PITP instabilities lectures 7/2016

Mathematical Derivation of MTI

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \bar{v} = 0$$

$$\rho \frac{\partial \bar{v}}{\partial t} + \rho (\bar{v} \cdot \nabla) \bar{v} = \frac{(\nabla \times B) \times \bar{B}}{4\pi} - \nabla P + \rho \vec{g}$$

$$- \nabla \cdot \Pi$$

$$\frac{\partial \bar{B}}{\partial t} = \nabla \times (\bar{v} \times \bar{B})$$

$$\rho T \frac{\partial s}{\partial t} + \rho (\bar{v} \cdot \nabla) s = - \nabla \cdot \vec{Q}$$

Key physics
non-adiabatic

$$\bar{Q} = - \chi \hat{b} (\hat{b} \cdot \nabla T)$$

heat flows along b

Weak B-field $\Rightarrow$ drop Lorentz force

Boussinesq approx: keep buoyancy $\delta \rho \, \bar{g}$
but otherwise set $\nabla \cdot \bar{v} = 0$
dropping cont. to $\delta \rho$ from pressure variations (soundwaves) ; only keeping buoyancy .. applies if timescale of instability >> sound crossing time

linear perturbation theory with

$$\delta \bar{q} \longrightarrow \bar{B}_0 \longrightarrow \vec{k}$$

key is heat flux

0 in our IC

$$\delta\vec{Q} = -\chi\,\delta\hat{b}\,(\cancel{\hat{b}_0\cdot\nabla T}) - \chi\,\hat{b}_0\,(\delta\hat{b}\cdot\nabla T) - i\chi\,\hat{b}_0(\hat{b}_0\cdot\bar{k})\,\delta T$$

$$\hat{b} = \frac{\bar{B}}{|\bar{B}|}$$

here $\nabla\cdot\bar{B} = 0 \Rightarrow$

$\bar{k}\cdot\delta\bar{B} = 0 \Rightarrow \delta B_x = 0$

$$\boxed{\delta\hat{b} = \delta\bar{B}/B_0}$$

$|\bar{B}| = (B_0^2 + \cancel{2\bar{B}_0\cdot\delta\bar{B}} + \cancel{\delta B^2})^{1/2}$

$= B_0$ to linear order

i.e. no change to $|\bar{B}|$

$$\frac{\partial\bar{B}}{\partial t} = \nabla\times(\bar{v}\times\bar{B}) = (\bar{B}\cdot\nabla)\bar{v} - (\bar{v}\cdot\cancel{\nabla})\bar{B}$$

0 in linear theory

$$-i\omega\,\delta\bar{B} = iB_0 k\,\delta\bar{v}$$

$$\delta\bar{v} = \frac{\partial\bar{\xi}}{\partial t} = -i\omega\bar{\xi} \Rightarrow \boxed{\frac{\delta\bar{B}}{B_0} = ik\,\bar{\xi} = \delta\hat{b}}$$

$$\delta\bar{Q} = \delta Q_x = -\chi i k \xi_z \frac{dT}{dz}$$

$$- i \chi k \delta T$$

$$\nabla \cdot \delta\bar{Q} = i k \delta Q_x = k^2 \chi \left(\xi_z \frac{dT}{dz} + \delta T \right)$$

rapid thermal conduction $\Rightarrow$ $k^2 \chi \to \infty$ (conduction time $\to 0$)

"shorting out" the heat flux $\Rightarrow$

$$\delta T + \xi_z \frac{dT}{dz} = 0 \Rightarrow \Delta T = 0 .$$

field lines isothermal

pressure equil $\quad \frac{\delta p}{p} = 0 = \frac{\delta \rho}{\rho} + \frac{\delta T}{T}$

rapid thermal conduction $\Rightarrow$

$$\frac{\delta \rho}{\rho} = \xi_z \frac{d \ln T}{dz}$$

$\frac{dT}{dz} < 0 \quad \xi_z > 0 \quad \delta\rho < 0 \quad$ unstable!

MTI

Note: by this reasoning $\frac{dT}{dz} > 0$ stable

HBI Physics

$$\frac{dT}{dz} > 0$$

hot

cold

$\vec{g}$ $\vec{B}$

need k_x + k_z

$$\bar{k} \cdot \delta \bar{B} = 0 \Rightarrow k_x \delta B_x + k_z \delta B_z = 0$$

$$\Rightarrow \delta B_z \neq 0$$

$$|\bar{B}| = \left(B_0^2 + 2\,\delta\bar{B}\cdot\bar{B}_0 + \dots\right)^{1/2}$$

$$\Rightarrow \delta|\bar{B}| = \delta B_z \neq 0$$

same trick as in MTI derivation

let $k_z^2 \chi \to \infty$ conduction time $\to 0$

$$\frac{\Delta T}{T} = 2\,\xi_z \frac{d\ln T}{dz} \qquad \frac{\delta\rho}{\rho} = -\,\xi_z \frac{d\ln T}{dz}$$

Lagrangian

$$\frac{dT}{dz} > 0 \quad \xi_z > 0 \quad \delta\rho < 0$$

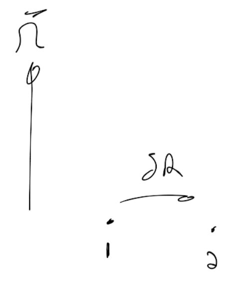
buoyantly unstable!

MTI $\Delta T/T = 0$ isothermal pert.

here $\Delta T/T > 0$ for rising fluid $\to$ hotter than surroundings $\to$ buoyant (taps into bg. heat flux)

Axssymmetric Hydro/MHD Instability Condition

Heuristic Derivation (do detailed linear pert. theory yourself to check this!!)



at position 1

$$\Omega^2(R_1)\,R_1 = GM(R_1)/R_1^2$$

at position 2

$$\Omega^2(R_2)\,R_2 = GM(R_2)/R_2^2$$

defines equil

& note $\Omega^2 R = \ell^2/R^3$

after perturbation

$$a_{net}(R_2) = a_{cent} - a_{grav}$$

$$= \frac{\ell^2(R_1)}{R_2^3} - \frac{GM(R_2)}{R_2^2}$$

assumed here that ℓ is conserved during perturbation which is why $\ell^2(R_1)$ shows up in cent. force at new position

Taylor expand

$$\ell^2(R_2) = \ell^2(R_1) + \frac{d\ell^2}{dR}\,\delta R$$

$$\Rightarrow \quad a_{net}(R_2) = \underbrace{\frac{\ell^2(R_2)}{R_2^3} - \frac{GM(R_2)}{R_2^2}}_{0 \text{ by equil. condition}} - \frac{1}{R^3}\frac{d\ell^2}{dR}\,\delta R$$

$$\Rightarrow \quad a_{net} = -\kappa^2\,\delta R \quad \text{as claimed}$$

this can be written as

$$\delta\ddot{R} + \kappa^2 \delta R = 0$$

unstable if $\kappa^2 < 0$

stable if $\kappa^2 > 0$

MHD calculation is the same except assume $\Omega = $ const. (tension $\Rightarrow$ co-rotation is heuristic argument)

instead of $l =$ const.

get $a_{net} = -\dfrac{d\Omega^2}{d\ln R}\,\delta R$

unstable if $\dfrac{d\Omega^2}{d\ln R} < 0$

MRI instability condition