Relativistic MHD.

1. When do we need rel. MHD?

2. Basic equations review. Conserved currents.

3. Numerical techniques. Constrained transport.

4. Dirty secrets.

5. Beyond ideal MHD. Radiation; plasma closure model.

When do we need to use relativistic MHD?

Should not do rel. MHD unless you have to, because it's very expensive, ~ 3-5 × more expensive than nonrelativistic MHD. Even in calculations that combine rel. MHD w/ solution of Einstein's equations, MHD sector dominates computational cost.

(1) wave speed is ~ c. $\boxed{\vec{v},\ \vec{v}_A,\ c_s}$

$\frac{|\vec{v}|}{c} \sim 1$ or Lorentz factor $\Gamma \equiv \frac{1}{\sqrt{1-(v/c)^2}}$ $\Gamma - 1 \gtrsim 1$.

$c_s \sim c$

$$c_{s,NR}^2 = \overbrace{\gamma p}^{\text{restoring force}} / \underbrace{\rho}_{\text{inertia}} \qquad \text{EOS: } \overset{\text{pressure}}{p} = (\gamma - 1)\overset{\text{internal energy}}{u}$$

$$c_{s,R}^2 = \gamma p \Big/ \left(\rho + \frac{u+p}{c^2}\right)$$

for rel. fluid, $\gamma = \frac{4}{3}$, $u = 3p$.

$$c_{s,R}^2 \Rightarrow \frac{\gamma p}{\rho + \frac{\gamma}{\gamma-1}\frac{p}{c^2}}$$

will soon set $c \to 1$.

$$\frac{p}{\rho} \to \infty \ ; \ \to (\gamma - 1)c^2 \to \frac{1}{3}c^2.$$

$$c_{s,R}^2 = \frac{c_{s,NR}^2}{1 + \frac{c_{s,NR}^2}{c^2}\frac{1}{\gamma - 1}}$$

Sound speed is directly related to temperature

$$p = nkT \qquad \text{ideal gas.}$$

$$n = \frac{\rho}{m} \leftarrow \text{mean molecular weight}$$

$$\text{so: } p \sim \rho c^2 \Rightarrow \frac{kT}{mc^2} \sim 1.$$

common to define dimensionless temperature

$$\Theta_i \equiv \frac{kT_i}{m_i c^2}$$ temperature in rest-mass units

electron $\Theta_e = 1 \iff 5.93 \times 10^9$ K

proton $\Theta_p = 1 \iff 1.09 \times 10^{13}$ K

$\Theta_e \gtrsim 1$ commonly achieved in low density electron-ion plasmas close to compact objects

~~[illegible]~~

$$v_{A,NR} = \frac{|\vec{B}|}{\sqrt{4\pi\rho}}$$ $\rho \to 0$, $v_{A,NR}$ can become large.

$$v_{A,R} = \frac{|B|}{\sqrt{4\pi\rho + B^2/c^2}} = \frac{v_{A,NR}}{\sqrt{1 + v_{A,NR}^2/c^2}}$$

$$v_{A,NR} \to \infty, \quad v_{A,R} \to c.$$

dimensionless parameter $\sigma \equiv \dfrac{B^2}{4\pi\rho c^2}$

$\sigma > 1 \Rightarrow$ rel. MHD.
$\sigma \gg 1$ in bh & NS magnetosph.

(2) geometry is nontrivial, i.e.

$$\phi \sim \frac{GM}{rc^2} \sim 1.$$

Equations of rel. MHD.

Write & integrate equations in conservative form.

$$\underset{\substack{\text{cons.}\\ \text{variable}}}{\partial_t \mathcal{U}} = \underset{\text{fluxes}}{-\nabla\cdot F} + \underset{\substack{\text{source}\\ \text{term.}}}{S}$$

Just like nonrel. MHD! But need to be careful with what is meant by $\mathcal{U}$, "$\nabla\cdot$".

Write down each of the conservation laws side by side w/ nonrel. equivalent

mass: $\partial_t \rho = -\nabla\cdot(\rho\vec{v}) \overset{\text{Cartesian}}{=} -\partial_i(\rho v_i)$ nonrel.

summation convention.

$\nabla_\mu(\rho u^\mu) = 0$ rel.

$\nabla_\mu \rightarrow$ covariant derivative (more later).

$\rho \rightarrow$ rest-mass density in comoving frame.

$\frac{dx^\mu}{d\tau} \equiv u^\mu \rightarrow$ 4-velocity, so in Minkowski space w/ Cartesian coordinates

proper time.

$$u^\mu = (\Gamma, \Gamma v_x, \Gamma v_y, \Gamma v_z). \Rightarrow$$ 3 independent components since $u^\mu u_\mu = -1$

this can be expanded into the form

really cons. of particle #.

$$\boxed{\frac{\partial}{\partial t}(\sqrt{-g}\,\rho u^t) = -\frac{\partial}{\partial x^i}(\sqrt{-g}\,\rho u^i)}$$

any coordinate system!

Raising & lowering indices: $u_\mu = g_{\mu\nu}u^\nu$

where $g = \mathrm{Det}(g_{\mu\nu})$ & $g_{\mu\nu}$ is the metric w/ indices down.

Example: Minkowski, ~~flat space~~, spherical t, r, θ, φ

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & r^2 & \\ 0 & & & r^2\sin^2\theta \end{pmatrix} \Rightarrow \sqrt{-g} = r^2\sin\theta.$$

Notice that the "conserved" variable is now

$$\sqrt{-g}\,\rho\, u^t \leftarrow \text{coupled to velocities.}$$

So that, for an isolated system, $\int d^3x \sqrt{-g}\,\rho\, u^t$ is constant.

This equation is covariant in the sense that it applies for any set of coordinates t, x^1, x^2, x^3.

Another example: Minkowski, quasi-spherical coord. w/ $x^1 \to \ln r$. Then $dr \to r\,dx$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & r^2 & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix}, \quad \sqrt{-g} = r^3 \sin\theta = e^{3x^1}\sin\theta.$$

and $u^1 = \dfrac{dx^1}{d\tau} \leftarrow$ proper time

$$= \frac{d\ln r}{d\tau} = \frac{1}{r}\frac{dr}{d\tau} = \frac{1}{r} u^r .$$

energy-momentum conservation

non rel. - two sets of equations.

$$\partial_t\,(\text{energy density}) = -\partial_i\,(\text{energy flux})$$

ener:

$$\partial_t\left(\frac{1}{2}\rho v^2 + u + \frac{B^2}{8\pi}\right)$$

$$= -\partial_i\left(\frac{1}{2}\rho v^2 v_i + (u+p)\,v_i + \underbrace{\frac{1}{4\pi}\left(B^2 v_i - (B\cdot v)B_i\right)}_{\text{Poynting flux.}}\right)$$

mom:

$$\partial_t(\rho v_i) = -\partial_j \Pi_{ij} \leftarrow \text{stress tensor.}$$

where for an ideal fluid

$$\Pi_{ij} = \rho v_i v_j + p\,\delta_{ij} + \frac{B^2}{8\pi}\delta_{ij} - \frac{B_i B_j}{4\pi}$$

= flux of i-momentum in j direction.

In relativity these equations are combined in the simple form

$$\nabla_\mu T^{\mu\nu} = 0$$

$T^{\mu\nu}$ = stress-energy tensor.

This equation can be expanded in the form

$$\boxed{\partial_t\left(T^t_{\ \mu}\sqrt{-g}\right) = -\partial_i\left(\sqrt{-g}\,T^i_{\ \mu}\right) + \sqrt{-g}\,T^\kappa_{\ \lambda}\Gamma^\lambda_{\mu\kappa}}$$

(4 equations). Here $\Gamma \equiv$ connection coefficient,

$$\Gamma \propto \partial_\alpha g_{\mu\nu}$$

and includes gravity, centrifugal force, Coriolis force (in rotating frame) etc. See any rel. textbook for computation of Γ, which has $4\times 4\times 4 = 64$ components, only 40 of which are independent since $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$. Γ is computed from the metric by numerical differentiation in <u>harm</u>.

Now we need to specify $T_{\mu\nu}$. Ideal MHD stress tensor can be written

$$T_{\mu\nu} = (\rho + u + p + b^2)u_\mu u_\nu + \left(p + \frac{b^2}{2}\right)g_{\mu\nu} - b_\mu b_\nu$$

where $c \to 1$.

$\Rightarrow$ in fluid frame, Cart coord.

$$\begin{pmatrix} \rho + u + b^2/2 & & & 0 \\ & p + b^2/2 & & \\ & & p + b^2/2 & \\ 0 & & & p - b^2/2 \end{pmatrix}$$

The 4-vector b^μ represents magnetic field
$b^i \rightarrow$ component of B_i (up to factors of $\sqrt{4\pi}$ that we have defined away for convenience) in the plasma frame, where $b^t \rightarrow 0$.

$\Rightarrow$ 3 independent components, since $b^\mu u_\mu = 0$.
$\Leftrightarrow b^t = 0$ in plasma frame.

New feature: term $\rho v_i v_j \rightarrow (\rho + u + p + b^2) u_i u_j$
so internal energy & magnetic field contribute to inertia of the plasma.

<u>magnetic field evolution</u>

ex. for student.

nonrel: $$\frac{\partial B_i}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}) = -\partial_j (u_j B_i - u_i B_j)$$

$$\partial_i B_i = 0$$

rel: $$\nabla_\mu (u^\mu b^\nu - u^\nu b^\mu) = 0. \Rightarrow$$ follows from source-free Maxwell + ideal MHD condition $u_\mu F^{\mu\nu} = 0$.

this can be expanded using an identity to
$$\partial_\mu [\sqrt{-g} (u^\mu b^\nu - u^\nu b^\mu)] = 0$$

or
$$\partial_t [\sqrt{-g}(u^t b^i - u^i b^t)] = -\partial_j [\sqrt{-g} (u^j b^i - u^i b^j)]$$

useful to define $\tilde{B}^i = b^i u^t - u^i b^t$; use as prim. variable.

$$\boxed{\partial_t (\tilde{B}^i \sqrt{-g}) = -\partial_j \left[\sqrt{-g} (u^j b^i - u^i b^j) \right]}$$

and the time component reduces to

$$\boxed{\partial_i\left(\sqrt{-g}\,\tilde{B}^i\right) = 0.}$$

OK, done.

Conserved variables are density $\sqrt{-g}\,\rho u^t$

E-M density $\sqrt{-g}\left((\rho+u+p+b^2)u^t u_\mu + (p+b^2/2)g^t_\mu - b^t b_\mu\right)$

magnetic field $\sqrt{-g}\,\tilde{B}^i$.

While the "primitive" variables are

$$\rho,\ u,\ \tilde{B}^i,\ u^i$$

although it is convenient to define

$$\boxed{\tilde{u}^i \equiv \underbrace{(g^i_\mu + n^i n_\mu)}_{\text{projection tensor.}} u^\mu}$$

$n_\mu \equiv$ normal observer 4-velocity.

$n_i = 0.$

which has the advantage that <u>all</u> values of $\tilde{u}^i$ are physical.

Conserved currents

Wrote energy-mom. equation w/ 1 index up & one index down for a reason.

Briefly, many metrics of interest have symmetries that give rise to conserved currents

Example: Kerr. axisymmetry, $\partial_\varphi \to 0$
t, r, θ, φ stationarity, $\partial_t \to 0$

this gives rise to conserved energy and angular momentum.

Symmetry $\Rightarrow$ existence of Killing vector ξ_μ satisfying Killing's equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$$

If symmetry is related to some ignorable coordinate A, then $\xi^A = 1$, $\xi^{!A} = 0$ is a Killing vector. (exercise for student). Then in Kerr metric $\xi^\mu_{(t)} = (1,0,0,0)$ is a Killing vector, as is $\xi^\mu_{(\varphi)} = (0,0,0,1)$.

Then in general if ξ^μ is a Killing vector,

$$J^\mu = \xi^\nu T^\mu{}_\nu$$

is a conserved current. Proof:

$$\nabla_\mu J^\mu = \nabla_\mu (T^\mu{}_\nu \xi^\nu) = \cancel{(\nabla_\mu T^\mu{}_\nu)} \xi^\nu + T^{\mu\nu} \nabla_\mu \xi_\nu$$
$$= -T^{\mu\nu} \nabla_\nu \xi_\mu \quad \text{(Killing)}$$

but since $T^{\mu\nu} = T^{\nu\mu}$, this is $\varnothing$. QED.

Therefor $J^\mu_{(\varphi)} = \xi^\nu_{(\varphi)} T^\mu{}_\nu = T^\mu{}_\varphi$ is a conserved current in the Kerr metric, and $\Gamma^\mu{}_{\varphi\nu} T^\nu{}_\mu = 0$.

$\Rightarrow$ Writing eqs. w/ one index down makes

Source terms vanish for φ & t components in the Kerr metric.

Numerical Techniques

Now many schemes for special & general rel. MHD, including Whisky, athena++, ECHO, IllinoisGRMHD (with Einstein solver), pluto, RAISHIN, harm, GRHydro. Kyoto group code, and LSU group code. Focus on harm.

Most of these codes use generalizations of standard nonrel. finite-volume techniques. harm, which many of you played with on Monday afternoon, is no exception.

harm evolves primitive ~~conserved~~ variables $\mathbb{P}$ as follows

$$\mathbb{P}^n \to \mathbb{U}^n, \mathbb{F}^n \longrightarrow \mathbb{U}^{n+1/2} \to \mathbb{P}^{n+1/2} \to \mathbb{F}^{n+1/2}$$

$$\to \mathbb{U}^{n+1} \to \mathbb{P}^{n+1}$$

Use simplest possible fluxes — local lax Friedrichs,
- → no solution of Riemann problem
- → only need estimate for fastest wave speed.
- → diffusive in comparison to more sophisticated techniques.
- → simple in comparison ...

Need to maintain $\nabla \cdot B = 0$ constraint; possible strategies include
- hope for the best — $\nabla \cdot B$ random walks away from $\emptyset$ w/ trunc. error steps.
- constrained transport

  $\nabla \cdot B$ random walks away from $\emptyset$ but w/ roundoff error size steps.

  used by discriminating numericists everywhere

- divergence cleaning
  solve elliptic equation for "closest" $\vec{B}$
  config that satisfies $\nabla \cdot B = 0$

  $\nabla \cdot B = \rho_M$ = monopole density.

$$\nabla \cdot (\vec{B} + \underbrace{\delta B}_{\nabla \phi}) = \rho_M$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla^2 \phi = \rho_M \quad \text{— solve!}$$

$$\vec{B} = B - \vec{\nabla} \phi$$

- hyperbolic divergence cleaning Dedner et al 2002.

$$\partial_t B = \ldots \quad - \nabla \psi$$

  ψ satisfies

$$\partial_t \psi = - c_h^2 \nabla \cdot B - \frac{\psi}{\tau}$$

  c_h = constant propagation speed for ψ field.

  monopoles propagate & damp.

- evolve vector potential; differentiate to get $\vec{B}$; differentiate again to get force. Used in IllinoisGRMHD.

harm uses constrained transport in a form suitable for zone-centered B fields.

Distinctive feature of rel. hydro/MHD in conservative form: nonlinear equations must be solved for transformation $\mathbb{U}^{n+1} \to \mathbb{P}^{n+1}$

Example: density conserved variable is

$$D = \sqrt{-g}\, \rho\, u^t$$

to recover ρ, we need $u^t(\tilde{u}^i)$. Density, velocity, magnetic fields are strongly coupled.

This process is now fairly well understood (in the sense that there are practical algorithms for inversion). Variable inversion $\mathbb{U} \to \mathbb{P}$ is not the dominant expense. in harm.

Still some open questions, e.g. is it possible to test whether $\mathbb{U}$ is physical without performing variable inversion?

<u>Dirty</u> <u>secrets</u>: floors and fixups.

Sometimes evolution takes $\mathbb{U}$ to a bad place.
Example: $\rho < 0$ required.

To prevent this, impose constraints on solution. When constraints, e.g. $\rho < \rho_{SMALL}$, violated impose "floor" that simply adds in new density.

harm imposes conditions

$$\rho > \rho_{MIN}$$
$$u > u_{MIN}$$
$$-n^\mu u_\mu < \Gamma_{max}$$

Sometimes this is not enough and $\mathbb{U}$ is spoiled so that inversion is impossible. Then a "fixup" is required. This should be done in a way that does not compromise solution globally, if possible.

harm resorts, first, to replacing $\mathbb{P}$ by average value from nearby zones.

On to code testing.