The Boltzmann Equation and the One-Fluid Hydromagnetic Equations in the Absence of Particle Collisions

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(*Communicated by S. Chandrasekhar, F.R.S.—Received* 9 *March* 1956)

Starting from the Boltzmann equation for a completely ionized dilute gas with no inter-particle collision term but a strong Lorentz force, an attempt is made to obtain one-fluid hydromagnetic equations by expanding in the ion mass to charge ratio. It is shown that the electron degrees of freedom can be replaced by a macroscopic current, but true hydrodynamics still does not result unless some special circumstance suppresses the transport of pressure along magnetic lines of force. If the longitudinal transport of pressure is ignored, a set of self-contained one-fluid hydromagnetic equations can be found even though the pressure is not a scalar.

1. Introduction

In some cases of physical interest, such as in gas discharges and certain astrophysical problems, one may have to deal with plasmas so dilute that the mean free path for particle collisions is long compared to any other dimensions in the problem. The particles interact with each other but only through long-range fields which can be described macroscopically. Another way to express this situation is to say that all the important impact parameters are larger than the Debye radius. In such a case the usual collision mechanism which generates randomness and allows a description of the system in terms of the hydrodynamic variables, pressure, density and mass velocity, is absent. However, it is often assumed (Alfvén 1950; Spitzer 1955) that the presence of a strong magnetic field adequately replaces the randomizing tendency of collisions so that hydrodynamical concepts again are valid. The purpose of this paper is to investigate this assumption by attempting to derive hydromagnetic equations, starting from the Boltzmann equation with no collision term but a large Lorentz force.

The attempt turns out to be not entirely successful. It is true that the Larmor radius of a particle may in a sense be considered a collision mean free path, but 'collisions' occur only for motion perpendicular to the magnetic field. Motion parallel to the field is relatively free so there is no reason, in general, for odd moments of the velocity distribution to be small, the condition necessary for hydrodynamics to apply (Chapman & Cowling 1952). Our conclusion, then, will be that a strictly hydrodynamic description of the problem is possible only when some special circumstance suppresses the third moment of the velocity distribution. When such is not the case one must go outside the framework of hydrodynamics, that is to say, to the Boltzmann equation itself.

As a by-product of this investigation we are able to show that the electron degrees of freedom can almost always be replaced by a macroscopic current, so the problem is really that of one fluid—the ions. This result is independent of the difficulty with the odd moments. Further, we write down, for what we believe is the first time,* hydromagnetic equations in the general case of a non-scalar pressure tensor.

2. The expansion of the Boltzmann function

With no collisions the Boltzmann equation for the ions is

$$\frac{\partial f}{\partial t}+(\mathbf{v}\,.\,\mathrm{grad})f+\frac{e}{M}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\,.\,\mathrm{grad}_{\mathbf{v}}f=0, \tag{1}$$

where f is the distribution function, depending on position $\mathbf{r}$, velocity $\mathbf{v}$ and time t. The ion charge is e and the mass M, while $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. (Rationalized Gaussian units with $c = 1$ will be employed.) The symbol $\mathrm{grad}_{\mathbf{v}}$ means gradient in the velocity space.

The usual derivation of hydrodynamics from the Boltzmann equation (Chapman & Cowling 1952) depends on an expansion in powers of the collision mean free path. That is, the collision term is assumed to dominate in the equation and the other terms are treated as perturbations. We hope that in our case the Lorentz force will play a role analogous to that of the collision term, so we make an expansion in powers of M/e, which is equivalent to an expansion in powers of the Larmor radius. This procedure can also be stated as an adiabatic approximation, since it depends on the Larmor frequency eB/M being large compared to other frequencies in the problem.

If we expand the Boltzmann function in powers of M/e,

$$f=f_0+f_1+f_2+\dots, \tag{2}$$

then the equations satisfied by the sequence of functions f are

$$(\mathbf{E}+\mathbf{v}\times\mathbf{B})\,.\,\mathrm{grad}_{\mathbf{v}}\,f_0=0, \tag{3}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\,.\,\mathrm{grad}\right)f_0+\frac{e}{M}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\,.\,\mathrm{grad}_{\mathbf{v}}f_1=0, \tag{4}$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\,.\,\mathrm{grad}\right)f_1+\frac{e}{M}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\,.\,\mathrm{grad}_{\mathbf{v}}f_2=0. \tag{5}$$

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It is not obvious that this sequence of equations can be satisfied. For example, it is difficult if not impossible to fulfil (3) unless the electric field is perpendicular to the magnetic field. Fortunately, such a condition is satisfied to a very good approximation because of the presence of the electrons. Owing to their small mass the electrons are highly mobile, so any electric field which develops parallel to the magnetic field will cause violent electron motion and cannot long persist. A semi-quantitative restatement of this argument is to say that if the electron plasma frequency is very large compared to the ion Larmor frequency, then plasma oscillations may be neglected. The plasma frequency is

$$\omega_P \sim (ne^2/m)^{\frac{1}{2}}, \tag{6}$$

* Certain aspects of non-scalar pressure tensors are discussed in Spitzer (1955); however, we are not aware of any systematic discussions of the general case.

where n is the number of electrons per unit volume and m the electron mass. The ion Larmor frequency, on the other hand, is

$$\omega_L = eB/M. \tag{7}$$

Thus the ratio ω_P/ω_L is given by

$$\frac{\omega_P}{\omega_L} \sim \left(\frac{M}{m}\frac{nM}{B^2}\right)^{\frac{1}{2}}, \tag{8}$$

which is usually very large in problems of interest, not only because $M/m \sim 10^3$ but also because the mass energy density nM is nearly always much greater than the field energy density B^2.

Assuming that the electric and magnetic fields are mutually perpendicular we introduce a vector $\boldsymbol{\alpha}$ perpendicular to both $\mathbf{B}$ and $\mathbf{E}$ by the definition

$$\mathbf{E} = -\boldsymbol{\alpha} \times \mathbf{B}, \tag{9}$$

so that

$$\boldsymbol{\alpha} = \mathbf{E} \times \mathbf{B}/B^2. \tag{10}$$

Equation (3) may then be written in the form

$$(\mathbf{V} \times \mathbf{B}) . \operatorname{grad}_{\mathbf{v}} f_0 = 0, \tag{11}$$

where

$$\mathbf{V} = \mathbf{v} - \boldsymbol{\alpha}. \tag{12}$$

The solution of equation (11) is straightforward. The function f_0 may vary in the plane containing the vectors $\mathbf{V}$ and $\mathbf{B}$ but may not vary in the direction $\mathbf{V} \times \mathbf{B}$. Thus the most general solution of (11) is

$$f_0\{(\mathbf{v} - \boldsymbol{\alpha})^2, \mathbf{v} . \mathbf{B}, \mathbf{r}, t\}. \tag{13}$$

By inspection of (13) it is clear that the component of the average ion velocity perpendicular to $\mathbf{B}$ calculated from f_0 alone is $\boldsymbol{\alpha}$. Furthermore, $\boldsymbol{\alpha}$ is also the average electron velocity perpendicular to $\mathbf{B}$, since the above arguments about f_0 are independent of mass and charge. Very often $\boldsymbol{\alpha}$ is referred to as the 'electric drift velocity'. There are other drift velocities perpendicular to $\mathbf{B}$, but these are of higher order in the mass to charge ratio and arise from f_1, f_2, etc. For the electrons these other drifts can be completely ignored. For the ions we shall take into account drifts arising from f_1.

It is possible to show systematically that the remaining equations (4), (5), etc., may be satisfied, but this demonstration will be deferred to a subsequent paper which actually attempts to solve for the Boltzmann function. Here we shall consider only the first few moments of the Boltzmann equation in an attempt to get hydromagnetic equations.

3. The hydromagnetic equations

Integrating equation (4) over all velocity space yields a continuity equation:

$$\frac{\partial n_0}{\partial t} + \operatorname{div}(n_0 \mathbf{u}_0) = 0, \tag{14}$$

where

$$n_0 = \int \mathrm{d}\mathbf{v} f_0 \tag{15}$$

and
$$n_0\mathbf{u}_0 = \int d\mathbf{v}\,\mathbf{v} f_0. \tag{16}$$

Next we multiply (4) by $\mathbf{v}$ and integrate to obtain

$$\rho_0 \frac{du_0}{dt} = -\operatorname{div} \overleftrightarrow{\mathbf{P}}_0 + \left[e\int d\mathbf{v}(\mathbf{v}-\mathbf{u}_0)f_1\right] \times \mathbf{B}, \tag{17}$$

where
$$\rho_0 = Mn_0 \quad \text{and} \quad \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_0 . \operatorname{grad}, \tag{18}$$

and $\overleftrightarrow{\mathbf{P}}_0$ is a pressure tensor defined by

$$\overleftrightarrow{\mathbf{P}}_0 = M\int d\mathbf{v}(\mathbf{v}-\mathbf{u}_0)(\mathbf{v}-\mathbf{u}_0)f_0. \tag{19}$$

The restriction (13) on the functional form of f_0 implies that the pressure tensor must be of the form

$$\overleftrightarrow{\mathbf{P}}_0 = P_n \mathbf{nn} + P_s(\overleftrightarrow{\mathbf{1}} - \mathbf{nn}), \tag{20}$$

where $\mathbf{n}$ is a unit vector pointing along the magnetic field and $\overleftrightarrow{\mathbf{1}}$ signifies the unit dyadic. In other words, the pressure tensor is diagonal in a local rectangular co-ordinate system one of whose axes points along $\mathbf{B}$. In the plane perpendicular to $\mathbf{B}$ the pressure is a scalar of magnitude P_s. The pressure 'along' $\mathbf{B}$ is P_n, which in general need not equal P_s.

Equation (17) is peculiar in that it should be regarded as determining the behaviour only of the component of $\mathbf{u}_0$ parallel to $\mathbf{B}$. Recall that the component of $\mathbf{u}_0$ perpendicular to $\mathbf{B}$ is the electric drift $\boldsymbol{\alpha}$, which is determined by the electromagnetic field. The component of equation (17) perpendicular to the magnetic field properly should be regarded as a condition on the function f_1. Before taking a third moment of equation (17) let us consider the Maxwell equations, which must be added to the Boltzmann equation in order to make the problem determinate. Remembering that $\mathbf{E} = -\mathbf{u}_0 \times \mathbf{B}$, the Maxwell equations become

$$-\operatorname{div}(\mathbf{u}_0 \times \mathbf{B}) = e\int d\mathbf{v}(f_0 + f_1 - f_e), \tag{21}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{22}$$

$$-\frac{\partial}{\partial t}(\mathbf{u}_0 \times \mathbf{B}) = \operatorname{curl} \mathbf{B} - e\int d\mathbf{v}\,\mathbf{v}(f_0 + f_1 - f_e) \tag{23}$$

and
$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{u}_0 \times \mathbf{B}), \tag{24}$$

if we keep only terms up to first order in the ion mass to charge ratio and denote the Boltzmann function for the electrons by f_e. It will now be shown that one may eliminate both f_1 and f_e in favour of a single macroscopic quantity which will be called $\mathbf{j}_1$.

We define $\mathbf{j}_1$ as the difference between the total current and the charge density times the velocity $\mathbf{u}_0$. That is to say,

$$\mathbf{j}_1 = e\int d\mathbf{v}\mathbf{v}(f_0 + f_1 - f_e) - e\int d\mathbf{v}\mathbf{u}_0(f_0 + f_1 - f_e). \tag{25}$$

Remembering the definition (16) of $\mathbf{u}_0$, the terms containing f_0 cancel and we find

$$\mathbf{j}_1 = e\int \mathrm{d}\mathbf{v}(\mathbf{v}-\mathbf{u}_0)f_1 - e\int \mathrm{d}\mathbf{v}(\mathbf{v}-\mathbf{u}_0)f_e. \tag{26}$$

The way in which the function f_1 occurs in $\mathbf{j}_1$ is exactly the same as the way in which f_1 occurs in the hydrodynamic equation (17). Furthermore, the part of $\mathbf{j}_1$ due to f_e is parallel to $\mathbf{B}$, since we have seen that the average electron velocity perpendicular to $\mathbf{B}$ is $\boldsymbol{\alpha}$, which is also the component of $\mathbf{u}_0$ perpendicular to $\mathbf{B}$. Thus the second term on the right-hand side of (17) may be written $\mathbf{j}_1 \times \mathbf{B}$. If we note further that $\mathbf{j}_1$ itself can be determined by equations (23) and (21), that is, by

$$\mathbf{j}_1 = \operatorname{curl}\mathbf{B} + \frac{\partial}{\partial t}(\mathbf{u}_0 \times \mathbf{B}) + \mathbf{u}_0 \operatorname{div}(\mathbf{u}_0 \times \mathbf{B}), \tag{27}$$

then we see that f_1 and f_e have indeed been successfully eliminated.

To recapitulate, we have found the following equations for $\mathbf{u}_0$, ρ_0 and $\mathbf{B}$:

$$\rho_0 \frac{\mathrm{d}\mathbf{u}_0}{\mathrm{d}t} = -\operatorname{div}\overleftrightarrow{\mathbf{P}}_0 + \operatorname{curl}\mathbf{B}\times\mathbf{B} + \left[\frac{\partial}{\partial t}(\mathbf{u}_0\times\mathbf{B})\right]\times\mathbf{B} + (\mathbf{u}_0\times\mathbf{B})\operatorname{div}(\mathbf{u}_0\times\mathbf{B}), \tag{28}$$

$$\frac{\partial \rho_0}{\partial t} + \operatorname{div}(\rho_0 \mathbf{u}_0) = 0, \tag{29}$$

and

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{u}_0 \times \mathbf{B}). \tag{30}$$

What is missing is an equation for the pressure. We have succeeded in the task of eliminating the electron variables, but we do not have a useful hydrodynamics until an equation for the pressure in terms of $\mathbf{u}_0$, ρ_0 and $\mathbf{B}$ is obtained. We attempt to get this equation by the standard procedure (Chapman & Cowling 1952) of multiplying (4) by $(\mathbf{v}-\mathbf{u}_0)(\mathbf{v}-\mathbf{u}_0)$ and integrating over velocity space. Separate equations for the scalar quantities P_n and P_s are obtained if, in addition, one takes the scalar product with the dyadics $\mathbf{nn}$ and $\overleftrightarrow{\mathbf{1}}$, respectively. The result is

$$\frac{\mathrm{d}P_n}{\mathrm{d}t} = -P_n \operatorname{div}\mathbf{u}_0 - 2P_n \mathbf{n}.(\mathbf{n}.\operatorname{grad})\mathbf{u}_0 - \operatorname{div}(q_n + q_s)\mathbf{n} - 2\mathbf{n}.\operatorname{grad} q_s, \tag{31}$$

$$\frac{\mathrm{d}P_s}{\mathrm{d}t} = -2P_s \operatorname{div}\mathbf{u}_0 + P_s \mathbf{n}.(\mathbf{n}.\operatorname{grad})\mathbf{u}_0 - \operatorname{div}(q_s\mathbf{n}) - q_s \operatorname{div}\mathbf{n}, \tag{32}$$

where the new quantities q_n and q_s are components of the pressure-transport tensor (Chapman & Cowling 1952). Precisely, if we introduce the totally symmetric third-rank tensor $q^{(0)}_{ijk}$ by the definition

$$q^{(0)}_{ijk} = M\int \mathrm{d}\mathbf{v}(v_i - u_{0,i})(v_j - u_{0,j})(v_k - u_{0,k})f_0, \tag{33}$$

then the known functional form (13) of f_0 implies that $q^{(0)}_{ijk}$ can be written

$$q^{(0)}_{ijk} = q_n n_i n_j n_k + q_s(\delta_{ij} n_k + \delta_{ik} n_j + \delta_{jk} n_i). \tag{34}$$

The presence of the pressure-transport terms in (31) and (32) in general prevents the closing of the hydromagnetic equations. A new equation is needed to determine

q_s and q_n, and this equation will bring in a fourth moment of the velocity distribution. The sequence does not terminate unless some moment vanishes. When collisions dominate the Boltzman equation, f_0 is isotropic in velocities, so all the odd moments vanish. Here the Lorentz force has given an isotropic velocity distribution only in the plane perpendicular to $\mathbf{B}$. There is no reason in general for moments involving an odd power of the velocity along $\mathbf{B}$ to vanish.

It is possible of course that some special circumstance may cause the unwanted pressure-transport terms in (31) and (32) to be small. For example, if the problem is essentially two-dimensional with no important variations along the magnetic lines, then the pressure-transport terms, all of which involve derivatives along the lines, may be dropped. The remainder of this paper will discuss the hydromagnetic problem in such a case. In general, however, one must go back to the Boltzmann equation (4) to have a complete description of the problem.

4. The hydromagnetic problem with no pressure transport

With neglect of the terms involving q_n and q_s, equations (31) and (32) determine the behaviour of the pressure as a function of the variables $\mathbf{B}$, ρ_0 and $\mathbf{u}_0$. In fact the equations may be cast in the following very simple form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P_n B^2}{\rho_0^3}\right) = 0, \tag{35}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{P_s}{\rho_0 B}\right) = 0. \tag{36}$$

These pressure laws can be given an interpretation in terms of the known behaviour of individual charged particles in a strong magnetic field. For example, equation (36) says that if one moves along with a fluid element the average transverse energy per particle is proportional to B. This is not at all surprising in view of the well-known constancy of the magnetic moment of an individual particle in the adiabatic approximation. The result (35) can also be discussed in terms of individual ions, although not in such a simple way as (36).

The pressure equations (35) and (36), together with equations (28), (29) and (30) for $\mathbf{u}_0$, ρ_0 and $\mathbf{B}$, respectively, completely determine the hydromagnetic problem. They are quite similar to the one-fluid equations which have been used in the past (Alfvén 1950; Spitzer 1955) but have a greater generality. In particular, no assumption has been made here that the charge density is small, and the pressure need not be scalar. More important, we feel that a systematic derivation based on the mass to charge ratio has been achieved. This is not entirely academic because one understands now that the hydrodynamic velocity, pressure and density in these equations contain only effects independent of the ion mass to charge ratio. The so-called 'pressure drift', for example, is not contained in $\mathbf{u}_0$. It does contribute to $\mathbf{j}_1$ and thus has not been ignored, but the only transverse component of $\mathbf{u}_0$ is the electric drift.

It is easy to show that the quantity

$$E = \frac{1}{2}\int \mathrm{d}r[\rho_0 u_0^2 + (2P_s + P_n) + B^2 + (\mathbf{u}_0 \times \mathbf{B})^2], \tag{37}$$

is conserved by the hydromagnetic equations; E may therefore be interpreted as the total energy of the system and can be used as the basis of discussions of plasma stability such as have been given by Frieman (1955). The only new element here is the non-scalar pressure, but our pressure equations (35) and (36) maintain the essential feature required by Frieman's approach: they are holonomic constraints, so P_n and P_s can be eliminated in favour of a fluid displacement variable.

In conclusion, we emphasize again that a strictly hydrodynamic approach to the problem is appropriate only when some special circumstance suppresses the effects of pressure transport along the magnetic lines. A subsequent paper will discuss attempts to solve the Boltzman equation when pressure transport cannot be ignored.

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