \frac{\mathbf{J}_1 \times \mathbf{B}_0}{c} + \frac{\mathbf{J}_0 \times \mathbf{B}_1}{c} - \nabla p_1$$

$$K(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \frac{1}{2} \int d^3 r \ \rho_0 \dot{\boldsymbol{\xi}}^* \cdot \dot{\boldsymbol{\xi}}$$

$$\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = -\frac{1}{2} \int d^3 r \ \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})$$

Variational principle formulation

$$\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = -\frac{1}{2}\int d^3r\ \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})$$

$$K(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \frac{1}{2}\int d^3r\ \rho_0 \dot{\boldsymbol{\xi}}^* \cdot \dot{\boldsymbol{\xi}}$$

$$\Omega^2(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = \frac{\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi})}{K(\boldsymbol{\xi}^*, \boldsymbol{\xi})} \qquad \delta\Omega^2 = 0$$

$$\frac{\delta W(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi}) + \delta W(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi})}{K(\delta\boldsymbol{\xi}^*, \boldsymbol{\xi}) + K(\boldsymbol{\xi}^*, \delta\boldsymbol{\xi})} = \frac{\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi})}{K(\boldsymbol{\xi}^*, \boldsymbol{\xi})} \equiv \omega^2$$

$$\Longrightarrow -\omega^2 \rho_0 \boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi})$$

MHD Energy Principle

$$\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = -\frac{1}{2}\int d^3r \; \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})$$

$$-\omega^2 \rho_0 \boldsymbol{\xi} = \mathbf{F}(\boldsymbol{\xi}) \qquad \boldsymbol{\xi} = \sum_n a_n \boldsymbol{\xi}_n \exp(-i\omega_n t)$$

$$\Rightarrow \quad \delta W = \sum_n |a_n|^2 \omega_n^2$$

$\delta W < 0 \Rightarrow \omega_m^2 < 0$ **instability**

$\delta W \geq 0$ **for all** $\boldsymbol{\xi} \Rightarrow \omega_n^2 \geq 0$ **stability**

MHD Energy Principle solution

$$\delta W(\boldsymbol{\xi}^*, \boldsymbol{\xi}) = -\frac{1}{2}\int d^3r \; \boldsymbol{\xi}^* \cdot \mathbf{F}(\boldsymbol{\xi})$$

$$\boldsymbol{\xi} = \sum_n a_n \boldsymbol{\xi}_n$$

$$\Longrightarrow \quad \delta W = \sum_{m,n} a_m a_n \delta W(\boldsymbol{\xi_m}, \boldsymbol{\xi_n})$$

Numerical minimization of a_n determines stability

Mission accomplished?

- Even tokamaks not exactly axisymmetric: islands and stochasticity
- Very hard to obtain MHD equilibria for stellarators; needed for optimization
- What about corrections to ideal MHD?
- Occasional large MHD instabilities (ELMs, disruptions)
- Even with ‘good’ magnetic surfaces and MHD-quiescent plasma, still have transport (subject of Friday lecture)