Problem 1

The numerical exercise we will consider is the solution of ~~as~~ partial integro-differential equation of the form

$$\frac{\partial g(v,x)}{\partial t} + v\frac{\partial g(v,x)}{\partial x} + v\frac{\partial \varphi(x)}{\partial x} = 0\,; \quad \int dv\, e^{-v^2}\frac{g}{\sqrt{\pi}} = \frac{T_i}{T_e}\varphi$$

If you're interested in what this equation describes, read on. I will also discuss some of the ways one might treat this equation numerically.

Let's start with the equation

$$\frac{\partial \delta f_s}{\partial t} + v_z\frac{\partial \delta f_s}{\partial z} + \frac{q_s}{m_s}\,\delta \underline{E}_{\parallel}\cdot\frac{\partial F_{0s}}{\partial \underline{v}} = 0$$

This equation is the limit of gyrokinetics (described earlier in lectures) in which: the ratio of the gyroradius to the perpendicular wavelength of the fluctuation goes to zero, the magnetic field ($\underline{B} = B\hat{z}$) is straight and homogenious, and the plasma equilibrium is homogenious. If we assume the plasma has enough collisions to maintain a near-thermal equilibrium, then

$$F_{0s} = \frac{n_{0s}}{\pi^{3/2} v_{ts}^3}\exp\left(-\frac{v^2}{v_{ts}^2}\right), \quad v_{ts} = \sqrt{\frac{2T_s}{m_s}}$$

Recalling that $\underline{E} = -\nabla\phi$ in GK and using the above form for F_{0s}, we get

$$\frac{\partial \delta f_s}{\partial t} + v_z\frac{\partial \delta f_s}{\partial z} + \frac{q_s}{T_s}\frac{\partial \phi}{\partial z}v_z F_{0s} = 0 \quad \text{— (1)}$$

This is essentially a 1D equation in v-space because only the parallel speed v_z appears explicitly outside the source term. Let's take advantage of this and integrate out the other v-space dimensions: $(g = \int dv_x \int dv_y \, \delta f)$

$$\int dv_x \int dv_y \,(1) = \frac{\partial g_s}{\partial t} + v_z \frac{\partial g_s}{\partial z} + \frac{q_s}{T_s} v_z \frac{\partial \phi}{\partial z} \frac{n_0 e^{-v_z^2/v_{ts}^2}}{\pi^{1/2} v_{ts}} = C$$

Normalize: $\tilde{g}_s = \dfrac{g_s \pi^{1/2} v_{ts}}{n_0 e^{-v_z^2/v_{ts}^2}}$, $\tau = t \dfrac{v_{ti}}{L}$, $x = \dfrac{z}{L}$, $v = \dfrac{v_z}{v_{ti}}$,

$$\varphi = \frac{e\phi}{T_i}.$$

The equation for the ions with these normalizations becomes

$$\frac{\partial \tilde{g}_i}{\partial \tau} + v \frac{\partial \tilde{g}_i}{\partial x} + v \frac{\partial \varphi}{\partial x} = 0, \quad \text{—— (2)}$$

where I have taken the ions to be hydrogenic. We can close the system by assuming the electrons have a Boltzmann response; i.e. that $E_\parallel$ balances $\nabla_\parallel p$:

$$\delta n_e = \frac{e\phi}{T_e} n_0$$

Quasineutrality then gives $\delta n_i = \delta n_e = \dfrac{e\phi}{T_e} n_0$

$$\Rightarrow \int dv_z \, g_i = \frac{e\phi}{T_e} n_0$$

$$\Rightarrow \int dv \, e^{-v^2} \frac{\tilde{g}_i}{\pi^{1/2}} = \frac{T_i}{T_e} \varphi \quad \text{—— (3)}$$

From here on, will drop species subscript and tilde.

Eqs. (1) and (3) form a closed system which supports Landau damped ion acoustic waves. This is the system we want to solve numerically.

There are essentially 3 different choices that must be made when solving this system numerically, and each choice affects the others. They are:

1) implicit or explicit time stepping
2) spectral or grid-based in x
3) spectral or grid-based in v

$\boxed{\text{Simplest (stable) scheme: explicit in time, grid-based in } x \text{ and } v}$

~~Define~~ $\cancel{g = \frac{\delta f_i}{F_0},\ \tau = t\frac{v_{ti}}{L},\ v = \frac{v}{v_{ti}},\ X = \frac{z}{L},\ \varphi = \frac{e\phi}{T_i}}$

$\cancel{\Rightarrow \frac{\partial g}{\partial \tau} + v\frac{\partial g}{\partial x} + v\frac{\partial \varphi}{\partial x} = 0}$

$$\frac{g_{i,j}^{n+1} - g_{i,j}^{n}}{\Delta\tau} + v_j\frac{g_{i+1,j}^{n} - g_{i,j}^{n}}{\Delta x} + v_j\frac{\varphi_{i+1}^{n} - \varphi_i^{n}}{\Delta x} = 0 \qquad (v<0)$$

$$\frac{g_{i,j}^{n+1} - g_{i,j}^{n}}{\Delta\tau} + v_j\frac{g_{i,j}^{n} - g_{i-1,j}^{n}}{\Delta x} + v_j\frac{\varphi_{i}^{n} - \varphi_{i-1}^{n}}{\Delta x} = 0 \qquad (v>0)$$

$$\frac{T_e}{T_i}\frac{\Delta v}{\sqrt{\pi}}\sum_j e^{-v_j^2} g_{i,j} = \varphi_i$$

More complicated scheme that is more fun:
Implicit in time, spectral in x and v

Expand $g = \sum_{m,k'} \hat{g}_{m,k'} H_m(v)\, e^{ik'x}$ — (4)

m^{th} Hermite polynomial

Hermite polynomials are orthogonal:

$$\int dv\, H_m(v) H_n(v)\, e^{-v^2} = \sqrt{\pi}\, 2^n n!\, \delta_{m,n}$$

and have useful recurrence relation:

$$v H_m = \frac{H_{m+1}}{2} + m H_{m-1}.$$

With the definition (4) it makes sense to multiply (2) by $H_n e^{-v^2} e^{-ikx}$ and integrate over v and x:

ik $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \delta_{m,1} ik q$

$$\sqrt{\pi}\, 2^m m!\, \frac{\partial \hat{g}_{m,k}}{\partial t} + \left[\sqrt{\pi}\, 2^m (m+1)!\, \hat{g}_{m+1,k} + \sqrt{\pi}\, 2^{m-1} m!\, \hat{g}_{m-1,k}\right] + \sqrt{\pi} \cdots = 0$$

where I used $v = \frac{H_1(v)}{2}$.

$$\hat{\varphi}_k = \frac{T_e}{T_i}\, \hat{g}_{0,k}$$

$$\Longrightarrow \frac{\partial \hat{g}_{1,k}}{\partial \tau} + 2ik\, \hat{g}_{2,k} + \frac{1}{2} ik\, \hat{g}_{0,k} + \frac{ik}{2} \frac{T_e}{T_i}\, \hat{g}_{0,k} \quad (m=1)$$

$$\frac{\partial \hat{g}_{m,k}}{\partial \tau} + ik(m+1)\, \hat{g}_{m+1,k} + \frac{ik}{2}\, \hat{g}_{m-1,k} = 0 \quad (m \geq 1)$$

Performing the implicit differentiation in time:

$$\frac{\hat{g}^{n+1}_{m,k} - \hat{g}^{n}_{m,k}}{\Delta\tau} + ik(m+1)\hat{g}^{n+1}_{m+1,k} + \frac{ik}{2}\hat{g}^{n+1}_{m-1,k} = 0$$

$$\Rightarrow \quad A\, \underline{g}^{n+1}_{k} = \underline{g}^{n}_{k},$$

~~with $A_{pq} = \delta_{pq} + ik$~~

~~with $A =$~~

$$A_{pq} = \delta_{pq} + ik\Delta\tau(q+1)\delta_{p,q+1} + \frac{ik}{2}\Delta\tau\, \delta_{p,q-1}$$

This is a tridiagonal matrix. ~~which~~ The associated linear system can be solve in $\mathcal{O}[M]$ operations, with M the number of Hermite polynomials retained.

We know how to treat m=0, 1. What to do for m large?

Problem 2

The GK system of eqns are of the form

$$\frac{\partial g}{\partial t} + L_g[g] + L_\phi[\phi] + NL[g,\phi] = 0$$

where L_g and L_ϕ are linear operators
and NL is a quadratic nonlinearity operator
These operators depend on the 2D v-space, but do not involve any differentiation with respect to velocity variables.

$\phi = \phi[g]$ is an 2D integral operator in velocities.

What are the advantages/disadvantages of PIC vs grid approaches to the problem?

What changes (if anything) if we introduce a derivative in v-space to the eqn?