Given the wide range of backgrounds in the audience, my approach has been to produce a set of lecture notes in which there is something for everyone. Some material will be basic for many of the students. Some material will be at an appropriate level. And some material will be quite advanced. While some of you may not be ready to follow everything in the notes right now, the idea is that some day soon you will be. When that day comes, why not have some material at your disposal? Consider the following quote from one of my favorite books, Doctor Faustus by Thomas Mann:

“To play the good family doctor who warns about reading something prematurely, simply because it would be premature for him his whole life long — I’m not the man for that. And I find nothing more tactless and brutal than constantly trying to nail talented youth down to its ‘immaturity,’ with every other sentence a ‘that's nothing for you yet.’ Let him be the judge of that! Let him keep an eye out for how he manages.”

M. Kunz
Princeton University
July 2016

DISCLAIMER: These notes provide a biased and incomplete survey of plasma physics. They have been written solely with astrophysical applications in mind, and are designed to accompany 2 lectures of just 90 minutes each. As such, they miss a lot. There are many subtleties that are glossed over (e.g. derivation of Vlasov-Landau kinetic equation), for example. The purpose of these notes are twofold:

(1) Ensure that you all, as PiTP summer students, are equipped to understand the subsequent lectures, which invoke plasma physics as an application;

(2) Interest you and provide some basis for advanced topics in plasma physics, which are part of the essential toolkit for modern research in plasma (astro) physics.

Enjoy and learn something new.

Matthew Kunz
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July 2016

M. Kunz Notes for PiTP Summer School 2016

Outline: lecture #1: Magnetohydrodynamics

- lengthscales, timescales, velocities, dimensionless parameters of interest — quick examples thereof
- What is a plasma?
- What is a fluid?
- Mass conservation, momentum conservation, energy conservation
- Lagrangian (comoving) derivative and curvilinear coordinates
- Induction equation, Lundquist theorem, flux freezing, Alfvén's theorem
- MHD in a straight field with homogeneous plasma : waves, linear theory, and the difference bet. δ and Δ.
- Reduced MHD
- Single-fluid treatment — what's been assumed? What's been dropped? Where is $\vec{E}$? what velocity is $\vec{u}$ exactly?
- Multi-fluid MHD: generalized Ohm's law
- Ohmic Dissipation, Ambipolar Diffusion, Hall Effect

Lecture #2: Kinetics

- particle motion w/o collisions, guiding-center motion
- Adiabatic invariance; Chew, Goldberger & Low equations; pressure anisotropy
- Braginskii-MHD for weakly collisional plasmas: anisotropic conduction and viscosity
- What if collisions aren't strong enough?
- Vlasov-Landau Equation + moments

(• Landau damping)

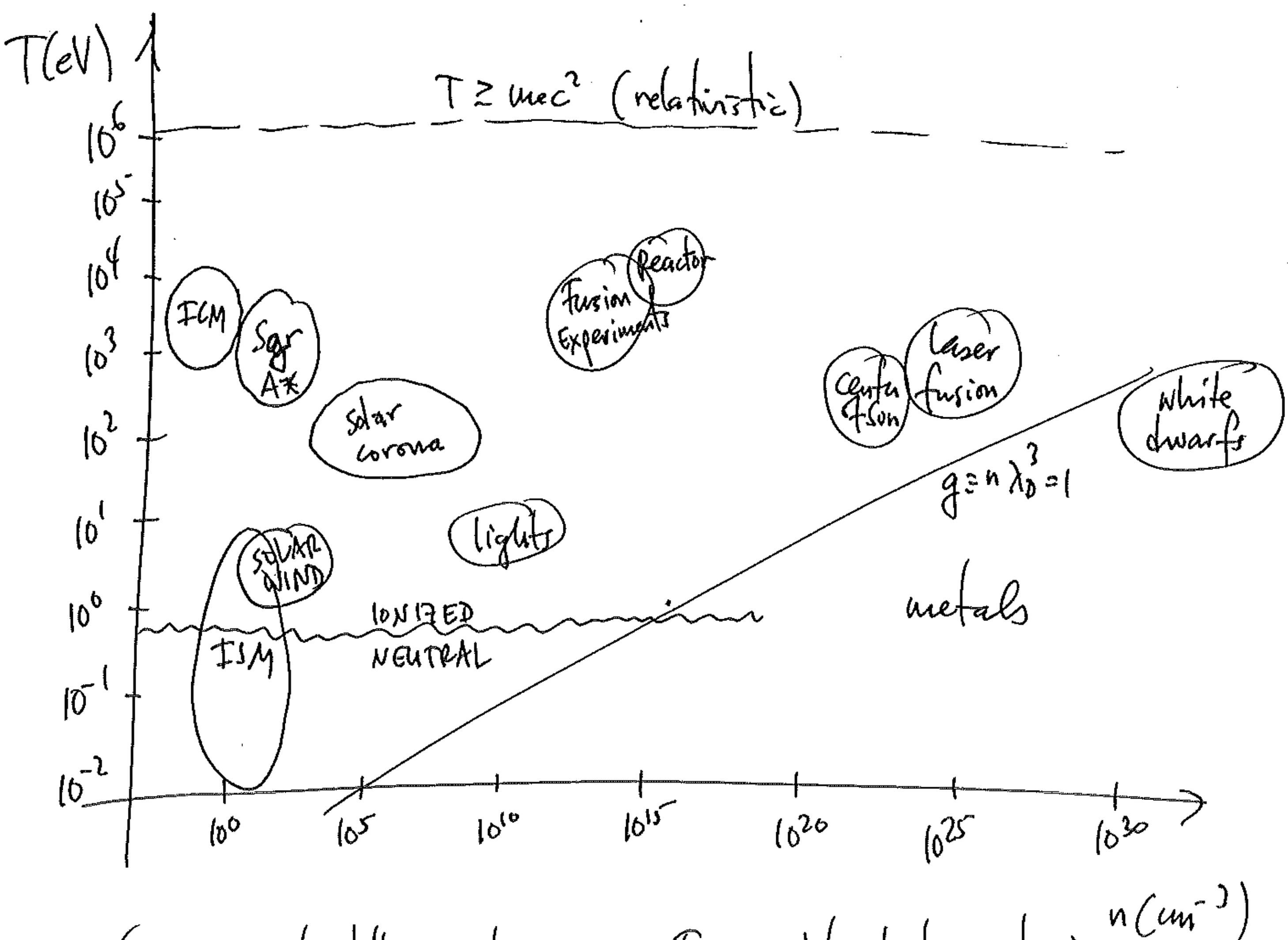
- Ordering parameters
- Kinetic MHD derivation ($k\rho_i \ll 1$, $\frac{\omega}{\Omega_i} \ll 1$, $k\lambda_{mfp} \sim 1$)
- Barnes damping and linear KMHD
- Firehose + Mirror Instabilities
- Gyrokinetics (ordering + derivation)

Magnetohydrodynamics (MHD)

If you wish to review fluid dynamics, I recommend Acheson's book "Elementary Fluid Dynamics". It's quite readable.

What is a plasma? Lengthscales, timescales, velocities, etc.

Before embarking on the wondrous world of MHD, it certainly would help to know what it is we're trying to describe with these equations. This, in itself, is no easy task, as plasmas are rich and diverse. Just look at this plot:



(ISM: interstellar medium ; ICM: intracluster medium)

You see that it's difficult to define what a plasma is!
The best definition I can offer is "significantly ionized gas that displays collective behavior". But what is "significantly"?
Indeed, protoplanetary disks with ~~tat~~ degrees of ionization $\lesssim 10^{-10}$ are still thought of as an MHD plasma (albeit nonideal).
That depends on the evolutionary timescales in the system.
The discriminating quantity is really the "plasma parameter":

$$g \equiv n\lambda_D^3 \sim \#\text{ of electrons in a Debye sphere}$$

$$\sim \frac{\langle KE\rangle}{\langle PE\rangle} \sim \frac{\frac{3}{2}T}{e^2/\lambda_D} \sim \frac{\lambda_{mfp}}{\lambda_D} \gg 1 \quad (\text{PE is small})$$

Need collective electrostatic interactions $\gg$ binary collisions, so that treatment doesn't involve ~~computing~~ counting pairwise interactions (λ_D = Debye length $\equiv \sqrt{\frac{T_e}{4\pi e^2 n_e}} \simeq 7.4\,\text{m}\sqrt{\frac{T_{eV}}{n_{cm^{-3}}}}$)

The best thing we can do is give some #s of T, n, B in representative plasmas and see what these say about lengthscales, timescales, etc...

* values are approximate *

	solar wind @ 1 au (earth location)	ICM @ ~100 kpc ("cooling radius")	galactic center @ 0.1 pc ("Bondi radius")	JET device (~1 meter)	ISM ("warm")
T	10 eV	8×10^3 eV	2×10^3 eV	10^4 eV	1 eV
n	10 cm^{-3}	5×10^{-3} cm^{-3}	100 cm^{-3}	10^{14} cm^{-3}	1 cm^{-3}
B	100 μG	1 μG	10^3 μG	3×10^4 G	5 μG

(NB: earth has $\sim\frac{1}{2}$G field and 1eV $\sim 10^4$ K)

Note that all of these have $g \gg 1$ (e.g. ICM has $g \sim 10^{15}$!)

~~Also~~ There are some interesting scales we can derive from these #'s. First, thermal speed: measures random motions of particles: $v_{th} \equiv \sqrt{\frac{2T}{m}}$ think of a Maxwell-Boltzmann distribution $\sim \frac{n}{\pi^{3/2} v_{th}^3} \exp\left(\frac{-v^2}{v_{th}^2}\right)$.

The thermal pressure of a species can be written

$$P = nT = \frac{1}{2} m n v_{th}^2 .$$

We'll see later that this need not be a scalar quantity.

Second, Alfvén speed: $v_A \equiv \frac{B}{\sqrt{4\pi m n}}$ measures speed of propagation of magnetic disturbance/wave

	SW	ICM	GC	JET	ISM
v_{thi}	40 km/s	1000 km/s	600 km/s	600 km/s	10 km/s
v_{Ai}	70 km/s	30 km/s	200 km/s	4000 km/s	10 km/s
$\beta_i \equiv \frac{v_{thi}^2}{v_{Ai}^2}$	$\sim 0.3-1$	$\sim 10^3$	~ 10	~ 0.02	~ 1

where subscript "i" means ion species (protons here)

Note that β_i is completely different in terrestrial plasmas as it is in most astrophysical plasmas. This is because most all astrophysical plasmas are confined by gravity, whereas most all terrestrial plasmas are confined by magnetic fields (by design).

So, when is a plasma a "fluid"? This introduces the mean free path between collisions: $\lambda_{mfp} = \frac{v_{th}}{\tau_{coll}}$, where τ_{coll} is some interparticle collision timescale. Some examples of the latter:

(1) $$\tau_{sn} = \frac{1}{a_{sHe}} \frac{1 + m_s/m_{H_2}}{n_{H_2} \langle \sigma w \rangle_{sH_2}}$$ for collisions between $s = \underbrace{i}_{\text{ions}}, \underbrace{e}_{\text{electrons}}, \underbrace{g_-, g_+, g_0}_{\text{grains of different charge}}$

and neutrals (both H_2 and He)

with ,

$$\left.\begin{aligned} a_{sHe} &= 1.14 \quad (s=i) \\ &= 1.16 \quad (s=e) \\ &= 1.28 \quad (s=g_+, g_-, g_0), \end{aligned}\right\} \text{ accounts for He cross section}$$

and
$$\left.\begin{aligned} \langle \sigma w \rangle_{iH_2} &= 1.69 \times 10^{-9}\ \mathrm{cm^3\,s^{-1}} \\ \langle \sigma w \rangle_{eH_2} &= 1.3 \times 10^{-9}\ \mathrm{cm^3\,s^{-1}} \\ \langle \sigma w \rangle_{gH_2} &= \underbrace{\pi a^2}_{a\text{: grain size}} \left(8T/\pi m_{H_2} \right)^{1/2} \end{aligned}\right\} \text{ collision rates}$$

(2) $$\tau_{ii} = \frac{3\sqrt{m_i}\, T_i^{3/2}}{4\sqrt{\pi}\, n_i \underbrace{\lambda_i}_{\text{Coulomb log.}} e^4} = 2.09 \times 10^7\ \mathrm{sec}\ \frac{T_{eV}^{3/2}}{n_{cm^{-3}} \lambda_i}$$ for collisions bet. ions and ions

(3) $$\tau_{ie} = \frac{3\sqrt{m_e}\, T_e^{3/2}}{4\sqrt{2\pi}\, n_e \lambda_e e^4} = 3.44 \times 10^5\ \mathrm{sec}\ \frac{T_{eV}^{3/2}}{n_{cm^{-3}} \lambda_e}$$ for i-e collisions

This ought to be compared to the other lengthscales in the system, such as those of the gradient lengthscales or fluctuations of interest, and perhaps the Larmor radius of the particle species:

$$\rho = \frac{v_{th}}{\Omega} = \sqrt{\frac{2T}{m}}\,\frac{mc}{qB}$$

representing the average gyroradius of particles of mass m, temperature T, in a field of strength B.

	SW	ICM	GC	JET	ISM
L	$\lesssim$ 1 au	~10 kpc – 100's kpc	$\lesssim$ 0.1 pc	~1 m	~1 pc – 100's kpc
λ_{mfp}	~0.1–1 au	~0.1–10 kpc	~0.01 pc	~10 km	~10^{-7} pc
ρ_i	~10^{-7} au	~1 npc	~1 ppc	~0.2 cm	~10^{-11} pc
Ω_i	~1 Hz	~0.01 Hz	~10 Hz	~300 MHz	~0.05 Hz

A "fluid" is when $\lambda_{mfp} \lll L$. A magnetized fluid has $\rho_i, \lambda_{mfp} \lll L$. Note that there are weakly collisional, magnetized fluids (GC, ICM, SW), with $\lambda_{mfp} \lesssim L$ and $\rho_i \lll L$. (near the Schwarzschild radius @ the galactic center, $\lambda_{mfp} \gg L$ too... not a fluid!)

MHD equations

these will be derived rigorously later in these notes, but can be immediately written down using some reasoning:

(1) mass conservation: $\boxed{\frac{\partial \varrho}{\partial t} + \nabla\cdot(\varrho\vec{u}) = 0}$ where ϱ = mass density, $\vec{u}$ = fluid velocity

I use \varrho (ϱ) for density to differentiate it from the Larmor radius \rho (ρ).

indeed, integrating over a fluid element's volume V:

$$\int \frac{\partial}{\partial t}\varrho \, dV + \int \vec{\nabla}\cdot(\varrho\vec{u})\, dV$$

$$= \frac{dM}{dt} + \underbrace{\oint \rho\vec{u}\cdot d\vec{S}}_{\text{flow out of mass volume through surface } S} = 0$$

(2) momentum conservation: this is just Newton's 2nd law of motion:

$$\varrho\frac{D\vec{u}}{Dt} = \varrho\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) = \vec{F} \leftarrow \text{force},$$

where $\frac{D}{Dt}$ is a comoving time derivative (Lagrangian derivative) following the motion of the fluid element — more on this later.

What kind of forces are we interested in? Well, it's astrophysics, so gravity: $-\varrho\vec{\nabla}\Phi = \vec{g}\varrho$. Also, a pressure exerts a force if it is not spatially uniform:

$$[P(x-dx/2, y, z, t) - P(x+dx/2, y, z, t)]\, dy\, dz = -\frac{\partial P}{\partial x} dV$$

nothing particularly special about x direction $\rightarrow -\vec{\nabla}P\, dV$.

Also, this is MHD, so the Lorentz force ought to enter, whose force per unit volume is $\frac{\vec{J}^e \times \vec{B}}{c}$, assuming the gas is everywhere (quasi-)neutral. With $\vec{J}^e = \frac{c}{4\pi}\vec{\nabla}\times\vec{B}$ (i.e. $\frac{v}{c} \lll 1$), this is $\frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi} = \frac{\vec{B}\cdot\vec{\nabla}\vec{B}}{4\pi} - \vec{\nabla}\frac{B^2}{8\pi}$. More on this later.

$$\Rightarrow \quad \varrho \frac{D\vec{u}}{Dt} = -\vec{\nabla}P - \varrho\vec{\nabla}\Phi + \frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi}$$

$$\boxed{\varrho \frac{D\vec{u}}{Dt} = -\vec{\nabla}\left(P + \frac{B^2}{8\pi}\right) - \varrho\vec{\nabla}\Phi + \frac{\vec{B}\cdot\vec{\nabla}\vec{B}}{4\pi}.}$$

(ideal MHD has no viscosity)

(3) energy conservation

The thermal energy is governed by conservation of entropy (in a lossless ideal fluid):

$$s = \frac{S}{N} = \frac{\ln P\varrho^{-\gamma}}{\gamma - 1},$$

up to an additive constant, where γ = adiabatic index ($= 1 + 2/f$, where f = # of deg. of freedom of a particle) and N is # of particles.

$$\Rightarrow \boxed{\frac{Ds}{Dt} = \frac{1}{\gamma-1}\frac{D\ln P\rho^{-\gamma}}{Dt} = 0} \quad \text{(adiabatic)}$$

If there are energy gains ~~of~~ or losses, then

$$\boxed{\frac{P}{\gamma-1}\frac{D\ln P\rho^{-\gamma}}{Dt} = \dot{Q}}$$ volumetric heating/cooling rate

Note that, with $P \equiv \rho c_s^2$, the adiabatic conservation law becomes

$$\frac{D}{Dt}\left[\ln \rho^{1-\gamma} + \ln c_s^2\right] = (1-\gamma)\frac{D\ln\rho}{Dt} + \gamma\frac{\ln c_s^2}{Dt}$$

$$= (1-\gamma)\left[-\vec{\nabla}\cdot\vec{u}\right] + \frac{D\ln c_s^2}{Dt} = 0$$

$$\Rightarrow \frac{Dc_s^2}{Dt} = -\frac{P}{\rho(\gamma-1)}\vec{\nabla}\cdot\vec{u} \Rightarrow$$ incompressible motions guarantee that T remains fixed for a fluid element.

Lagrangian Derivative and Curvilinear Coordinates

The derivative $\frac{D}{Dt}$ deserves some discussion, especially looking ahead to the other lectures on astrophysical applications in spherical and cylindrical geometries. It represents the time rate-of-change of a fluid quantity <u>in the frame of the fluid</u>. Thus, the $\vec{u}\cdot\vec{\nabla}\vec{u}$ "advection" term. This term can get quite complicated, since there is a differential operator acting on a vector quantity.

For example, in cylindrical (R, φ, z) coordinates:

$$\vec{u}\cdot\vec{\nabla}\vec{u} = \vec{u}\cdot\vec{\nabla}\left(u_R\hat{R} + u_\varphi\hat{\varphi} + u_z\hat{z}\right)$$

$$= (\vec{u}\cdot\vec{\nabla}u_R)\hat{R} + (\vec{u}\cdot\vec{\nabla}u_\varphi)\hat{\varphi} + (\vec{u}\cdot\vec{\nabla}u_z)\hat{z} + \frac{u_\varphi^2}{R}\frac{\partial}{\partial\varphi}(+\hat{\varphi}) + \frac{u_R u_\varphi}{R}\frac{\partial\hat{R}}{\partial\varphi}$$

$$= \underbrace{(\vec{u}\cdot\vec{\nabla}u_i)\,\hat{e}_i}_{\text{summation over } i \text{ implied here}} - \frac{u_\varphi^2}{R}\hat{R} + \frac{u_R u_\varphi}{R}\hat{\varphi} \quad \text{since } \frac{\partial\hat{\varphi}}{\partial\varphi} = -\hat{R},\ \frac{\partial\hat{R}}{\partial\varphi} = \hat{\varphi}$$

summation over i implied here

(Likewise, in spherical coordinates (r, ϑ, φ):

$$\vec{u}\cdot\vec{\nabla}\vec{u} = \left(u_r\frac{\partial}{\partial r} + \frac{u_\vartheta}{r}\frac{\partial}{\partial\vartheta} + \frac{u_\varphi}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\right)\left(u_r\hat{r} + u_\vartheta\hat{\vartheta} + u_\varphi\hat{\varphi}\right)$$

$$= (\vec{u}\cdot\vec{\nabla}u_i)\hat{e}_i - \frac{u_\vartheta^2 + u_\varphi^2}{r}\hat{r} + \left(\frac{u_\vartheta u_r}{r} - \frac{\cot\vartheta\, u_\varphi^2}{r}\right)\hat{\vartheta}$$

$$+ \left(\frac{\cot\vartheta\, u_\varphi u_\vartheta}{r} + \frac{u_\varphi u_r}{r}\right)\hat{\varphi}$$

(prove this.))

The final two terms in the cylindrical version of $\vec{u}\cdot\vec{\nabla}\vec{u}$ should look familiar from work on rotating frames. Indeed, let us write $\vec{u} = \vec{v} + R\Omega\hat{\varphi}$ and substitute into $\vec{u}\cdot\vec{\nabla}\vec{u}$:

$(\Omega = \Omega(R, z))$

$$\vec{u}\cdot\vec{\nabla}\vec{u} = \left[(\vec{v} + R\Omega\hat{\varphi})\cdot\vec{\nabla} v_i\right]\hat{e}_i + \left[(\vec{v}+R\Omega\hat{\varphi})\cdot\vec{\nabla}(R\Omega)\right]\hat{\varphi}$$

$$- \frac{(v_\varphi + R\Omega)^2}{R}\hat{R} + \frac{v_R(v_\varphi+R\Omega)}{R}\hat{\varphi}$$

$$= \left[(\vec{v}\cdot\vec{\nabla} v_i)\hat{e}_i - \hat{R}\frac{v_\varphi^2}{R} + \frac{v_R v_\varphi}{R}\hat{\varphi}\right] + \Omega\frac{\partial v_i}{\partial\varphi}\hat{e}_i$$

$$+ v_R\left(R\frac{\partial\Omega}{\partial R} + \Omega\right)\hat{\varphi} + v_z R\frac{\partial\Omega}{\partial z}\hat{\varphi} - 2v_\varphi\Omega\hat{R}$$

$$- R\Omega^2\hat{R} + v_R\Omega\hat{\varphi}$$

$$= \left[\left(\vec{v}\cdot\vec{\nabla} + \Omega\frac{\partial}{\partial\varphi}\right)v_i\right]\hat{e}_i + \left[v_R\frac{\kappa^2}{2\Omega}\hat{\varphi} + v_z R\frac{\partial\Omega}{\partial z}\hat{\varphi} - 2\Omega v_\varphi\hat{R}\right]$$

$$+ \left[\frac{v_R v_\varphi}{R}\hat{\varphi} - \frac{v_\varphi^2}{R}\hat{R}\right] - \left[R\Omega^2\hat{R}\right]$$

= [advection by flow and rotation] + [Coriolis force plus "tidal" terms due to differential rotation] + [curvature terms due to cylindrical geometry — dropped in shearing box] + [centrifugal force — usually balances disk gravity]

Induction Equation, Lundquist + Alfvén theorems

Faraday's law: $\frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla}\times\vec{E}$

Up to now, we've said nothing of $\vec{E}$, which is a subtlety in ideal MHD that we'll return to. For now, consider the following

Ohm's law: $\vec{E} = -\frac{\vec{u}\times\vec{B}}{c} + \eta\vec{J} = -\frac{\vec{u}\times\vec{B}}{c} + \frac{\eta c}{4\pi}\vec{\nabla}\times\vec{B}$

The $\eta\vec{J}$ part makes perfect sense ("Ohm's law"), but what of the $\frac{\vec{u}\times\vec{B}}{c}$ term? This accounts for a frame transformation from the lab frame to the comoving fluid frame, in which $\vec{E}=\eta\vec{J}$; $\frac{\vec{u}\times\vec{B}}{c}$ is just the frame transformation:

$$\underbrace{\vec{E}'}_{\text{fluid}} = \underbrace{\vec{E}}_{\text{lab}} + \frac{\vec{u}\times\vec{B}}{c} = \underbrace{\eta\vec{J}}_{\text{resistivity}}$$

$$\Rightarrow \boxed{\frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla}\times\left[-\frac{\vec{u}\times\vec{B}}{c} + \frac{\eta c}{4\pi}\vec{\nabla}\times\vec{B}\right] = \vec{\nabla}\times(\vec{u}\times\vec{B}) - \vec{\nabla}\times\left(\frac{\eta c^2}{4\pi}\vec{\nabla}\times\vec{B}\right)}$$

if η = constant, this $= \frac{c^2\eta}{4\pi}\nabla^2\vec{B}$ ← diffusion!

we'll now show that this represents the advection of magnetic field by the flow.

First, note that

$$\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{B}) = \nabla\cdot\left[\vec{\nabla}\times(\ldots)\right] = 0 \rightarrow$$ no monopoles if not existed initially.

Also, let's compare advection and diffusion:

$$\frac{|\vec{\nabla}\times(\vec{u}\times\vec{B})|}{|\eta\vec{\nabla}^2\vec{B}|} \sim \frac{\frac{u}{\ell}B}{\frac{\eta}{\ell^2}B} = \boxed{\frac{u\ell}{\eta} \equiv Rm}$$ magnetic Reynolds #.

examples of Rm:

liquid metals	$\sim 10^{-3} \ldots 10^{-1}$
planet interiors	$\sim 100 \ldots 300$
solar convection zone	$\sim 10^{6} \ldots 10^{9}$
ISM	$\sim 10^{18}$
ICM	$\sim 10^{29}$

with $Rm \ggg 1$ in most astrophysical systems, we can often safely neglect η, to get Ideal MHD. But this is only true to a point. Eventually, scales will almost always be developed with $\nabla^2\vec{B}$ large, in which case it doesn't matter how small is η.

(Aside: Another useful dimensionless # that you might encounter, especially in studies of reconnection, is the Lundquist number $S \equiv \frac{\ell v_A}{\eta}$. It measures the ratio of diffusion time to Alfvén crossing time.)

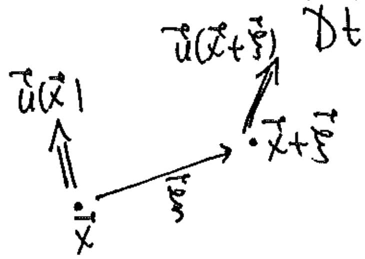
What does $\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B})$ imply? With the help of a vector identity and the continuity equation (i.e. mass conservation),

$$\frac{\partial \vec{B}}{\partial t} = \underbrace{-\vec{u}\cdot\vec{\nabla}\vec{B}}_{\text{"advection"}} + \underbrace{\vec{B}\cdot\vec{\nabla}\vec{u}}_{\text{"stretching"}} - \underbrace{\vec{B}\,\vec{\nabla}\cdot\vec{u}}_{\text{"compression"}} + \cancel{\vec{u}\,\vec{\nabla}\cdot\vec{B}}^{\,0}$$

$$\rightarrow \frac{D\vec{B}}{Dt} = \vec{B}\cdot\vec{\nabla}\vec{u} + \frac{\vec{B}}{\rho}\frac{D\rho}{Dt} \Rightarrow \boxed{\frac{D}{Dt}\frac{\vec{B}}{\rho} = \frac{\vec{B}}{\rho}\cdot\vec{\nabla}\vec{u}}$$

Note that the evolution equation of an infinitesimal Lagrangian displacement of a fluid element $\vec{\xi}(t)$ is

$$\frac{D\vec{\xi}(t)}{Dt} = \vec{u}(\vec{x}+\vec{\xi}) - \vec{u}(\vec{x}) \approx \vec{\xi}\cdot\vec{\nabla}\vec{u}$$

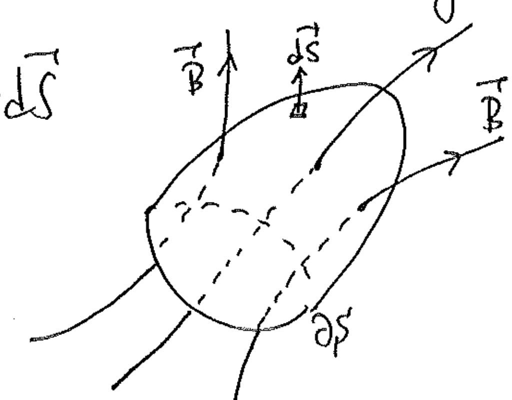


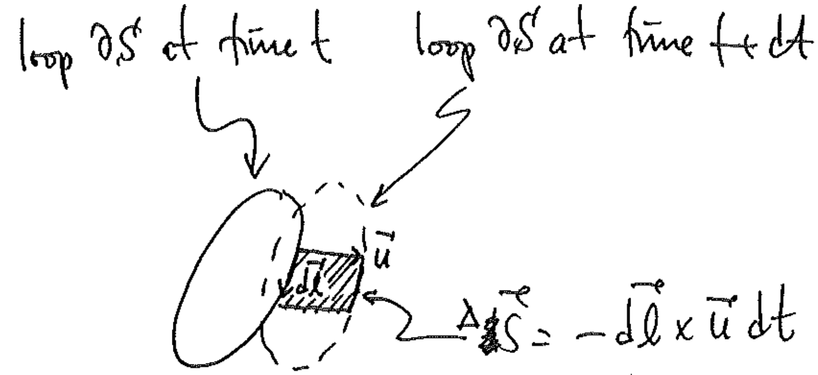
SAME EQUATION!

$\Rightarrow$ Lundquist theorem: fluid elements that lie on a field line initially will remain on that field line.

$\rightarrow$ in ideal MHD, fluid flow drags field lines along with it.

Alternatively, one can ~~phas~~ phrase this as a conservation law by ~~defini~~ defining the magnetic flux $\Phi_B \equiv \oint_S \vec{B}\cdot d\vec{S}$





moving surface ... does flux change through this surface?

$$\Phi_B(t) = \int_{S(t)} \vec{B}(t) \cdot d\vec{S}$$

$$\Phi_B(t+dt) = \int_{S(t+dt)} \vec{B}(t+dt) \cdot d\vec{S} = \int_{S(t)} \vec{B}(t+dt) \cdot d\vec{S} + \int_{\Delta S(t)} \vec{B}(t+dt) \cdot d\vec{S}$$

$$= \underbrace{\int_{S(t)} \vec{B}(t) \cdot d\vec{S}}_{= \Phi_B(t)} + dt \int_{S(t)} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \int_{\Delta S(t)} \vec{B}(t) \cdot (-d\vec{l} \times \vec{u})\, dt + O(dt^2)$$

$$= -dt \int_{\Delta S(t)} (\vec{u} \times \vec{B}) \cdot d\vec{l}$$

$$= -dt \oint_{S(t)} \left[\vec{\nabla} \times (\vec{u} \times \vec{B})\right] \cdot d\vec{S}$$

$$\Rightarrow \frac{d\Phi_B}{dt} = \frac{\Phi_B(t+dt) - \Phi_B(t)}{dt} = \int_{S(t)} \left[\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times (\vec{u} \times \vec{B})\right] \cdot d\vec{S} = 0.$$

$\Rightarrow$ Flux is conserved through any fluid element

(Alfvén's theorem)

(NB: there is an analogous result in hydrodynamics for a fluid flow described by a vorticity $\vec{\omega} = \vec{\nabla} \times \vec{u}$, which satisfies $\frac{\partial \vec{\omega}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{\omega})$.)

Before moving on to MHD waves, a comment is in order about the Lorentz force now that we know fluid elements carry field lines around with them. Recall

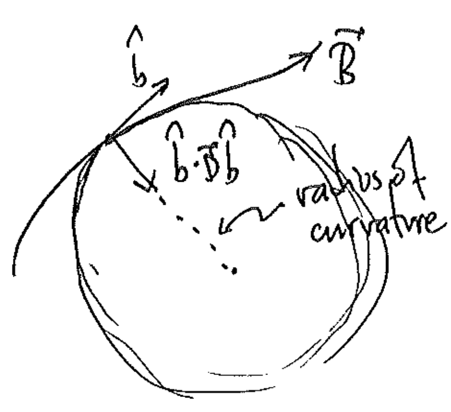
$$\vec{F}_M = \frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi} = \frac{\vec{B}\cdot\vec{\nabla}\vec{B}}{4\pi} - \vec{\nabla}\frac{B^2}{8\pi}$$

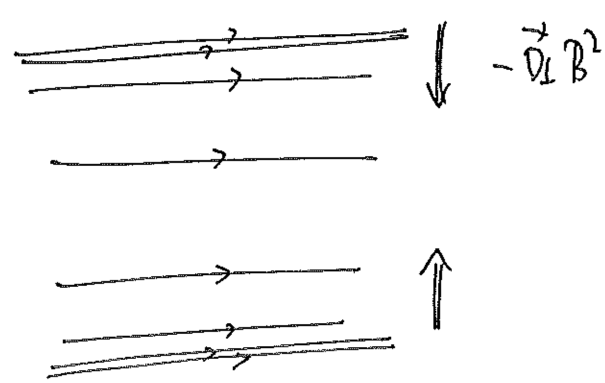
Because $\vec{\nabla}\cdot\vec{B}=0$, we can write $\vec{F} = -\vec{\nabla}\cdot\underbrace{\left[\frac{B^2}{8\pi}\overleftrightarrow{I} - \frac{\vec{B}\vec{B}}{4\pi}\right]}_{\text{"Maxwell stress"}}$

This suggests elasticity. To see this, write the magnetic unit vector $\hat{b} \equiv \frac{\vec{B}}{B}$. Then $\vec{B}\cdot\vec{\nabla}\vec{B} = B\hat{b}\cdot\nabla(B\hat{b}) = B^2\,\hat{b}\cdot\vec{\nabla}\hat{b} + \hat{b}\hat{b}\cdot\vec{\nabla}\frac{B^2}{2}$

$$\Rightarrow \vec{F}_M = \underbrace{\frac{B^2}{4\pi}\hat{b}\cdot\vec{\nabla}\hat{b}}_{\substack{\text{curvature}\\ \text{force}\\ (\text{since } \hat{b}\cdot\vec{\nabla}\hat{b}\\ \text{is curvature}\\ \text{of field line})}} - \underbrace{\underbrace{(\overleftrightarrow{I}-\hat{b}\hat{b})\cdot\vec{\nabla}}_{=\vec{\nabla}_\perp}\frac{B^2}{8\pi}}_{\substack{\text{perpendicular}\\ \text{magnetic}\\ \text{pressure force}}}$$

← wants to straighten, wants to be uniform in strength ... gives an MHD fluid some elasticity.





MHD waves and linear theory

let us summarize our dissipationless "ideal MHD" eqns:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

$$\rho\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\vec{u} = -\vec{\nabla}\left(P + \frac{B^2}{8\pi}\right) - \rho\vec{\nabla}\Phi + \frac{\vec{B}\cdot\vec{\nabla}\vec{B}}{4\pi}$$

$$\frac{P}{\gamma-1}\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\ln P\rho^{-\gamma} = 0$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B})$$

Now, let us consider a uniform, stationary MHD fluid, threaded by a uniform magnetic field. To orient our coordinate system, we will use $\vec{B} = B_0\hat{z}$, with the directions $\perp$ to the field being x and y. We perturb the fluid with small displacements, which we take (freely) to be sinusoidal:

$$\rho = \rho_0 + \delta\rho\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$$
$$\vec{B} = B_0\hat{z} + \vec{\delta B}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$$
$$\vec{u} = \phi + \vec{\delta u}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$$
$$P = P_0 + \delta P\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$$

Small? What's "small"? By "small", I mean that all nonlinearities ($\propto O(\delta^2)$) will be dropped. The result is <u>linear theory</u>.

Before we do this, note that, when computing actual observed quantities, we should take the real part (e.g. $e^{i\theta} \to \cos\theta$, $ie^{i\theta} \to -\sin\theta$, etc.)

First, let's do the simplest thing: $\vec{k} = k\hat{z}$. My notation is usually "$k_\parallel$" in this case, to remind me that k is parallel to the guide field. This notation is used in a lot of plasma physics, but less so in astronomy.
Our linearized MHD eqns. are then

$$-i\omega \frac{\delta\rho}{\rho_0} + ik_\parallel \delta u_\parallel = 0$$

$$-i\omega \vec{\delta u} = -i\frac{k_\parallel \hat{z}}{\rho_0}\left(\delta p + \frac{B_0 \delta B_\parallel}{4\pi}\right) + \frac{ik_\parallel B_0}{4\pi\rho_0}\vec{\delta B}$$

$$-i\omega \frac{\vec{\delta B}}{B_0} = ik_\parallel \vec{\delta u} - \hat{z}\, ik_\parallel \delta u_\parallel \longrightarrow \delta B_\parallel = 0 \quad (\text{as is required by } \vec{k}\cdot\vec{\delta B} = 0)$$

Note that we don't need to know $\delta\rho$ or δp to solve for the perpendicular ($\perp$) dynamics:

$$\left.\begin{aligned} -i\omega \vec{\delta u_\perp} &= \frac{ik_\parallel B_0}{4\pi\rho_0}\vec{\delta B_\perp} \\ -i\omega \frac{\vec{\delta B_\perp}}{B_0} &= ik_\parallel \vec{\delta u_\perp} \end{aligned}\right\} \left(\omega^2 - k_\parallel^2 V_A^2\right)\frac{\vec{\delta B_\perp}}{B_0} = 0$$

$$\boxed{\begin{aligned} &\omega = \pm k_\parallel V_A \\ &\text{with } V_A \equiv \frac{B_0}{\sqrt{4\pi\rho_0}} \end{aligned}}$$

These are "Alfvén waves", which are polarized across the guide field and which propagate at speed V_A, the "Alfvén speed". These waves are not associated with any motion along the field nor any changes in density.

Using $\frac{\delta p}{p_0} = \gamma \frac{\delta\rho}{\rho_0}$, the other modes are sound waves: $\boxed{\omega = \pm k_\parallel C_s}$, with $C_s \equiv \left(\frac{\gamma P_0}{\rho_0}\right)^{1/2}$ being the "sound speed".

The fifth mode is $\omega=0$, and corresponds to a relabeling of fluid elements. It's called the "entropy mode" — found by R. Kulsrud.

Now, let's let $\vec{k}=k_\parallel\hat{z}+\vec{k}_\perp$ — a more general wavevector. Then our linearized equations are

$$\text{(a)}\quad -i\omega\frac{\delta\rho}{\rho_0}+ik_\parallel\delta u_\parallel+i\vec{k}_\perp\cdot\delta\vec{u}_\perp=0$$

$$-i\omega\,\delta\vec{u}=-\frac{i\vec{k}}{\rho_0}\left(\delta p+\frac{B_0\,\delta B_\parallel}{4\pi}\right)+\frac{ik_\parallel B_0}{4\pi\rho_0}\,\delta\vec{B}$$

$$-i\omega\frac{\delta\vec{B}}{B_0}=ik_\parallel\,\delta\vec{u}-\hat{z}\left(ik_\parallel\delta u_\parallel+i\vec{k}_\perp\cdot\delta\vec{u}_\perp\right)$$

$$\text{(b)}\quad -i\omega\frac{\delta\vec{B}_\perp}{B_0}=ik_\parallel\,\delta\vec{u}_\perp \quad\text{and}\quad \text{(c)}\quad -i\omega\frac{\delta B_\parallel}{B_0}=-i\vec{k}_\perp\cdot\delta\vec{u}_\perp$$

$$\text{(d)}\quad -i\omega\,\delta\vec{u}_\perp=-\frac{i\vec{k}_\perp}{\rho_0}\left(\delta p+\frac{B_0\,\delta B_\parallel}{4\pi}\right)+\frac{ik_\parallel B_0}{4\pi\rho_0}\,\delta\vec{B}_\perp \quad\text{and}$$

$$\text{(e)}\quad -i\omega\,\delta u_\parallel=-\frac{ik_\parallel}{\rho_0}\left(\delta p+\frac{B_0\,\delta B_\parallel}{4\pi}\right)+\frac{ik_\parallel B_0}{4\pi\rho_0}\,\delta B_\parallel .$$

$$\vec{k}_\perp\cdot(d)+k_\parallel(e)\Rightarrow\ -i\omega\left(\vec{k}_\perp\cdot\delta\vec{u}_\perp+k_\parallel\delta u_\parallel\right)=-\frac{ik^2}{\rho_0}\left(\delta p+\frac{B_0\,\delta B_\parallel}{4\pi}\right)+\emptyset$$

$$\text{use (a)}:\quad \cancel{+i}\frac{\cancel{i}\omega}{\cancel{i}}\left(\frac{\cancel{i}\omega\,\delta\rho}{\rho_0}\right)=\cancel{+i}\frac{\cancel{i}k^2}{\rho_0}\left(\delta p+\frac{B_0\,\delta B_\parallel}{4\pi}\right)$$

here we've lost the entropy mode

$$\text{use }\frac{\delta p}{p_0}=\gamma\frac{\delta\rho}{\rho_0}:\quad (\omega^2-k^2c_s^2)\frac{\delta\rho}{\rho_0}=k^2V_A^2\frac{\delta B_\parallel}{B_0}$$

Now, (d) with (b) gives: $$(\omega^2-k_\parallel^2V_A^2)\frac{\delta\vec{B}_\perp}{B_0}=-k_\parallel\vec{k}_\perp\left(c_s^2\frac{\delta\rho}{\rho_0}+V_A^2\frac{\delta B_\parallel}{B_0}\right)$$

$$(\omega^2-k_\parallel^2V_A^2)\frac{\delta\vec{B}_\perp}{B_0}=-k_\parallel\vec{k}_\perp\frac{\delta B_\parallel}{B_0}\left[c_s^2\frac{k^2V_A^2}{\omega^2-k^2c_s^2}+V_A^2\right]$$

Note that the parallel and perpendicular components are now coupled!

$$\left(\omega^2 - k_\parallel^2 v_A^2\right)\frac{\delta \vec{B}_\perp}{B_0} = -k_\parallel \vec{k}_\perp v_A^2 \left[\frac{\omega^2}{\omega^2 - k^2 c_s^2}\right]\frac{\delta B_\parallel}{B_0}$$

Before we go any further, note that, if $c_s^2/v_A^2 \gg 1$, then we have $\omega^2 - k_\parallel^2 v_A^2 \simeq 0$, so we get back something like an Alfvén wave in this limit. Proceeding by using $\frac{\delta B_\parallel}{B_0} = -\frac{\vec{k}_\perp \cdot \delta\vec{B}_\perp}{k_\parallel B_0}$) we have

$$\left[\overleftrightarrow{I}\left(\omega^2 - k_\parallel^2 v_A^2\right) - \vec{k}_\perp \vec{k}_\perp v_A^2 \frac{\omega^2}{\omega^2 - k^2 c_s^2}\right] \cdot \frac{\delta \vec{B}_\perp}{B_0} = 0.$$

Taking the determinant and setting it to zero gives the dispersion relation

$$\left(\omega^2 - k_\parallel^2 v_A^2\right)\left[\omega^2 - k_\parallel^2 v_A^2 - k_\perp^2 v_A^2 \frac{\omega^2}{\omega^2 - k^2 c_s^2}\right] = 0.$$

you'll often see this written as

$$\left[\omega^4 - \omega^2 k^2\left(c_s^2 + v_A^2\right) + k_\parallel^2 v_A^2 k^2 c_s^2\right]$$

But I like it like this because you can take β limits easier.

Note that we recover the Alfvén wave solution $\omega = \pm k_\parallel v_A$. Now we also have $\omega^2 = \frac{k^2\left(c_s^2 + v_A^2\right)}{2} \pm \sqrt{\frac{k^4\left(c_s^2 + v_A^2\right)^2}{4} - k_\parallel^2 v_A^2 k^2 c_s^2}$.

These are the "magnetosonic" modes — the ⊕ solution being the "fast wave" and the ⊖ solution being the "slow wave".

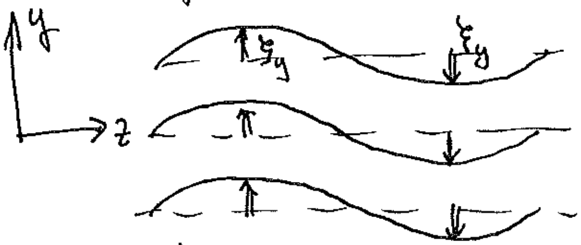
Note that, in the high-β limit, we have

$$\underbrace{\omega_+^2 \simeq k^2 c_s^2}_{\text{sound!}} \quad \text{and} \quad \underbrace{\omega_-^2 \simeq k_\parallel^2 v_A^2}_{\text{Alfvén!}}$$

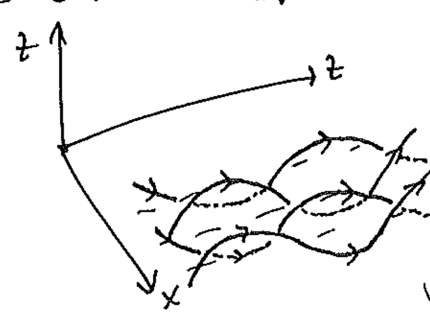
The difference between the slow mode here and an actual shear Alfvén wave is the latter involves no compressive fluctuations, being polarized with $\delta B_\parallel$ exactly $= 0$. This is sometimes called a "pseudo-Alfvén" wave.

Here are some pictures of these waves: ($\vec{\xi}$ is displacement)

Alfvén:

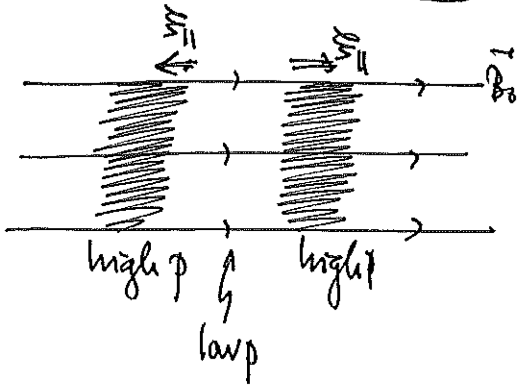


(NB: can have $k_\perp \neq 0$

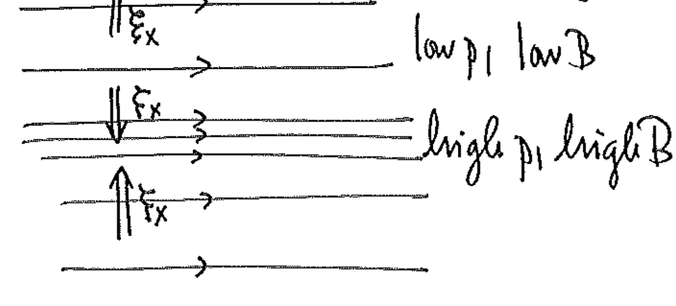


)

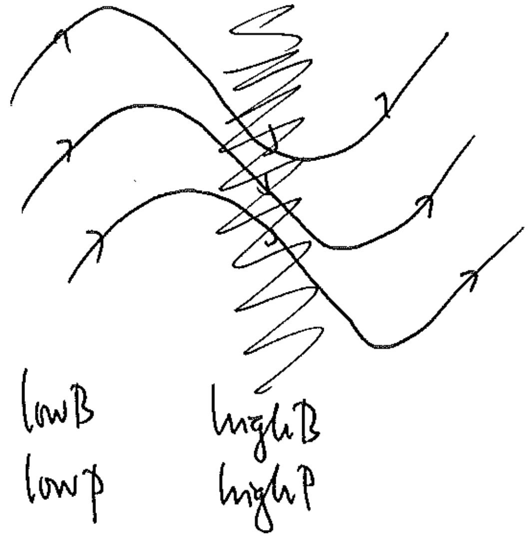
Sound:



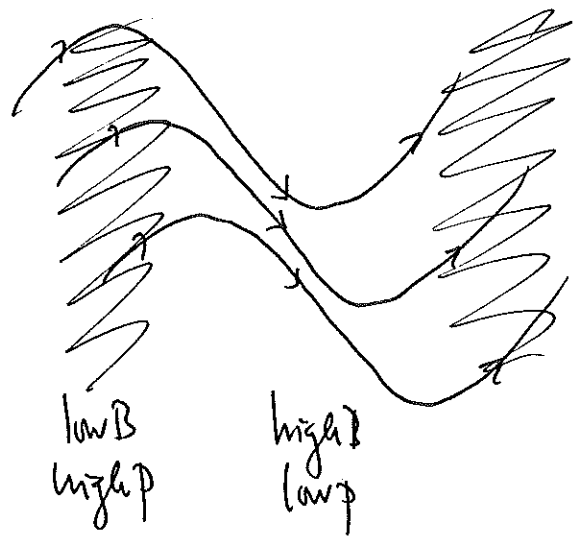
fast w/ $k_\parallel = 0$:



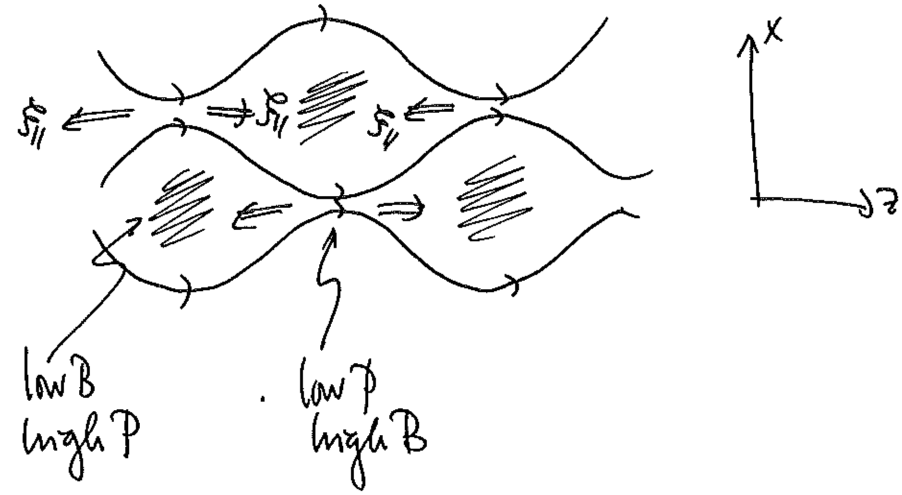
Fast:



Slow:



Slow with $k_\parallel/k_\perp \ll 1$:



Now, this last limit, $k_\parallel/k_\perp \ll 1$, is quite useful for studies of Alfvénic turbulence. What are all waves in this limit? Alfvén is the same: $\pm k_\parallel v_A$. Magnetosonic waves become

$$\omega^2 \approx \frac{k_\perp^2 (c_s^2+v_A^2)}{2}\left[1 \pm \left(1 - \frac{2k_\parallel^2 v_A^2 k_\perp^2 c_s^2}{k_\perp^4 (c_s^2+v_A^2)^2}\right)\right]$$

$\oplus$ FAST: $k_\perp^2 (c_s^2+v_A^2)$

$\ominus$ SLOW: $k_\parallel^2 v_A^2 \left(\dfrac{c_s^2}{c_s^2+v_A^2}\right)$

Let's look at the slow mode in this limit. Recall from our linear calculation that

$$\delta p = \delta\rho\, c_s^2 = \rho_0 c_s^2 \left(\frac{k^2 v_A^2}{\omega^2 - k^2 c_s^2}\right)\frac{\delta B_\parallel}{B_0} \Rightarrow \frac{\delta p}{p_0} + \left(\frac{k^2 v_A^2}{k^2 c_s^2 - \omega^2}\right)\frac{\delta B_\parallel}{B_0} = 0.$$

With $k_\perp \gg k_\parallel$ and $\omega^2 \approx k_\parallel^2 v_A^2 \left(\frac{c_s^2}{c_s^2 + v_A^2}\right)$, this becomes

$$\frac{\delta p}{p} + \frac{k_\perp^2 v_A^2 \,(\delta B_\parallel / B_0)}{k_\perp^2 c_s^2 - \frac{k_\parallel^2 v_A^2 c_s^2}{c_s^2 + v_A^2}} \approx \underbrace{\frac{\delta p}{p} + \frac{v_A^2}{c_s^2}\frac{\delta B_\parallel}{B_0} \approx 0}_{\text{pressure balance!}}$$

Slow modes w/ $k_\parallel \ll k_\perp$ are pressure-balanced structures. Indeed, in the solar wind, we observe lots of pressure-balanced ~~w~~ modes. Also, note that $\delta B_\parallel / \delta B_\perp \sim k_\perp / k_\parallel \gg 1$, so slow modes mainly have $\delta B_\parallel$. OK. Enough linear theory. It turns out that having $k_\parallel / k_\perp \ll 1$ is the gateway to nonlinear theory of Alfvén waves. This is called...

Reduced MHD

Reduced MHD is a nonlinear system of fluid equations used to describe anisotropic fluctuations in magnetized plasmas at lengthscales $L \gg \rho_i$ and frequencies $\omega \ll \Omega_i$. It was initially used to model elongated structures in tokamaks (Kadomtsev & Pogutse 1974; Strauss 1976, 1977) but has since become a standard paradigm ~~f~~ in astrophysical contexts such as solar-wind turbulence (Zank & Matthaeus 1992a,b; Bhattacharjee & Ng & Spangler 1998) and the solar corona (Oughton, Dmitruk & Matthaeus 2003; Perez & Chandran 2013). It results from applying the following ordering to the MHD equations:

$$\frac{u_\perp}{c_s} \sim \frac{u_\parallel}{c_s} \sim \frac{\delta B_\perp}{B_0} \sim \frac{\delta B_\parallel}{B_0} \sim \frac{\delta n}{n} \sim \frac{\delta p}{p_0} \sim \frac{\omega}{k_\perp c_s} \sim \frac{k_\parallel}{k_\perp} \equiv \epsilon \ll 1 ,$$

with $c_s \sim v_A$ (i.e., $\beta \sim 1$). Subsidiary limits in high & low β can be taken afterwards. Let us proceed.

continuity: $\frac{\partial n}{\partial t} + \nabla\cdot(n\vec{u}) = 0$

$$\frac{\partial}{\partial t}\delta n + \delta n\, \vec{\nabla}\cdot\vec{u} + n_0 \vec{\nabla}\cdot\vec{u} + \vec{u}\cdot\vec{\nabla}\,\delta n = 0$$

(relative to $k_\perp c_s$)

- $\frac{\partial}{\partial t}\delta n$: $\sim \omega\,\delta n \sim \epsilon^2$
- $\delta n\, \vec{\nabla}\cdot\vec{u}$: $= \delta n\, \nabla_\perp\cdot u_\perp + \delta n\, \nabla_\parallel u_\parallel \sim \epsilon^2 + \epsilon^3$
- $n_0 \vec{\nabla}\cdot\vec{u}$: $= n_0 \nabla_\perp\cdot u_\perp + n_0 \nabla_\parallel u_\parallel \sim \epsilon + \epsilon^2$
- $\vec{u}\cdot\vec{\nabla}\,\delta n$: $= u_\perp\cdot\nabla_\perp\,\delta n + u_\parallel \nabla_\parallel \delta n \sim \epsilon^2 + \epsilon^3$

lowest-order: $\boxed{\vec{\nabla}_\perp\cdot\vec{u}_\perp = 0}$

This implies that $\vec{u}_\perp$ can be written as a stream function: $\boxed{\vec{u}_\perp = \hat{z}\times\vec{\nabla}_\perp\phi}$

Similarly, for $\vec{\nabla}\cdot\delta\vec{B} = 0 \longrightarrow \boxed{\frac{\delta\vec{B}_\perp}{\sqrt{4\pi\rho_0}} = \hat{z}\times\vec{\nabla}_\perp\Psi}$

Thus, Alfvénic fluctuations can be described in terms of two scalar functions. These are governed by momentum eqn. &

induction equation.

First, induction:
$$\underset{\sim\epsilon^2}{\frac{\partial}{\partial t}\delta\vec{B}} + \underset{\sim\epsilon^2}{\vec{u}_\perp\cdot\vec{\nabla}_\perp\delta\vec{B}} + \underset{\sim\epsilon^3}{u_\parallel\nabla_\parallel\delta\vec{B}}$$

$$= \underset{\sim\epsilon^2}{B_0\frac{\partial\vec{u}}{\partial z}} + \underset{\sim\epsilon^2}{\delta\vec{B}_\perp\cdot\vec{\nabla}_\perp\vec{u}} + \underset{\sim\epsilon^3}{\delta B_\parallel\nabla_\parallel\vec{u}} - \underset{\sim\epsilon^2}{B_0\hat{z}\,\vec{\nabla}\cdot\vec{u}} - \underset{\sim\epsilon^3}{\delta\vec{B}\,\vec{\nabla}\cdot\vec{u}}$$

$$\epsilon^2:\quad \left(\frac{\partial}{\partial t} + \vec{u}_\perp\cdot\vec{\nabla}_\perp\right)\frac{\delta\vec{B}}{B_0} = \frac{\partial\vec{u}}{\partial z} + \frac{\delta\vec{B}_\perp\cdot\vec{\nabla}_\perp}{B_0}\vec{u} - \hat{z}\,\vec{\nabla}\cdot\vec{u}$$

take $\perp$ component:
$$\frac{D}{Dt}\frac{\delta\vec{B}_\perp}{B_0} = \underbrace{\left(\frac{\partial}{\partial z} + \frac{\delta\vec{B}_\perp\cdot\vec{\nabla}_\perp}{B_0}\right)}_{=\hat{b}\cdot\vec{\nabla} = \left(\hat{z} + \frac{\delta\vec{B}_\perp}{B_0}\right)\cdot\vec{\nabla}}\vec{u}_\perp$$

$$\Rightarrow \left[\frac{\partial}{\partial t} + \left(\hat{z}\times\vec{\nabla}_\perp\Phi\right)\cdot\vec{\nabla}_\perp\right]\frac{\delta\vec{B}_\perp}{B_0} = \left(\hat{z} + \frac{\delta\vec{B}_\perp\cdot\vec{\nabla}_\perp}{B_0}\right)\left(\hat{z}\times\vec{\nabla}_\perp\Phi\right)$$

Define $\{\Phi, f\} \equiv \frac{\partial\Phi}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial\Phi}{\partial y}\frac{\partial f}{\partial x}$ "Poisson bracket".

Then we have, after some algebra,
$$\boxed{\frac{\partial\Psi}{\partial t} + \{\Phi, \Psi\} = v_A\frac{\partial\Phi}{\partial z}}$$

Now, momentum equation's perpendicular component is
$$(\rho_0 + \delta\rho)\left(\underset{\sim\epsilon^2}{\frac{\partial}{\partial t}} + \underset{\sim\epsilon^2}{u_\perp\cdot\nabla_\perp} + \underset{\sim\epsilon^3}{u_\parallel\nabla_\parallel}\right)\vec{u}_\perp = -\vec{\nabla}_\perp\left(\underset{\sim\epsilon}{\delta p} + \underset{\sim\epsilon}{\frac{B_0\delta B_\parallel}{4\pi}} + \underset{\sim\epsilon^2}{\frac{|\delta\vec{B}|^2}{8\pi}}\right) + \underset{\sim\epsilon^2}{\frac{B_0}{4\pi}\frac{\partial}{\partial z}\delta\vec{B}_\perp}$$
$$+ \underset{\sim\epsilon^2}{\frac{\delta\vec{B}_\perp\cdot\vec{\nabla}_\perp}{4\pi}\delta\vec{B}_\perp} + \underset{\sim\epsilon^3}{\frac{\delta B_\parallel\nabla_\parallel\delta\vec{B}_\perp}{4\pi}}$$

$\delta\rho$: always negligible

$O(\epsilon)$: $\vec{\nabla}_\perp \left(\delta p + \frac{B_0 \delta B_\parallel}{4\pi} \right) = 0$ ($\perp$ ~~pa~~ pressure balance)

$$\rightarrow \boxed{\frac{\delta p}{p_0} = -\gamma \frac{V_A^2}{C_s^2} \frac{\delta B_\parallel}{B_0}}$$

$O(\epsilon^2)$: $$\frac{D}{Dt} \vec{u}_\perp = -\frac{\vec{\nabla}_\perp (2^{nd}\text{-order pressure})}{\rho_0} + \frac{B_0}{4\pi\rho_0} \frac{\partial \delta \vec{B}_\perp}{\partial z} + \frac{\delta \vec{B}_\perp \cdot \vec{\nabla}_\perp \, \delta \vec{B}_\perp}{4\pi\rho_0}$$

to elimate pressure term, take $\vec{\nabla}_\perp \times$ of this eqn:

$$\vec{\nabla}_\perp \times \left(\frac{\partial}{\partial t} + u_\perp \cdot \nabla_\perp \right) \underbrace{\vec{u}_\perp}_{\hat{z} \times \vec{\nabla}_\perp \Phi} = V_A^2 \, \vec{\nabla}_\perp \times \frac{\partial}{\partial z} \underbrace{\frac{\delta \vec{B}_\perp}{B_0}}_{\frac{1}{V_A} \hat{z} \times \vec{\nabla}_\perp \Psi} + V_A^2 \, \vec{\nabla}_\perp \times \left(\frac{\delta \vec{B}_\perp}{B_0} \cdot \vec{\nabla}_\perp \frac{\delta \vec{B}_\perp}{B_0} \right)$$

algebra... $\frac{D}{Dt} \nabla_\perp^2 \Phi = V_A \hat{b} \cdot \vec{\nabla} \nabla_\perp^2 \Psi$, or

$$\boxed{\frac{\partial}{\partial t} \nabla_\perp^2 \Phi + \{ \Phi, \nabla_\perp^2 \Phi \} = V_A \frac{\partial}{\partial z} \nabla_\perp^2 \Psi + \{ \Psi, \nabla_\perp^2 \Psi \}}$$

This is essentially a vorticity equation for $\vec{u}$.

Note that the Alfvénic fluctuations satisfy a closed set of equations:

$$\boxed{\begin{aligned} &\frac{D}{Dt} \nabla_\perp^2 \Phi = V_A \hat{b} \cdot \vec{\nabla} \nabla_\perp^2 \Psi \qquad \text{w/} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \{ \Phi, \ldots \} \\ &\frac{D\Psi}{Dt} = V_A \frac{\partial \Phi}{\partial z} \qquad\qquad\qquad\quad \hat{b} \cdot \vec{\nabla} = \frac{\partial}{\partial z} + \{ \Psi, \ldots \} \end{aligned}}$$

It's a straightforward exercise to obtain equations for the compressive fluctuations. They are

$$\frac{D}{Dt}\left(\frac{\delta B_\parallel}{B_0} - \frac{\delta\rho}{\rho_0}\right) = \hat{b}\cdot\vec{\nabla}u_\parallel$$

$$\frac{Du_\parallel}{Dt} = v_A^2\,\hat{b}\cdot\vec{\nabla}\,\frac{\delta B_\parallel}{B_0}$$

From entropy conservation, we also have

$$\frac{D}{Dt}\left(\frac{\delta p}{p_0} - \gamma\frac{\delta\rho}{\rho_0}\right) = 0,$$

which, when combined with pressure balance, gives

$$\frac{D}{Dt}\left(\frac{\delta\rho}{\rho_0} + \frac{v_A^2}{c_s^2}\frac{\delta B_\parallel}{B_0}\right) = 0.$$

Note that these equations are <u>linear</u> in the frame of the nonlinear Alfvén waves. Thus, compressive fluctuations propagate along disturbed field lines and are advected perpendicularly by the E×B flows.

Finally, Elsässer variables. There is a convenient simplification that makes clear the basis of theories of Alfvén-wave turbulence. Define the Elsässer potentials

$$\zeta^\pm \equiv \Phi \pm \Psi.$$

Then $\Phi = \frac{\zeta^+ + \zeta^-}{2}$ and $\Psi = \frac{\zeta^+ - \zeta^-}{2}$ and so we have

induction: $\frac{\partial}{\partial t}\left(\frac{\zeta^+ - \zeta^-}{2}\right) + \left\{\frac{\zeta^+ + \zeta^-}{2}, \frac{\zeta^+ - \zeta^-}{2}\right\} = v_A \frac{\partial}{\partial z} \frac{\zeta^+ + \zeta^-}{2}$

momentum: $\frac{\partial}{\partial t} D_\perp^2 \left(\frac{\zeta^+ + \zeta^-}{2}\right) + \left\{\frac{\zeta^+ + \zeta^-}{2}, D_\perp^2 \frac{\zeta^+ + \zeta^-}{2}\right\}$

$$= v_A \frac{\partial}{\partial z} D_\perp^2 \frac{\zeta^+ - \zeta^-}{2} + \left\{\frac{\zeta^+ - \zeta^-}{2}, D_\perp^2 \frac{\zeta^+ - \zeta^-}{2}\right\}$$

Noting that $\{\zeta^\pm, \zeta^\pm\} = 0$ and taking $D_\perp^2$ of the former equation, we have

$$\frac{\partial}{\partial t} D_\perp^2 (\zeta^+ - \zeta^-) + \frac{D_\perp^2}{2}\Big(\{\zeta^-, \zeta^+\} - \{\zeta^+, \zeta^-\}\Big) = v_A \frac{\partial}{\partial z} D_\perp^2 (\zeta^+ + \zeta^-)$$

$$\frac{\partial}{\partial t} D_\perp^2 (\zeta^+ + \zeta^-) + \frac{1}{2}\Big(\{\zeta^+, D_\perp^2 \zeta^-\} + \{\zeta^-, D_\perp^2 \zeta^+\}\Big)$$

$$= v_A \frac{\partial}{\partial z} D_\perp^2 (\zeta^+ - \zeta^-) + \frac{1}{2}\Big(-\{\zeta^+, \zeta^-\} - \{\zeta^-, \zeta^+\}\Big)$$

Adding and subtracting these two gives, finally,

$$\boxed{\frac{\partial}{\partial t} D_\perp^2 \zeta^\pm \mp v_A \frac{\partial}{\partial z} D_\perp^2 \zeta^\pm = -\frac{1}{2}\Big[\{\zeta^+, D_\perp^2 \zeta^-\} + \{\zeta^-, D_\perp^2 \zeta^+\} \mp D_\perp^2 \{\zeta^+, \zeta^-\}\Big]}$$

LHS is linear wave physics.
RHS is nonlinear wave couplings.

NB: only counterpropagating wave packets interact! more on this in Eliot's talk.

Lagrangian vs. Eulerian perturbations

In these notes, I am exclusively using "Eulerian" perturbations, denoted by a "δ". This measures the change in a quantity <u>at fixed position</u>: $\delta\vec{u} = \vec{u}(\vec{r}) - \vec{u}_0(\vec{r})$. This is fine when dealing with stationary equilibria. But sometimes it is useful to compute the change in a quantity comoving with the flow. This is a "Lagrangian" perturbation, denoted by a "Δ". It measures the change in a particular fluid element as it undergoes a displacement $\vec{\xi}$: $\Delta\vec{u} = \vec{u}(\vec{r}+\vec{\xi}) - \vec{u}_0(\vec{r})$. To linear order, the two are related by

$$\Delta = \delta + \vec{\xi}\cdot\vec{\nabla};$$

you can see that the difference matters in a stratified plasma. Here are a few handy things:

$$\Delta\vec{u} = \frac{\partial\vec{\xi}}{\partial t} + \underbrace{\vec{u}}_{\text{Background flow}}\cdot\vec{\nabla}\vec{\xi} = \delta\vec{u} + \vec{\xi}\cdot\vec{\nabla}\vec{u} \Rightarrow \delta\vec{u} = \frac{\partial\vec{\xi}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{\xi} - (\vec{\xi}\cdot\vec{\nabla})\vec{u}$$

You can also think of δ and Δ as difference operators, e.g., $\delta\left(\frac{1}{\rho}\right) = \frac{-\delta\rho}{\rho^2}$. But be careful! δ and $\frac{\partial}{\partial x}$ commute, but Δ and $\frac{\partial}{\partial \vec{x}}$ don't! Eulerian perturbations are less prone to misunderstanding, since $\Delta \neq 0$ doesn't necessarily indicate a physical change.

Single-fluid MHD, $\vec{E}$, and $\vec{u}$

You may have noticed that we've been a bit sloppy. What fluid velocity is $\vec{u}$, exactly? Where is $\vec{E}$ in the momentum equation? I thought we had ions and electrons ... where are they? There is much that is often glossed over in presentations of MHD about what exactly is being assumed. Some knowledge under our belts, we now clean this up. First, what is $\vec{u}$?

Technically, $\vec{u} = \frac{\sum_s m_s n_s \vec{u}_s}{\sum_s m_s n_s}$, where s is the species index.

But is this the same $\vec{u}$ that's in the induction equation? What if one of the species is a neutral species ... why would field lines be frozen into a neutral species? And we talked about $\vec{E}'$ being the electric field in the $\vec{u}$ frame. But why that particular frame? And I said nothing of Poisson's equation. Why didn't we obtain $\vec{E}$ from that? You do that in your E&M course ... why not here?

We start with a discussion of the latter issue (viz., where $\vec{E}$ comes from) and then delve in the former topic — what is $\vec{u}$ and why. This will lead to multi-fluid MHD.

So why doesn't $\nabla\cdot\vec{E} = \overset{4\pi!}{\Sigma_s} q_s n_s$ determine our electric field? This is because we are operating at scales larger than the Debye length, where quasi-neutrality holds. Indeed

$$\frac{\nabla\cdot\vec{E}}{4\pi e\delta n} \sim \frac{kE}{4\pi e\delta n} \sim \underbrace{\frac{k^2\varphi}{4\pi e\delta n} \sim \frac{k^2 T}{4\pi e^2 n}}_{\text{Boltzmann relation}} \sim \frac{k^2 m_e v_{the}^2}{4\pi e^2 n} \sim \frac{k^2 v_{the}^2}{\omega_{pe}^2} \sim k^2\lambda_D^2 \ll 1,$$

electrostatic potential

and so $\Sigma_s q_s n_s = 0$ holds to a very good approximation. This is because of Debye shielding: the screening of electrostatic fields by mobile charges on typical distances $\sim \lambda_D = \sqrt{\frac{T_e}{4\pi e^2 n_e}}$. Indeed, the Coulomb potential $\varphi \sim \frac{e^{-r/\lambda_D}}{r}$

Instead, $\vec{E}$ is obtained by enforcing quasi-neutrality on the momentum equation of the species comprising the plasma.

Moving on to $\vec{u}$...

If we were to have obtained our fluid equations from taking moments of the Vlasov-Landau kinetic equ., as is done later in these notes, instead of arguing for our fluid equations, all this, would become clear. The result would be that

(stuff about $\vec{u}$ and $\vec{E}$)

$$\frac{\partial n_s}{\partial t} + \vec{\nabla}\cdot(n_s\vec{u}_s) = 0$$

$$m_s n_s\left(\frac{\partial \vec{u}_s}{\partial t} + \vec{u}_s\cdot\vec{\nabla}\vec{u}_s\right) = -\vec{\nabla}\cdot\overleftrightarrow{P}_s + q_s n_s\left(\vec{E} + \frac{\vec{u}_s\times\vec{B}}{c}\right) + \vec{R}_s,$$

in concert with $\frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla}\times\vec{E}$, for each species s; here, $\overleftrightarrow{P}_s$ is the pressure tensor $\left(= p_s\overleftrightarrow{I}, \text{ for an isotropic fluid}\right)$ and $\vec{R}_s$ is the friction force on species s. We get our "single-fluid" MHD equations by summing these two over s:

- $\Sigma_s\left[\frac{\partial n_s}{\partial t} + \nabla\cdot(n_s\vec{u}_s) = 0\right] \rightarrow \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$

- $\Sigma_s\left[m_s n_s\left(\frac{\partial \vec{u}_s}{\partial t} + \vec{u}_s\cdot\vec{\nabla}\vec{u}_s\right) = -\vec{\nabla}\cdot\overleftrightarrow{P}_s + q_s n_s\left(\vec{E} + \frac{\vec{u}_s\times\vec{B}}{c}\right) + \vec{R}_s\right]$

$$\rightarrow \rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right) = -\vec{\nabla}\cdot\left[\Sigma_s\overleftrightarrow{P}_s + \Sigma_s m_s n_s\,\delta\vec{u}_s\,\delta\vec{u}_s\right]$$

$$+ \underbrace{\cancel{\Sigma_s q_s n_s\vec{E}}}_{0 \text{ by quasi-neutrality}} + \underbrace{\cancel{\Sigma_s \frac{q_s n_s\vec{u}_s\times\vec{B}}{c}}}_{=\frac{\vec{j}\times\vec{B}}{c}} + \underbrace{\cancel{\Sigma_s\vec{R}_s}}_{0 \text{ by Newton's 3rd law}}$$

where $\delta\vec{u}_s = \vec{u}_s - \vec{u}$

Evidently, we're assumed quasi-neutrality $\sum_s q_s n_s = 0$ and zero interspecies drifts $\sum_s m_s n_s \delta\vec{u}_s \delta\vec{u}_s = 0$. The latter can be argued for if $\delta u_s \ll v_{th,s}$, but more can actually be said (see below). The former is good for scales l satisfying $l/\lambda_D \ggg 1$.

What about $\vec{E} + \frac{\vec{u}\times\vec{B}}{c} = \eta\vec{J}$? We can obtain $\vec{E}$ from any one of the momentum equations:

$$\vec{E} + \frac{\vec{u}_s\times\vec{B}}{c} = + \frac{\nabla\cdot\overleftrightarrow{P}_s}{q_s n_s} + \frac{m_s}{q_s}\left(\frac{\partial\vec{u}_s}{\partial t} + \vec{u}_s\cdot\nabla\vec{u}_s\right) - \frac{\vec{R}_s}{q_s n_s}$$

"Generalized Ohm's law"

Estimate size of each term, relative to $\vec{u}_s\times\vec{B}/c$ advection term:

$$\left|\frac{\nabla\cdot\overleftrightarrow{P}_s}{q_s n_s}\right| \Big/ \left|\frac{\vec{u}_s\times\vec{B}}{c}\right| \sim \frac{n_s T_s}{l q_s n_s}\frac{c}{u_s B} \sim \left(\frac{\rho_s}{l}\right)\left(\frac{v_{th,s}}{u_s}\right) \lll 1$$

$$\left|\frac{\vec{R}_s}{q_s n_s}\right| \Big/ \left|\frac{\vec{u}_s\times\vec{B}}{c}\right| \sim \frac{\nu_{coll,s} m_s n_s u_s c}{q_s n_s u_s B} \sim \frac{\nu_{coll,s}}{\Omega_s} \lll 1$$

$$\left|\frac{m_s}{q_s}\frac{D\vec{u}_s}{Dt}\right| \Big/ \left|\frac{\vec{u}_s\times\vec{B}}{c}\right| \sim \frac{m_s u_s}{\tau q_s}\frac{c}{u_s B} \sim (\tau\Omega_s)^{-1} \lll 1$$

τ: some evolutionary timescale

All of these are tiny in MHD, which means $\boxed{\vec{E} + \frac{\vec{u}_s\times\vec{B}}{c} \simeq 0}$

This implies that all species drift across field @ the (species-independent!) $\vec{E}\times\vec{B}$ velocity: $\boxed{\vec{u}_{s\perp} = \vec{u}_\perp = \frac{c\vec{E}\times\vec{B}}{B^2}}$.

Thus, $\boxed{\delta\vec{u}_s \longrightarrow \delta u_{\parallel s}\hat{b} \text{ only!}}$

This also tells us that flux-freezing can be broken by pressure-gradients effects, inertial terms, and collisions.

(NB: Note that if $P_s = P_s(n_s)$, flux-freezing is not broken: $\vec{E} \sim \nabla \frac{\ln n_s}{q_s}$, whose curl vanishes!)

Multi-fluid MHD: Ambipolar Diffusion, Ohmic Dissipation, Hall Effect

What if one of the species were neutral? e.g. atomic/molecular neutral hydrogen or neutral helium, or neutral grains, etc. What if the charged particles only comprised a small percentage of the total population? Then our mean velocity is $\vec{u} \equiv \frac{\sum_s m_s n_s \vec{u}_s}{\sum_s m_s n_s} \approx \vec{u}_n$ ← "neutral". A neutral velocity in the induction equation? a neutral fluid subject to the Lorentz force? what's going on?

Our discussion of so-called "non-ideal MHD" will be in the context of molecular clouds, protostellar cores, protoplanetary disks, and other similarly cold, dense, poorly ionized fluids. These are all quite collisional, so much so that, even though the neutrals do not execute Larmor motion, they are (almost) frozen into the field via collisions with magnetized species.

Let us proceed:

Our neutral fluid satisfies the following equations:

$$\frac{\partial n_n}{\partial t} + \vec{\nabla}\cdot\left(n_n\vec{u}_n\right) = 0$$

$$m_n n_n\left(\frac{\partial}{\partial t} + \vec{u}_n\cdot\vec{\nabla}\right)\vec{u}_n = -\vec{\nabla}p_n + m_n n_n\vec{g} + \sum_s \vec{F}_{ns},$$

where $\vec{F}_{ns}$ is the force on the neutrals (n) by collisions with species $s\neq n$.

What of the other species? It's a very good approx. in the systems mentioned above to ignore their inertia (verify!), and so their momentum equations are

$$0 = q_s n_s\left(\vec{E} + \frac{\vec{u}_s\times\vec{B}}{c}\right) + \vec{F}_{sn} + \sum_{s'}\vec{F}_{ss'},$$

where $\vec{F}_{ss'}$ is the collisional force on s due to ~~&~~ collisions with $s' \neq s \neq n$.

Summing this over s gives

$$0 = \frac{\vec{j}\times\vec{B}}{c} + \sum_s\vec{F}_{sn} = \frac{\vec{j}\times\vec{B}}{c} - \sum_s\vec{F}_{ns},$$

using $\sum_s q_s n_s = 0$ and $\sum_s q_s n_s \vec{u}_s = \vec{j}$. Adding this to the neutrals' momentum equation, we have

$$m_n n_n\left(\frac{\partial\vec{u}_n}{\partial t} + \vec{u}_n\cdot\vec{\nabla}\vec{u}_n\right) = -\vec{\nabla}p_n + m_n n_n\vec{g} + \frac{(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi};$$

the neutrals feel the Lorentz force!

Surely the fact that the neutrals are <u>neutral</u> shows up somewhere! It does... cue the induction equation:

$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B})$. That $\vec{u}$ certainly shouldn't be $\vec{u}_n$.

How about this... define $\vec{u}_f$ as the velocity of the frame to which the flux is frozen: $\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}_f\times\vec{B})$. Then write

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\Big[\underbrace{(\vec{u}_f-\vec{u}_e)}_{①}\times\vec{B} + \underbrace{(\vec{u}_e-\vec{u}_i)}_{②}\times\vec{B} + \underbrace{(\vec{u}_i-\vec{u}_n)}_{③}\times\vec{B} + \vec{u}_n\times\vec{B}\Big].$$

Let's write ①, ②, and ③ in terms of things we know.*

①: $(\vec{u}_f-\vec{u}_e)\times\vec{B}$ is customarily written as $-\frac{\eta c^2}{4\pi}\vec{\nabla}\times\vec{B}$; i.e. the flux is frozen in the electrons but for resistivity in the form of Ohm's law. This is <u>Ohmic Dissipation</u>.

② $\vec{u}_e-\vec{u}_i = -\frac{1}{en_e}\vec{J}$ by quasi-neutrality

$\Rightarrow (\vec{u}_e-\vec{u}_i)\times\vec{B} = -\frac{\vec{J}\times\vec{B}}{en_e}$. This is the <u>Hall effect</u>.

③ For $\vec{u}_i-\vec{u}_n$, return to the summed-charged momentum equation: $\frac{\vec{J}\times\vec{B}}{c} = \sum_s \vec{F}_{ns} = \vec{F}_{ni}+\vec{F}_{ne}$, where

$\vec{F}_{ni} = \frac{\rho_n}{\tau_{ni}}(\vec{u}_i-\vec{u}_n)$ and $\vec{F}_{ne} = \frac{\rho_n}{\tau_{ne}}(\vec{u}_e-\vec{u}_n)$

are the forces on the neutrals due to elastic collisions with ions and electrons. One can show that $\frac{\tau_{ni}}{\tau_{ne}} \sim \left(\frac{m_e}{m_n}\right)^{1/2}\left(\frac{n_e}{n_i}\right) \ll 1$;

---

* Here, we're considering a plasma of ions, electrons, and neutrals. A more general calculation can be found in Appendix B of Kunz + Mouschovias 2009, ApJ.

that is, the momentum exchange bet. a neutral and an electron is much less effective than that bet. a neutral and an ion.

Then,
$$\frac{\rho_n}{\tau_{ni}}\left(\vec{u}_i - \vec{u}_n\right) \simeq \frac{\vec{J}\times\vec{B}}{c}$$

$$\Rightarrow \quad \vec{u}_i - \vec{u}_n \simeq \frac{\tau_{ni}}{\rho_n}\frac{\vec{J}\times\vec{B}}{c}, \quad \tau_{ni} \text{ is sometimes written as } (\gamma\rho_i)^{-1},$$

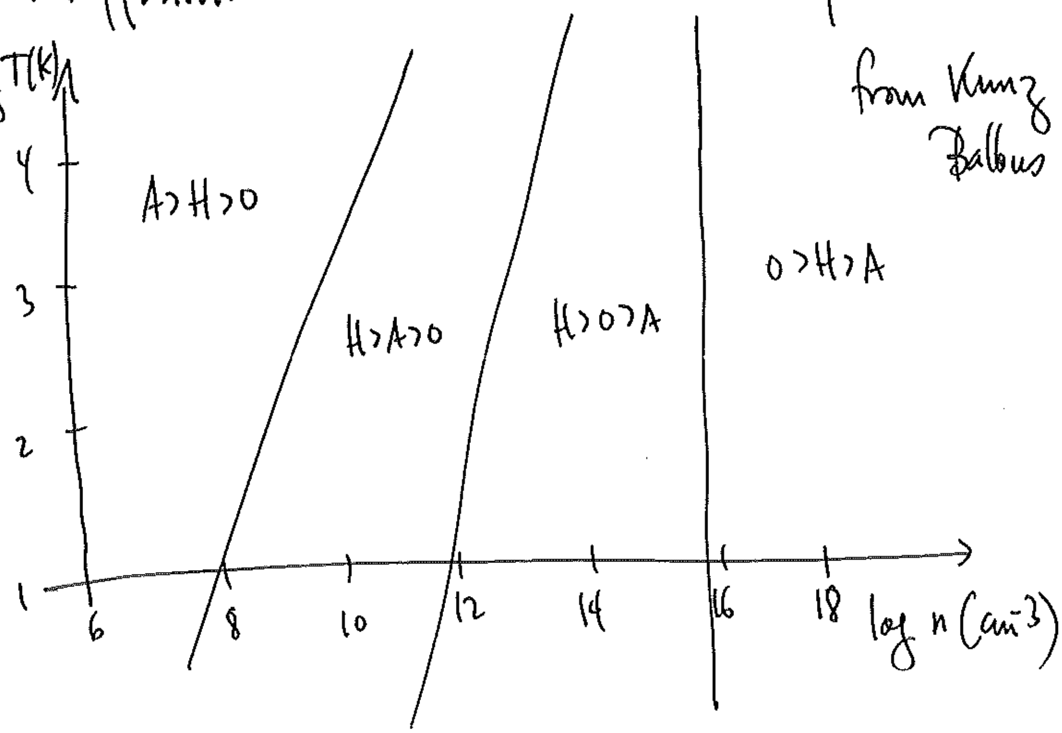
where γ is the drag coefficient. So,
$$(\vec{u}_i - \vec{u}_n)\times\vec{B} \simeq \frac{(\vec{J}\times\vec{B})\times\vec{B}}{c\gamma\rho_i\rho_n}.$$

This is Ambipolar Diffusion.

Our induction equation becomes

$$\boxed{\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{u}_n\times\vec{B}\right) - \vec{\nabla}\times\left[\underbrace{\frac{\eta c^2}{4\pi}\vec{\nabla}\times\vec{B}}_{O} + \underbrace{\frac{c(\vec{\nabla}\times\vec{B})\times\vec{B}}{4\pi e n_e}}_{H} - \underbrace{\frac{((\vec{\nabla}\times\vec{B})\times\vec{B})\times\vec{B}}{4\pi\gamma\rho_i\rho_n}}_{A}\right]}$$

The flux is frozen in the bulk (neutral) fluid but for Ohmic, Hall, and ambipolar diffusion. In an ion-electron-neutral plasma with $\frac{v_A}{c_s} = 0.1$,



from Kunz & Balbus 2004

let's compare each of these with the induction term, $\underset{\text{"I"}}{\vec{U}_n\times\vec{B}} \sim v_A B$:

$$\frac{I}{O} \sim \frac{l v_A B\, 4\pi}{\eta c^2 B} \sim \frac{4\pi}{c^2}\frac{l v_A}{\eta} = S \text{ (Lundquist \#)}$$

$$\frac{I}{H} \sim \frac{l v_A B\, 4\pi e n_e}{c B^2} \sim \frac{l\, 4\pi e n_e}{c\sqrt{4\pi \rho_n}} = \left(\frac{4\pi e^2 n_e}{m_i c^2}\right)^{1/2}\left(\frac{\rho_i}{\rho_n}\right)^{1/2} = \left(\frac{l}{d_i}\right)\left(\frac{\rho_i}{\rho_n}\right)^{1/2}$$

(d_i = ion skin depth)

Normally, Hall effect is important on ion-skin-depth scales, but in a poorly ionized plasma, this lengthscale is weighted by $(\rho_n/\rho_i)^{1/2} \gg 1$ — i.e. the skin depth is artificially enhanced. This is because the ions carry the inertia of collisionally coupled neutrals with them — a heavy load!

$$\frac{I}{A} \sim \frac{l v_A B\, 4\pi \gamma \rho_i \rho_n}{B^3} \sim \frac{l/v_A}{\tau_{ni}} \sim \frac{t_A}{\tau_{ni}} = \frac{\text{Alfven crossing time}}{\text{neutral-ion collision time}}$$

if $\frac{t_A}{\tau_{ni}} \ll 1$, then collisions are rare and Alfven wave damps on the neutrals; if $\frac{t_A}{\tau_{ni}} \gg 1$, then collisions are frequent and the entire plasma is well-coupled to the magnetic field.

* An exercise you should do is repeat the linear Alfven wave calculation ($\vec{k} = k_\parallel \hat{b}_0$) with each of these terms in place to see what happens. *
You'll find that Alfven waves damp at a different rate than slow modes, and that the Hall effect introduces wave dispersion.

As an Appendix to this part of the notes, I give a rigorous derivation of the generalized Ohm's law for a poorly ionized plasma:

Start w/ momentum equation for inertialess, charged species s:

$$q_s n_s\left(\vec{E} + \frac{\vec{u}_s \times \vec{B}}{c}\right) + \frac{\rho_s}{\tau_{sn}}\left(\vec{u}_n - \vec{u}_s\right) = 0$$

Introduce $\vec{w}_s \equiv \vec{u}_s - \vec{u}_n$ and $\vec{E}_n \equiv \vec{E} + \frac{\vec{u}_n \times \vec{B}}{c}$ as the electric field in the frame of the neutrals. Then the above equation may be written as

$$0 = \Omega_s \tau_{sn}\left(\frac{c}{B}\vec{E}_n + \vec{w}_s \times \hat{b}\right) - \vec{w}_s,$$

where $\hat{b} \equiv \vec{B}/B$ and $\Omega_s \equiv q_s B/m_s c$. Take the cross product of this with $\hat{b}$: $\vec{w}_s \times \hat{b} = \Omega_s \tau_{sn}\left(\frac{c}{B}\vec{E}_n \times \hat{b} - \vec{w}_{s\perp}\right)$

and put back in to find

$$(\Omega_s \tau_{sn})^2 \frac{c}{B}\vec{E}_n \times \hat{b} + (\Omega_s \tau_{sn})\frac{c}{B}\vec{E}_n = \vec{w}_s + (\Omega_s \tau_{sn})^2 \vec{w}_{s\perp}$$

Parallel component is: $\Omega_s \tau_{sn} \frac{c}{B}\vec{E}_{n\parallel} = \vec{w}_{s\parallel}$

$$\Rightarrow \vec{j}_\parallel = \Sigma_s n_s q_s \vec{u}_{s\parallel} = \Sigma_s n_s q_s \vec{w}_{s\parallel}$$

$$= \left(\Sigma_s \frac{q_s^2 n_s \tau_{sn}}{m_s}\right)\vec{E}_{n\parallel} \equiv \Sigma_s \sigma_s \vec{E}_{n\parallel} \equiv \sigma_\parallel \vec{E}_{n\parallel}$$

which is a simple Ohm's law!

Perpendicular components: $\left[\frac{(\Omega_s\tau_{sn})^2}{1+(\Omega_s\tau_{sn})^2}\right]\frac{c}{B}\vec{E}_n\times\hat{b} + \left[\frac{(\Omega_s\tau_{sn})}{1+(\Omega_s\tau_{sn})^2}\right]\frac{c}{B}\vec{E}_{n\perp}$

$$= \vec{w}_{s\perp}$$

$$\Rightarrow \vec{j}_\perp = \Sigma_s q_s n_s \vec{w}_{s\perp}$$

$$= \left[\Sigma_s \frac{\sigma_s}{1+(\Omega_s\tau_{sn})^2}\right]\vec{E}_{n\perp} + \left[\Sigma_s \frac{\sigma_s\Omega_s\tau_{sn}}{1+\Omega_s^2\tau_{sn}^2}\right]\vec{E}_n\times\hat{b}$$

$$\equiv \sigma_\perp \vec{E}_{n\perp} - \sigma_H \vec{E}_n\times\hat{b}$$

$$\Rightarrow \vec{j} = \sigma_\parallel \vec{E}_{n\parallel} + \sigma_\perp \vec{E}_{n\perp} - \sigma_H \vec{E}_n\times\hat{b}$$

This can be inverted to give

$$\vec{E} + \frac{\vec{u}_n\times\vec{B}}{c} = \overleftrightarrow{\eta}\,\vec{j}, \quad \text{where} \quad \eta_\parallel \equiv \frac{1}{\sigma_\parallel}$$

$$\eta_\perp \equiv \frac{\sigma_\perp}{\sigma_\perp^2+\sigma_H^2}$$

$$\eta_H \equiv \frac{\sigma_H}{\sigma_\perp^2+\sigma_H^2}$$

$$\to \vec{E} = -\frac{\vec{u}_n\times\vec{B}}{c} + \eta_\parallel\vec{j}_\parallel + \eta_\perp\vec{j}_\perp + \eta_H\vec{j}\times\hat{b}$$

$$= -\frac{\vec{u}_n\times\vec{B}}{c} + \underbrace{\eta_\parallel}_{\equiv\eta_{OD}}\vec{j} + \underbrace{(\eta_\perp-\eta_\parallel)}_{\equiv\eta_{AD}}\vec{j}_\perp + \eta_H\vec{j}\times\hat{b} \checkmark$$

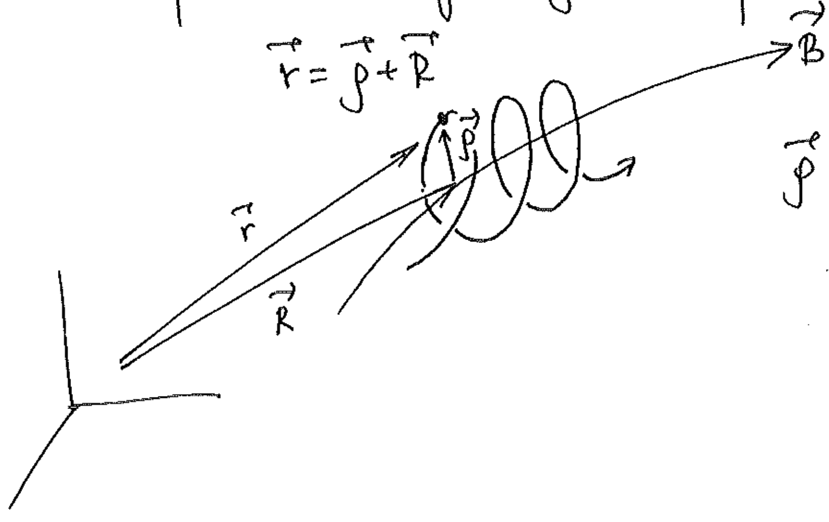
Kinetics

Up to this point, we've concerned ourselves with the evolution of infinitesimal <u>fluid elements</u>. In particular, we've assumed that particle-particle collisions occur so often that the particle distribution function in each of these fluid elements is Maxwellian. In this lecture, we relax these two approaches — we focus on particle dynamics and then on statistical treatments of these dynamics, and then we construct two fluid-like sets of equations that allow for departures from Maxwellian equilibria and evolution. Finally, we preview gyrokinetics — yet another reduction of the Vlasov-Landau-Maxwell equations — which is particularly useful in describing magnetized, low-frequency dynamics in weakly collisional plasmas. Derivations are provided, and you should try to reproduce them.

Particle motion

Any discussion of plasma kinetics begins with an investigation of particle motion. We'll skip a few steps and get right to the heart of all that follows, by decomposing the particle position into Larmor position and guiding-center position:

$$\vec{r} = \vec{\rho} + \vec{R}$$



$$\vec{\rho} = -\frac{\vec{v}\times\hat{b}}{\Omega}$$

let's begin with the simplest case: uniform $\vec{E}$ and $\vec{B}$. What's the trajectory of the particle?

$$\dot{\vec{R}} = \dot{\vec{r}} - \dot{\vec{\rho}} = \vec{v} + \frac{d\vec{v}}{dt} \times \frac{\hat{b}}{\Omega}$$

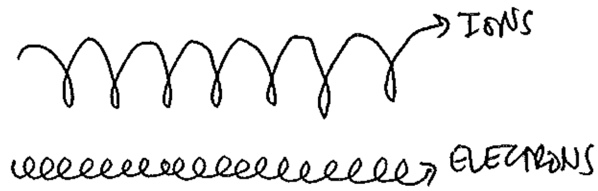
$$= \vec{v} + \frac{q}{m}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) \times \frac{\hat{b}}{\Omega}$$

$$= \vec{v} + \frac{q\vec{E}\times\hat{b}}{m\Omega} - \frac{q}{m}\frac{\vec{v}_\perp B}{c\Omega}$$

$$= \underbrace{v_\parallel \hat{b}}_{\text{parallel streaming of guiding center}} + \underbrace{\frac{c\vec{E}\times\vec{B}}{B^2}}_{\text{perpendicular drift}}$$

"E×B drift"

NB: charge independent!



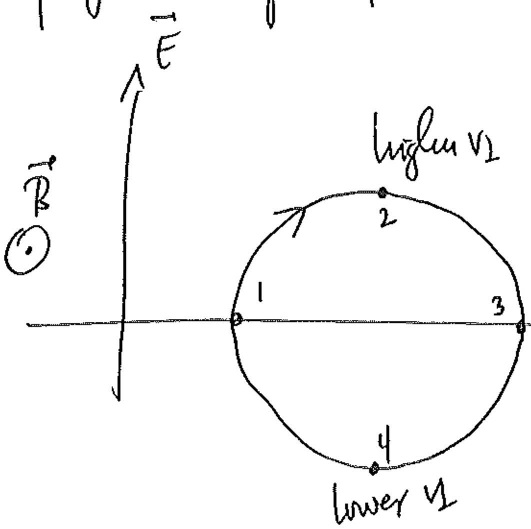
(no current generated by these drifts)

in general, for some force F, $\dot{\vec{R}} = v_\parallel \hat{b} + \frac{\vec{F}\times\vec{B}c}{qB^2}$

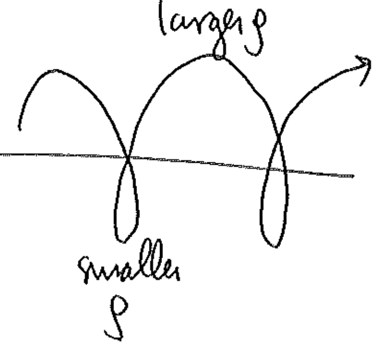
charge dependent!

physical origin of E×B drift:

recall $\rho = \frac{v_\perp}{\Omega} \propto \frac{v_\perp}{B}$



① → ② accel.
① → ③ decel.
② → ④ decel.
④ → ① accel.



Now let the magnetic field be non-uniform:

$$\dot{\vec{R}} = \dot{\vec{r}} - \dot{\vec{\rho}} = \vec{v} + \frac{d}{dt}\left(\frac{\vec{v}\times\hat{b}}{\Omega}\right)$$

$$= \vec{v} + \frac{d\vec{v}}{dt}\times\frac{\hat{b}}{\Omega} + \vec{v}\times\frac{d}{dt}\frac{\hat{b}}{\Omega}$$

$$= v_\parallel\hat{b} + \frac{c\vec{E}\times\vec{B}}{B^2} + \vec{v}\times\frac{d}{dt}\left(\frac{\hat{b}}{\Omega}\right)$$

parallel drift can result in ⊥ drift if $\hat{b}$ changes "curvature drift"

$\vec{E}\times\vec{B}$ drift from changing Larmor radius from E accel.

gradients in B field along particle orbit cause force to change around gyro-orbit.

$$\left\langle \vec{v}\times\vec{v}\cdot\vec{\nabla}\left(\frac{\hat{b}}{\Omega}\right)\right\rangle_{\underbrace{\vartheta}_{\text{gyrophase}}} = \varepsilon_{ijk}\left\langle v_j v_m \partial_m \frac{b_k}{\Omega}\right\rangle_\vartheta = \varepsilon_{ijk}\langle v_j v_m\rangle_\vartheta \partial_m \frac{b_k}{\Omega}$$

$$= \varepsilon_{ijk}\left[v_\parallel^2 b_j b_m + \frac{v_\perp^2}{2}(\delta_{jm} - b_j b_m)\right]\partial_m \frac{b_k}{\Omega}$$

$$= v_\parallel^2\, \hat{b}\times\left(\hat{b}\cdot\vec{\nabla}\frac{\hat{b}}{\Omega}\right)$$

$$+ \frac{v_\perp^2}{2}\left(\vec{\nabla}\times\frac{\hat{b}}{\Omega} - \hat{b}\times\hat{b}\cdot\vec{\nabla}\frac{\hat{b}}{\Omega}\right)$$

requires some algebra ... we can compute this easier. Picture a rapidly gyrating charge as a dipole with <u>magnetic moment</u>

$$\vec{\mu} = \frac{1}{2}q\frac{\vec{r}\times\vec{v}}{c} = -\frac{1}{2}q\frac{\rho v_\perp}{c}\hat{b} = -\frac{1}{2}m\frac{v_\perp^2}{B}\hat{b} \equiv -\mu\hat{b}$$

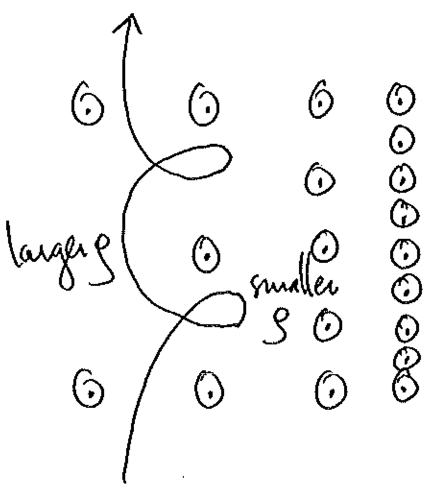
Force on a dipole $= \vec{\nabla}(\vec{\mu}\cdot\vec{B}) = -\mu\vec{\nabla}B$

Then $\vec{v}_{drift} = \frac{\vec{F}\times\vec{B}c}{qB^2} = -\frac{\mu\vec{\nabla}B\times\vec{B}c}{qB^2} = \frac{v_\perp^2}{2\Omega}\hat{b}\times\vec{\nabla}\ln B$ "grad-B drift"

The correction on $v_\parallel\hat{b}$ can be thought of as due to a centrifugal force as the field line bends $\rightarrow$ $F_c = \frac{mv_\parallel^2}{r_c}\hat{r}_c$, where r_c is the radius of curvature of the field line. With $\hat{r}_c = -r_c\,\hat{b}\cdot\vec{\nabla}\hat{b}$; then

$$\vec{F}_c = -mv_\parallel^2\,\hat{b}\cdot\vec{\nabla}\hat{b} \quad \text{and so}$$

$$\dot{\vec{R}} = \underbrace{v_\parallel\hat{b}}_{\text{(streaming along } \hat{b})} + \underbrace{\frac{c\vec{E}\times\vec{B}}{B^2}}_{(E\times B \text{ drift})} + \underbrace{\underbrace{\frac{v_\perp^2}{2\Omega}\hat{b}\times\vec{\nabla}\ln B}_{\text{(grad-B drift)}} + \underbrace{\frac{v_\parallel^2}{\Omega}\hat{b}\times(\hat{b}\cdot\vec{\nabla}\hat{b})}_{\text{(curvature drift)}}}_{\text{charge dependent!}}$$



$\rightarrow \vec{\nabla}B$

$\hat{b}\times\vec{\nabla}B$ is upwards

There are other drifts, such as the polarization drift due to the time rate-of-change of $\vec{E}\times\vec{B}$ drift: $\frac{\hat{b}}{\Omega}\times\frac{d\vec{u}_{E\times B}}{dt}$, which can be derived in a similar way. See §4 of Goldston & Rutherford's textbook, for example.

Adiabatic Invariance

Adiabatic invariants are related to exactly conserved Poincaré invariants. They are one of the most important concepts in the plasma physics of weakly collisional plasmas. These quantities emerge from the periodic motion induced by the magnetic fields and come from the action in Hamiltonian classical mechanics, $\simeq \oint p\,dq$ around a loop representing nearly periodic motion. The 1st adiabatic invariant of charged-particle motion in a magnetic field is μ, the magnetic moment — the periodic motion here is obviously the gyromotion of a particle about a magnetic field. The appropriate momentum "p" in the case is the particle's ang. momentum, $m\rho v_\perp$; the angular variable ϑ is our "q". If the orbit changes slowly, either because $\frac{d\ln B}{dt} \ll \Omega$, or because the particle is drifting slowly into a region of different field geometry, then the action changes very little. In the case of μ conservation, which we prove below, the small change in μ due to changes in B at some frequency ω is $\propto \exp(-\Omega/\omega)$. As $\frac{\Omega}{\omega}$ becomes ~~[illegible]~~ large, changes in μ are exponentially small. Since $\exp(-\Omega/\omega)$ cannot be expressed as a Taylor series, we say that μ is conserved to all orders. (Such a quantity is not precisely the μ that we've written above, but one can find such a μ order-by-order in ρ/l_B.) So... what is the change in $\frac{1}{2}mv_\perp^2/B \equiv \mu$ over one orbit?

$$\Delta\mu = \frac{\Delta\left(\frac{1}{2}mv_\perp^2\right)}{B} - \mu\frac{\Delta B}{B}$$

$$= \frac{\int_0^{2\pi/\Omega} \frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) dt}{B} - \frac{\mu\Delta B}{B}$$

$$= \underbrace{\frac{\int_0^{2\pi/\Omega} q\vec{v}_\perp\cdot\vec{E}_\perp\, dt}{B}}_{\frac{q}{B}\oint \vec{E}_\perp\cdot d\vec{l}_\perp = -\frac{q}{cB}\int\frac{\partial\vec{B}}{\partial t}\cdot d\vec{a} = \frac{q}{c}\frac{\Delta B}{B}\frac{\Omega}{2\pi}(\pi\rho^2) = \mu\frac{\Delta B}{B}} - \frac{\mu\Delta B}{B}$$

$$= \mu\frac{\Delta B}{B} - \mu\frac{\Delta B}{B} = 0. \checkmark$$

Application: magnetic mirroring

$$\varepsilon = \frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2 = \text{constant in a static field}$$

$$\text{and } \frac{1}{2}\frac{mv_\perp^2}{B} = \mu = \text{constant in a slowly varying field}$$

Then $\frac{1}{2}mv_{\parallel 0}^2 + \mu B_0 = \frac{1}{2}mv_\parallel^2 + \mu B = \varepsilon$

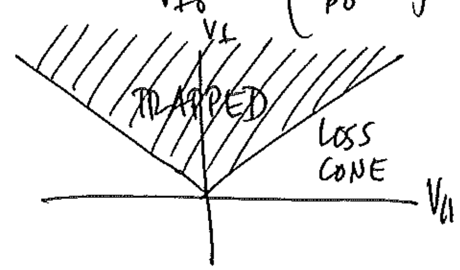
$$\rightarrow v_\parallel = \pm\left(\frac{2}{m}\right)^{1/2}\left[\frac{1}{2}mv_{\parallel 0}^2 + \mu(B_0 - B)\right]^{1/2}$$

$$= \pm\sqrt{\frac{2}{m}(\varepsilon - \mu B)}$$

$\Rightarrow B\uparrow \quad v_\perp \uparrow \quad v_\parallel \downarrow$ until $v_\parallel = 0$, then the particle "reflects" off strong-field region

criterion for reflection is $v_\parallel = 0 \rightarrow \frac{1}{2}mv_{\parallel 0}^2 + \mu(B_0 - B) \leq 0$

$$\rightarrow \frac{v_{\parallel 0}}{v_{\perp 0}} \leq \left(\frac{B}{B_0} - 1\right)^{1/2}$$ for containment, otherwise there is leakage



Collisions that break μ by pitch-angle scattering would of course promote leakage of particles.

Now, what if the ends of the mirror moved slowly? This leads to the 2nd adiabatic invariant, $J = \oint mv_\parallel \, dl$, which is due to the periodic motion of the guiding center as it bounces back and forth in a magnetic mirror. The integral is taken over the "bounce orbit", with the limits of the integral being the two turning points in the orbit (and back again). If the mirror shrinks, then $v_\parallel \uparrow$ if

$\tau_{bounce} \ll \tau_{mirror\ \dot{B}}$.

Note that both μ and J are of the form $\frac{W \sim \text{energy}}{\Omega \sim \text{frequency}}$

$\left(\frac{\frac{1}{2}mv_\perp^2}{\Omega}, \frac{\frac{1}{2}mv_\parallel^2}{v_\parallel / l}, \text{etc.}\right)$ general form of adiabatic invariants!

(think of $\frac{E}{\omega} = \hbar$ — Einstein — or Sommerfeld $\oint p \cdot dq = nh$)
Einstein @ Solvay conference in 1911 said that this was the general form of adiabatic invariant, and that this is what should thus be quantized.

Pressure Anisotropy and Double-Adiabatic Laws (sometimes referred to as CGL equations — although CGL is more general really)

So, we have a collection of charges, all of them approx. conserving μ and J. What does this mean for the gross ("fluid") properties of the plasma?

Compute expectation value of μ in a plasma comprised of particles satisfying some phase-space distribution function $f = f(x, v, t)$:

$$\langle \mu \rangle = \frac{\int \mu f \, d^3v}{\int f d^3v} = \frac{\frac{1}{B}\int \frac{1}{2} m v_\perp^2 f \, d^3v}{n} = \frac{p_\perp}{nB} = \frac{T_\perp}{B}.$$

Since μ's are individually conserved, we have $\boxed{\frac{T_\perp}{B} \simeq \text{constant}}$

$\boxed{\text{Double } B, \text{ double } T_\perp}$ What about J?

$$\langle J^2 \rangle = \frac{\int J^2 f d^3v}{\int f d^3v} = \frac{\int m^2 v_\parallel^2 l^2 f \, d^3v}{\int f d^3v} = \ldots \text{ ah... what's } l?$$

with mass and flux conserved in a changing magnetic mirror, we have $\ell \sim \frac{B}{n}$. Then

$$\langle J^2 \rangle = \frac{1}{n}\int m^2 v_\parallel^2 \frac{B^2}{n^2} f\, d^3v = m\frac{p_\parallel B^2}{n^3} \simeq \text{constant}$$

$$\Rightarrow \boxed{\frac{T_\parallel B^2}{n^2} \simeq \text{constant}}$$

These are the double-adiabatic equations. They can be written as

$$\boxed{\frac{d}{dt}\left(\frac{p_\perp}{nB}\right) = 0 \qquad \frac{d}{dt}\left(\frac{p_\parallel B^2}{n^3}\right) = 0}$$

SEE CHEW, GOLDBERGER, & LOW 1956

Note that, with $p = \frac{2}{3}p_\perp + \frac{1}{3}p_\parallel$ (more on this later), we have

$$\frac{d\ln p_\perp}{dt} = \frac{d\ln n}{dt} + \frac{d\ln B}{dt}, \qquad \frac{d\ln p_\parallel}{dt} = 3\frac{d\ln n}{dt} - 2\frac{d\ln B}{dt}$$

$$\Rightarrow \frac{2}{3}\frac{d\ln p_\perp}{dt} + \frac{1}{3}\frac{d\ln p_\parallel}{dt} = \frac{5}{3}\frac{d\ln n}{dt}$$

$$\Rightarrow \frac{d}{dt}\left[\ln p n^{-5/3}\right] = \frac{2}{3}\left(\frac{p_\perp - p_\parallel}{p}\right)\frac{d}{dt}\left[\ln B n^{-2/3}\right]$$

when $p_\perp - p_\parallel \simeq 0$ (collisional), then we recover the usual adiabatic eqn. of state $\frac{d}{dt}\ln p n^{-5/3} = 0$.

Also, using $p_\perp = p + \frac{1}{3}(p_\perp - p_\parallel)$

$p_\parallel = p - \frac{2}{3}(p_\perp - p_\parallel)$, we find

$$\frac{d(p_\perp - p_\parallel)}{dt} = p_\perp\left(\frac{d\ln n}{dt} + \frac{d\ln B}{dt}\right) - p_\parallel\left(3\frac{d\ln n}{dt} - 2\frac{d\ln B}{dt}\right)$$

$$= p\left[\frac{d\ln n}{dt} + \frac{d\ln B}{dt} - 3\frac{d\ln n}{dt} + 2\frac{d\ln B}{dt}\right]$$

$$+ \frac{(p_\perp - p_\parallel)}{3}\left[\frac{d\ln n}{dt} + \frac{d\ln B}{dt} + 6\frac{d\ln n}{dt} - 4\frac{d\ln B}{dt}\right]$$

$$= 3p\,\frac{d\ln B n^{-2/3}}{dt} + \frac{p_\perp - p_\parallel}{3}(-3)\,\frac{d\ln B n^{-7/3}}{dt}$$

$$\Rightarrow \left[\frac{d}{dt} + \frac{d\ln B n^{-7/3}}{dt}\right]\frac{(p_\perp - p_\parallel)}{3} = p\,\frac{d\ln B n^{-2/3}}{dt}$$

$\Rightarrow$ change B and/or n, produce pressure anisotropy.

Braginskii-MHD

Of course, collisions will always push the plasma back towards an isotropic Maxwellian with $p_\perp = p_\parallel = p$. Let us take this into consideration by letting $\Omega \gg \nu_{coll} \gg \omega$. Then adiabatic invariance still works because Ω is fast, but collisions will break it since they are faster than the rate at which B and n are changing. The result is that

$$p_\perp - p_\parallel \simeq \frac{3p}{\nu_{coll}} \frac{d\ln B n^{-2/3}}{dt},$$ which is a balance between adiabatic production of anisotropy and collisional relaxation.

Now, let's go all the way back to the ideal MHD induction equation: $\frac{\partial \vec{B}}{\partial t} = \vec{B}\cdot\vec{\nabla}\vec{u} - \vec{u}\cdot\vec{\nabla}\vec{B} - \vec{B}\vec{\nabla}\cdot\vec{u}$

Dot w/ $\frac{\vec{B}}{2B^2}$: $\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\ln B = \frac{d\ln B}{dt} = \hat{b}\hat{b}:\vec{\nabla}\vec{u} - \vec{\nabla}\cdot\vec{u}$.

Also, continuity equation gives $\frac{d\ln n}{dt} = -\vec{\nabla}\cdot\vec{u}$

So, $$p_\perp - p_\parallel \simeq \frac{3p}{\nu_{coll}}\left[\hat{b}\hat{b}:\vec{\nabla}\vec{u} - \vec{\nabla}\cdot\vec{u} - \frac{2}{3}\left(-\vec{\nabla}\cdot\vec{u}\right)\right]$$
$$= \frac{3p}{\nu_{coll}}\left[\hat{b}\hat{b} - \frac{1}{3}\overleftrightarrow{I}\right]:\vec{\nabla}\vec{u}$$

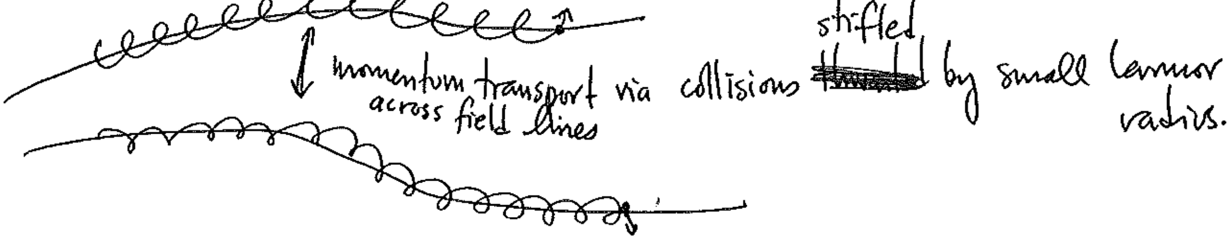
This pressure anisotropy enters the momentum equation as follows:
$$\rho\frac{d\vec{u}}{dt} = -\vec{\nabla}\left(p_\perp + \frac{B^2}{8\pi}\right) + \vec{\nabla}\cdot\left[\hat{b}\hat{b}\left(\frac{B^2}{4\pi} + p_\perp - p_\parallel\right)\right] + \dots$$
$$= \underbrace{-\vec{\nabla}\left(p + \frac{B^2}{8\pi}\right) + \frac{\vec{B}\cdot\vec{\nabla}\vec{B}}{4\pi}}_{\text{ideal MHD}} + \underbrace{\vec{\nabla}\cdot\left[\left(\hat{b}\hat{b} - \frac{1}{3}\overleftrightarrow{I}\right)\left(\hat{b}\hat{b} - \frac{1}{3}\overleftrightarrow{I}\right):\vec{\nabla}\vec{u}\,\frac{3p}{\nu_{coll}}\right]}_{\text{anisotropy viscosity}}$$

two derivatives of $\vec{u}$, a viscosity $\frac{p}{\nu_{coll}}$, and a tensor set by field direction (which is anisotropic)

This is "Braginskii viscosity", the restriction of momentum transport to the direction along the local field lines.

If you go read Braginskii (1965), and you should, you'll see lots of other "viscous" terms ... cross-field transport, gyroviscosity, etc. At long wavelengths in a plasma with $\frac{\omega}{\Omega} \ll 1$, the above term $\propto \left(\hat{b}\hat{b}:\vec{\nabla}\vec{u} - \frac{1}{3}\vec{\nabla}\cdot\vec{u}\right)$ is all that matters.

The moral is <u>parallel gradients of parallel velocities are subject to viscous dissipation</u>. This is because communication between particles across field lines is blocked by (small) Larmor gyrations, whereas particles are free to move along field lines but for collisions. In some sense, ρ_i plays the role of the mean free path $\perp$ to $\vec{B}$.



It's not just momentum transport that behaves this way. It's also heat transport. Particles can exchange information about temperature readily along field lines, but not so across them, for the same reason: <u>magnetic fields serve at conduits for transport</u>. Small Larmor radii restricts conduction to be primarily along the field: <u>parallel gradients of temperature are subject to diffusion.</u> When conduction is fast, field lines tend towards isotherms: $\hat{b}\cdot\vec{\nabla}T = 0$.

These properties are derived rigorously later in these notes, but for now I summarize the Braginskii-MHD equ's, to leading order in $\frac{\omega}{\nu_{coll}} \ll 1$:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho \vec{u}) = 0$$

$$\rho\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\vec{u} = -\vec{\nabla}\left(p + \frac{B^2}{8\pi}\right) + \frac{\vec{B}\cdot\vec{\nabla}\vec{B}}{4\pi}$$
$$+\vec{\nabla}\cdot\left[\left(\hat{b}\hat{b} - \tfrac{1}{3}\overleftrightarrow{I}\right)\left(\hat{b}\hat{b} - \tfrac{1}{3}\overleftrightarrow{I}\right) : \vec{\nabla}\vec{u}\ \frac{3p_i}{\nu_{coll,i}}\right]$$

$$\frac{p}{\gamma-1}\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\ln p\rho^{-5/3} = -\vec{\nabla}\cdot\vec{Q} + (p_\perp - p_\parallel)\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\ln B\rho^{-2/3}$$

$$\text{with } \vec{Q} = -\Big(\hat{b}\,\underbrace{\hat{b}\cdot\vec{\nabla}T}_{\parallel \text{ gradient of } T}\Big)\chi_e$$

heat flows along $\hat{b}$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times(\vec{u}\times\vec{B})$$

The labelling of $\frac{p_i}{\nu_{coll,i}}$ and χ_e with i (ions) and e (electrons) is because ions dominate the collisional momentum transport (by virtue of their heavier mass) and electrons dominate the collisional heat transport (by virtue of their lighter mass).

What if collisions aren't strong enough?

Strong collisions w/ $\frac{\omega}{\nu} \ll 1$ allowed us to write down an expression ~~s~~ for the heat flux and ~~viscosity~~ viscous flux in terms of field-line-oriented velocity gradients and temperature gradients. What if collisions aren't strong enough? Does anything else interfere w/ adiabatic invariance? If not, how far can the plasma depart from a Maxwellian? (In Braginskii, not far... deviations of just $O\left(\frac{\omega}{\nu}\right) \ll 1$ are allowed.) Can we construct a "fluid" model, even in the collisionless case? These questions will be addressed in what follows.

The first thing to realize is that a brute-force approach — ie., follow each particle as it evolves under a Hamiltonian — is not feasible. There is simply too much information. There are $\sim 10^{28}$ particles in this room alone. One data dump of $\vec{r}$ and $\vec{v}$ (velocity) for all these particles would be $\sim 5 \times 10^{17}$ TB (!!!) In any case, we're not really all that interested in every particle — we want bulk information — so what's the point? There's also a nasty sensitivity to initial conditions. Displace a single particle an infinitesimal amount, and you'll get a different answer for the system's evolution. We need a statistical approach. Now, there is an entire course at Princeton on deriving rigorously such a statistical treatment — AST554: Irreversible Processes in Plasma (taught over many years by Bernstein, Fisch, Hammett, Karney, Kaw, Kulsrud, Oberman, and a lot by John Krommes — I'm teaching it this spring). There is no time here, unfortunately, so I'm going to skip right to the answer:

Vlasov-Landau Equation: $$\frac{\partial f_s}{\partial t} + \vec{v}\cdot\vec{\nabla} f_s + \frac{q_s}{m_s}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \vec{v}} = \left(\frac{\partial f_s}{\partial t}\right)_c,$$

where $f_s = f_s(\vec{r},\vec{v},t)$ is the "one-particle" distribution function of species s. This equation is closed by the Maxwell equations:

$$\vec{\nabla}\cdot\vec{E} = 4\pi \sum_s q_s \int d^3v\, f_s(\vec{r},\vec{v},t)$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\sum_s q_s \int d^3v\, \vec{v}\, f_s(\vec{r},\vec{v},t)$$

The Landau collision operator has the form

$$\left(\frac{\partial f_s}{\partial t}\right)_{coll.} = \sum_{s'} \left(\frac{\partial f_s}{\partial t}\right)_{s',coll} ;$$

collisions with species s'

it depends on f_s as well as all other $f_{s'}$.

While I'll say nothing specifically of $(\partial f_s/\partial t)_{coll.}$, it is worth noting that it satisfies the following properties:

- $\int d^3v \left(\frac{\partial f_s}{\partial t}\right)_c = 0$ (conserves particle number each species individually)

- $\sum_s \int d^3v \; m_s \vec{v} \left(\frac{\partial f_s}{\partial t}\right)_c = 0$ (conserves total momentum)

  basically a statement of Newton's 3rd law; note the summation!

- $\sum_s \int d^3v \; \frac{m_s v^2}{2} \left(\frac{\partial f_s}{\partial t}\right)_c = 0$ (conserves total energy)

- $\frac{d}{dt}\left[-\Sigma_s \int d^3r \int d^3v \; f_s \ln f_s \right] = -\Sigma_s \int d^3r \int d^3v \left(\frac{\partial f_s}{\partial t}\right)_c \ln f_s \geq 0.$

  (Boltzmann's H theorem — entropy increases, with the above inequality becoming an exact equality for $f_s = f_{Ms} = \frac{n_s}{\pi^{3/2} v_{th_s}^3} \exp\left(\frac{-v^2}{v_{th_s}^2}\right)$)

- $\left(\frac{\partial f_s}{\partial t}\right)_c$ is a Fokker-Planck operator, and so it removes small-scale structure in velocity space through diffusion.

Moving on...

In these lecture notes, I will always be working under quasi-neutrality and in the non-relativistic regime. That is,

$$\sum_s q_s n_s = \sum_s q_s \int f_s \, d^3v = 0 \quad \text{and} \quad \left(\frac{v^2}{c^2}\right) \ll 1 \Rightarrow \frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla}\times\vec{E}$$

$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} \approx 0 = \vec{\nabla}\times\vec{B} - \vec{J}$$

are our "Maxwell" equations: Faraday & Ampère.

This is true at scales above the Debye scale: $k\lambda_D \lll 1$.

Note that we can obtain moment equations similar to our MHD equations by taking moments of the V-L equation:

- $\int d^3v \left[\frac{\partial f_s}{\partial t} + \vec{v}\cdot\vec{\nabla} f_s + \cancel{\vec{a}\cdot\vec{\nabla}_v f_s} = \cancel{C[f_s]} \right]$

0 int. by parts 0 cons. of #

$$\Rightarrow \boxed{\frac{\partial n_s}{\partial t} + \vec{\nabla}\cdot(n_s\vec{u}_s) = 0}$$

- $\int d^3v \, m_s\vec{v} \left[\frac{\partial f_s}{\partial t} + \vec{v}\cdot\vec{\nabla} f_s + \vec{a}\cdot\vec{\nabla}_v f_s = C[f_s] \right]$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}(m_s n_s\vec{u}_s) + \vec{\nabla}\cdot(m_s n_s\vec{u}_s\vec{u}_s) = -\vec{\nabla}\cdot\overleftrightarrow{P}_s + q_s n_s\left(\vec{E} + \frac{\vec{u}_s\times\vec{B}}{c}\right) + \vec{R}_s}$$

where $\overleftrightarrow{P}_s \equiv m_s \int (\vec{v}-\vec{u}_s)(\vec{v}-\vec{u}_s) f_s \, d^3v$ is the pressure tensor

and $\vec{R}_s \equiv \int m_s\vec{v}\, C[f_s]\, d^3v$ is the friction force

and so on. Note that the moment hierarchy never closes...

n_s depends on $\vec{u}_s$, $\vec{u}_s$ depends on $\overleftrightarrow{P}_s$, $\overleftrightarrow{P}_s$ will depend on

$\overleftrightarrow{Q}_s = m_s \int (\vec{v}-\vec{u}_s)(\vec{v}-\vec{u}_s)(\vec{v}-\vec{u}_s) f_s \, d^3v$, and so on... yuck. We'll return to this when we do KMHD.

A. Schekochihin has excellent lecture notes on Landau damping at www-thphys.physics.ox.ac.uk/people/AlexanderSchekochihin/KT/2014/sec3_Linear.pdf

Landau Damping

The cleanest example of how a kinetic system differs from a fluid is Landau damping — the collisionless damping of electrostatic fluctuations by means of wave-particle resonances. (There is an electromagnetic version of this — "Barnes" damping, or "transit-time" damping, which we'll come back to later.) I'm not going to go through this, because you can find it in just about any textbook (e.g., Hazeltine & Waelbroeck or Goldston & Rutherford). The essential physical feature is that, for a distribution function w/ $\partial f/\partial v < 0$ & in the presence of an electrostatic wave, (of wavenumber $k = \frac{2\pi}{\lambda}$) there are more particles w/ $v < \omega/k$ than with $v > \omega/k$, and so the slower particles (which comprise the majority of the plasma) get accelerated by the wave at the expense of the wave energy. This is a conservative (and reversible) transfer of free energy from the wave (the electrostatic fluctuation) to the particles. Equivalently, this is the process of <u>phase mixing</u>. During this transfer of free energy, the distribution function develops small scales in velocity space due to phase-space shear $\left(\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = 0\right)$. Ultimately, this structure triggers collisional relaxation, and entropy increases.

While I'm not doing the Landau calculation here, I will provide you with some handy formulae:

$$Z(\zeta) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}\,dt}{t-\zeta}$$

is the plasma dispersion function. For small argument, $Z(\zeta) \simeq i\sqrt{\pi}$. Also, $\frac{dZ}{d\zeta} = -2\left[1+\zeta Z\right] = \frac{-2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{t e^{-t^2}\,dt}{t-\zeta}$.

Likewise,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \frac{t^2 e^{-t^2}\,dt}{t-\zeta} = \zeta\left[1+\zeta Z\right]$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \frac{t^3 e^{-t^2}\,dt}{t-\zeta} = \frac{1}{2} + \zeta^2 (1+\zeta Z)$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \frac{t^4 e^{-t^2}\,dt}{t-\zeta} = \zeta\left[\frac{1}{2} + \zeta^2 (1+\zeta Z)\right]$$

and so on...

Ordering Parameters

Electrostatics is plenty rich, but electromagnetics is even richer. Kinetics gets very complicated, and it's best to specialize our equations to some situations of interest. To do so, we examine the following dimensionless parameters:

ratio of fluctuation frequency to Larmor frequency

$$(\Omega_s \tau)^{-1}, \quad \frac{\rho_s}{\ell}, \quad \frac{\omega}{\Omega_s}, \quad k_\parallel \rho_s, \quad k_\perp \rho_s, \quad k_\parallel \lambda_{mfp}, \quad k_\perp \lambda_{mfp}, \quad \frac{\nu_s}{\Omega_s}$$

ratio of coll. to Larmor freq.

ratio of Larmor radius to gradient lengthscale in plasma

ratio of macroscopic timescales to Larmor timescale

size of fluctuations along (∥) and across (⊥) the field relative to Larmor radius

size of fluctuations relative to collisional mean free path.

The Mach number $\frac{u_s}{v_{th_s}}$ and plasma beta parameter $\beta_s \equiv \frac{v_{th_s}^2}{v_A^2}$ are also of interest.

In these notes, we'll mostly be using the "high-flow" ordering $u_s \sim v_{th_s}$, as it's most useful to astrophysical plasmas. This is where Braginskii comes from (otherwise, you get a set of drift-kinetic equations describing drift waves — relevant to tokamaks and such — which takes $\frac{u_s}{v_{th_s}} \sim \frac{\rho_s}{\ell}$).

(MHD has $\frac{u}{v_{th}} \sim 1$ and $\frac{\nu}{\Omega}, \frac{\rho}{\ell}, \frac{\omega}{\Omega}, k\rho, k\lambda_{mfp} \to 0$.)

Various incarnations and reductions of kinetics results from having a small number $\epsilon \ll 1$ and ordering these quantities with respect to it. In these notes, we'll do two:

Kinetic MHD: $k_\parallel \rho_i, k_\perp \rho_i \ll 1$, $\frac{\omega}{\Omega_i} \ll 1$, $k \lambda_{mfp} \sim 1$

Gyrokinetics: $k_\parallel \rho_i \ll 1$, $k_\perp \rho_i \sim 1$, $\frac{\omega}{\Omega_i} \ll 1$, $k_\parallel \lambda_{mfp} \sim 1$

The ordering $k \lambda_{mfp} \sim 1$ just means that we're interested in both the collisional and collisionless regimes; subsidiary expansions in $k \lambda_{mfp}$ can be taken later (i.e. $\nu \ll \omega$, or $\nu \sim \omega$, can be taken later).

KMHD starts on the next page. You may want to take a look at Kulsrud 1983 ("MHD description of plasma"). The derivation isn't particularly pedagogical, but there is a really nice physical description, and it's always best to get an original paper directly from the source! ("KMHD" is often called "Kulsrud's KMHD".)

Derivation of Drift-Kinetic Eqn.

We start, of course, with the Vlasov-Landau equation:

$$(1) \qquad \frac{\partial f_s}{\partial t} + \vec{v}\cdot\vec{\nabla} f_s + \left[\frac{q_s}{m_s}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) + \vec{g}\right]\cdot\frac{\partial f_s}{\partial \vec{v}} = C[f_s]$$

The notation is standard: $f_s = f_s(t, \vec{r}, \vec{v})$ is the distribution function of species s $(=i, e, \ldots)$; q_s and m_s are the charge and mass of that species; $\vec{E}$ and $\vec{B}$ are the electric and magnetic fields; $\vec{g}$ is some externally imposed acceleration – e.g., gravity; and $C[f_s]$ is the collision operator, $\sim \nu_{coll} f_s$, where ν_{coll} is a collision frequency.

The idea here is to reduce this equation so that it describes only plasmas with fluctuations satisfying

$$\frac{\omega}{\Omega_s} \sim k\rho_s \equiv k\frac{v_{ths}}{\Omega_s} \sim \epsilon \ll 1;$$

i.e. frequencies ω that are small compared to the species' Larmor frequency $\Omega_s \equiv \frac{q_s B}{m_s c}$, and wavenumbers $(k \equiv 2\pi/\lambda)$ small compared to the inverse of the species' Larmor radius $\rho_s \equiv \frac{v_{ths}}{\Omega_s}$, where $v_{ths} \equiv \left(\frac{2T_s}{m_s}\right)^{1/2}$. We take

f_s to evolve on the fluctuation timescale, ω^{-1}, and have spatial structure on scales $k^{-1} \sim H \simeq \left(\frac{d\ln p}{dz}\right)^{-1}$ ← for example. Using this information — along with the assumption that $\nu_{coll} \sim \omega$ — we can order each of these terms in eqn. (1) and find out which are the dominant ones. This also involves expanding $f_s = f_{0s} + \epsilon f_{1s} + \ldots$

Before doing so, it helps to make the following change of variables:

$$\vec{w} = \vec{v} - \vec{u}_s(t,\vec{r}) \quad , \text{where } \vec{u}_s \equiv \frac{1}{n_s}\int f_s \vec{v}\, d^3\vec{v}$$

is the mean flow of species s. Using

$$\vec{\nabla}_r\Big|_{\vec{v}} = \vec{\nabla}_r\Big|_{\vec{w}} + (\vec{\nabla}\vec{w})_{\vec{v}} \cdot \frac{\partial}{\partial \vec{w}} = \vec{\nabla}_r\Big|_{\vec{w}} - (\vec{\nabla}\vec{u}_s)\cdot\frac{\partial}{\partial\vec{w}}$$

$$\frac{\partial}{\partial t}\Big|_{\vec{v}} = \frac{\partial}{\partial t}\Big|_{\vec{w}} + \frac{\partial \vec{w}}{\partial t}\Big|_{\vec{v}} \cdot \frac{\partial}{\partial \vec{w}} = \frac{\partial}{\partial t}\Big|_{\vec{w}} - \frac{\partial \vec{u}_s}{\partial t}\cdot\frac{\partial}{\partial\vec{w}} \quad ,$$

eqn. (1) becomes

$$(2) \quad \frac{\partial f_s}{\partial t} + \vec{u}_s\cdot\vec{\nabla} f_s + \vec{w}\cdot\vec{\nabla} f_s + \left[\frac{q_s}{m_s}\left(\vec{E} + \frac{\vec{u}_s\times\vec{B}}{c} + \frac{\vec{w}\times\vec{B}}{c}\right) + \vec{g} - \frac{\partial\vec{u}_s}{\partial t} - \vec{u}_s\cdot\vec{\nabla}\vec{u}_s - \vec{w}\cdot\vec{\nabla}\vec{u}_s\right]\cdot\frac{\partial f_s}{\partial\vec{w}} = C[f_s]$$

The notation is eased if we define the co-moving derivative,

$$\frac{D}{Dt_s} \equiv \frac{\partial}{\partial t} + \vec{u}_s \cdot \vec{\nabla}$$

for species s, and the electric field in that frame,

$$\vec{E}' \equiv \vec{E} + \frac{\vec{u}_s \times \vec{B}}{c}.$$

Then eqn. (2) is

$$(3) \qquad \frac{Df_s}{Dt_s} + \vec{w}\cdot\vec{\nabla} f_s + \left[\frac{q_s}{m_s}\left(\vec{E}' + \frac{\vec{w}\times\vec{B}}{c}\right) + \vec{g} - \frac{D\vec{u}_s}{Dt_s} - \vec{w}\cdot\vec{\nabla}\vec{u}_s\right]\cdot\frac{\partial f_s}{\partial \vec{w}} = C[f_s]$$

Let us proceed ~~#~~ with the ordering...

$$\frac{Df_s}{Dt_s} \sim \omega f_s \sim \epsilon \Omega_s f_{s0} + O(\epsilon^2)$$

$$\vec{w}\cdot\vec{\nabla} f_s \sim k v_{ths} f_s \sim k\rho_s \Omega_s f_s \sim \epsilon \Omega_s f_{s0} + O(\epsilon^2)$$

$$\frac{q_s \vec{E}'}{m_s}\cdot\frac{\partial f_s}{\partial \vec{w}} \sim \epsilon \frac{q_s}{m_s}\frac{v_{ths}B}{c}\frac{f_s}{v_{ths}} \sim \epsilon \Omega_s f_{s0} + O(\epsilon^2)$$

(NB: it will turn out that we must order $E' \sim \epsilon \frac{v_{ths} B}{c}$; I'll show why a posteriori)

$$\frac{q_s}{m_s}\frac{\vec{w}\times\vec{B}}{c}\cdot\frac{\partial f_s}{\partial \vec{w}} \sim \Omega_s f_{s0} + \Omega_s f_{1s}\epsilon$$

$$\vec{g}\cdot\frac{\partial f_s}{\partial \vec{w}} \sim \frac{v_{ths}^2}{H}\frac{f_s}{v_{ths}} \sim \omega f_s \sim \epsilon \Omega_s f_{s0} + O(\epsilon^2)$$

$$\frac{D\vec{u}_s}{Dt_s}\cdot\frac{\partial f_s}{\partial \vec{w}} \sim \omega f_s \sim \epsilon \Omega_s f_{s_0} + O(\epsilon^2)$$

$$\vec{w}\cdot\vec{\nabla}\vec{u}_s\cdot\frac{\partial f_s}{\partial \vec{w}} \sim k u_s f_s \sim \omega f_s \sim \epsilon \Omega_s f_{s_0} + O(\epsilon^2)$$

$$C[f_s] \sim \nu_{coll} f_s \sim \omega f_s \sim \epsilon \Omega_s f_{s_0} + O(\epsilon^2)$$

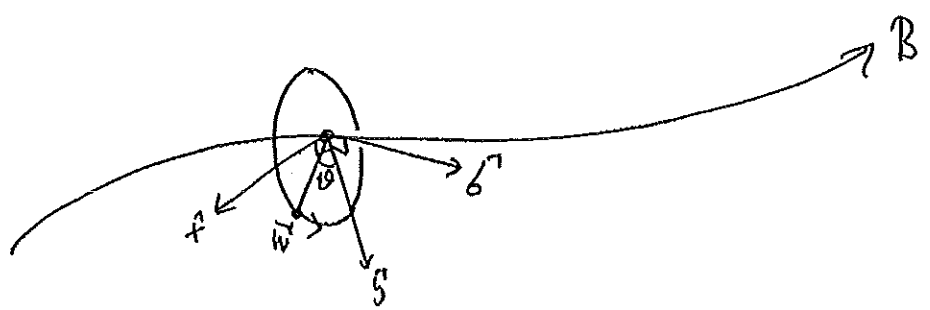
So, at zeroth order in ϵ, we have

$$\frac{q_s}{m_s}\frac{\vec{w}\times\vec{B}}{c}\cdot\frac{\partial f_{s_0}}{\partial \vec{w}} = 0.$$

The physical content of this may be extracted if we write

$$\vec{w} = w_\parallel \hat{b} + w_\perp\left(\sin\vartheta\,\hat{x} + \cos\vartheta\,\hat{y}\right)$$

where



defines a local coordinate system parallel and perpendicular to $\hat{b} \equiv \frac{\vec{B}}{B}$. Then, with $\frac{\partial}{\partial \vec{w}} = \hat{b}\frac{\partial}{\partial w_\parallel} + \frac{\vec{w}_\perp}{w_\perp}\frac{\partial}{\partial w_\perp} + \frac{(\vec{w}\times\hat{b})}{w_\perp^2}\cdot\frac{\partial}{\partial\vartheta}$, we have

$$(*)\quad \boxed{\frac{\partial f_{s_0}}{\partial \vartheta} = 0} \qquad \text{i.e. } f_{s_0} \text{ is "gyrotropic"}$$

Now, let's go to next (first) order:

$$(5)\quad \frac{D f_{0s}}{D t_s} + \vec{w}\cdot\vec{\nabla} f_{0s} + \left(\frac{q_s \vec{E}'}{m_s} + \vec{g} - \frac{D\vec{u}_s}{Dt_s} - \vec{w}\cdot\vec{\nabla}\vec{u}_s\right)\cdot\frac{\partial f_{0s}}{\partial \vec{w}}$$

$$+ \underbrace{\frac{q_s}{m_s}\frac{\vec{w}\times\vec{B}}{c}\cdot\frac{\partial f_{1s}}{\partial \vec{w}}}_{\Omega_s \frac{\partial f_{1s}}{\partial \vartheta}} = C[f_{0s}]$$

To lose the $\partial/\partial\vartheta$ term involving f_{1s}, which we don't know, we gyroaverage eqn. (5): $\frac{1}{2\pi}\oint d\vartheta\,(\dots)$. The last term on the LHS vanishes, by periodicity of Larmor orbits, and so we have

$$(6)\quad \frac{1}{2\pi}\oint d\vartheta\left\{\frac{D f_{0s}}{D t_s} + \vec{w}\cdot\vec{\nabla} f_{0s} + (\dots)\cdot\frac{\partial f_{0s}}{\partial \vec{w}} + \cancel{0} = C[f_{0s}]\right\}$$

Now, since we know $\frac{\partial f_{0s}}{\partial\vartheta} = 0$, we know that $\frac{\partial f_{0s}}{\partial \vec{w}} = \hat{b}\frac{\partial f_{0s}}{\partial w_\parallel} + \frac{\vec{w}_\perp}{w_\perp}\frac{\partial f_{0s}}{\partial w_\perp}$.

The next part is tricky. We now know that $(t, \vec{r}, \vec{w})$ is not the most useful coordinate system, since it contains unnecessary information — the gyrophase dependence of f_{0s}, which is actually independent of gyrophase! So, what we really seek is eqn. (6) in $(t, \vec{r}, w_\parallel, w_\perp)$ coordinates. That involves a change of variables, $(t, \vec{r}, \vec{w}) \to (t, \vec{r}, w_\parallel, w_\perp)$. The trick is that these coordinates change in time, as the magnetic-field direction changes. It's best if we didn't have <u>both</u> velocity variables changing, so it's actually easiest to work in $(w, w_\parallel)$ variables, where $w \equiv \sqrt{w_\parallel^2 + w_\perp^2}$. Then, only $w_\parallel = \vec{w}\cdot\hat{b}$ changes. To make progress, we must change our derivatives to the new coordinate system:

$$\frac{Df}{Dt_s}\bigg|_{\vec{w}} = \frac{Df}{Dt_s}\bigg|_{w, w_\parallel} + \underbrace{\frac{Dw_\parallel}{Dt_s}\bigg|_{\vec{w}}}_{\frac{D\hat{b}}{Dt_s}\cdot\vec{w}} \frac{\partial f}{\partial w_\parallel}\bigg|_w + \cancel{\frac{Dw}{Dt_s}\bigg|_{\vec{w}} \frac{\partial f}{\partial w}\bigg|_{w_\parallel}}^{\,0}$$

$$\vec{D}_r\big|_{\vec{w}} f = \vec{D}_r\big|_{w, w_\parallel} f + \underbrace{\vec{D}_r w_\parallel\big|_{\vec{w}}}_{(\vec{\nabla}\hat{b})\cdot\vec{w}} \frac{\partial f}{\partial w_\parallel}\bigg|_w + \cancel{\vec{D}_r w\big|_{\vec{w}} \frac{\partial f}{\partial w}\bigg|_{w_\parallel}}^{\,0}$$

and $\frac{\partial}{\partial \vec{w}} = \frac{\vec{w}}{w}\frac{\partial}{\partial w}\big|_{w_\parallel} + \hat{b}\frac{\partial}{\partial w_\parallel}\big|_w$. Also, we'll need to know the gyroaverages of $\vec{w}$ and $\vec{w}\vec{w}$:

$$\langle \vec{w} \rangle_\vartheta = \frac{1}{2\pi}\oint d\vartheta \left[w_\parallel \hat{b} + w_\perp(\sin\vartheta\,\hat{x} + \cos\vartheta\,\hat{y}) \right] = w_\parallel \hat{b}$$

$$\begin{aligned} \langle \vec{w}\vec{w} \rangle_\vartheta = \frac{1}{2\pi}\oint d\vartheta \Big[& w_\parallel^2 \hat{b}\hat{b} + w_\perp^2 \left(\sin^2\vartheta\,\hat{x}\hat{x} + \sin\vartheta\cos\vartheta(\hat{x}\hat{y} + \hat{y}\hat{x}) + \cos^2\vartheta\,\hat{y}\hat{y} \right) \\ & + w_\parallel w_\perp \left(\sin\vartheta\,\hat{b}\hat{x} + \sin\vartheta\,\hat{x}\hat{b} + \cos\vartheta\,\hat{b}\hat{y} + \cos\vartheta\,\hat{y}\hat{b} \right) \Big] \\ = {} & w_\parallel^2 \hat{b}\hat{b} + \frac{w_\perp^2}{2}\left(\overleftrightarrow{I} - \hat{b}\hat{b} \right) \end{aligned}$$

good. So, back to eqn (6), in $(w_\parallel, w)$ coordinates:

$$\Big\langle \frac{Df_{s0}}{Dt_3} + \frac{D\hat{b}}{Dt_3}\cdot\vec{w}\,\frac{\partial f_{s0}}{\partial w_\parallel} + \vec{w}\cdot\vec{\nabla} f_{s0} + \vec{w}\vec{w}:\vec{\nabla}\hat{b}\,\frac{\partial f_{s0}}{\partial w_\parallel}$$
$$+ \left(\frac{q_s\vec{E}'}{m_s} + \vec{g} - \frac{D\vec{u}_s}{Dt_3}\right)\cdot\frac{\vec{w}}{w}\frac{\partial f_{s0}}{\partial w} + \left(\frac{q_s\vec{E}'}{m_s} + \vec{g} - \frac{D\vec{u}_s}{Dt_3}\right)\cdot\hat{b}\,\frac{\partial f_{s0}}{\partial w_\parallel}$$
$$- \frac{\vec{w}\vec{w}:\vec{\nabla}\vec{u}_s}{w}\frac{\partial f_{s0}}{\partial w} - \vec{w}\hat{b}:\vec{\nabla}\vec{u}_s\,\frac{\partial f_{s0}}{\partial w_\parallel} + \Omega_s\frac{\partial f_{s0}}{\partial \vartheta} = C[f_{s0}]\Big\rangle_\vartheta$$

With $\langle f_{s0}\rangle_\vartheta = f_{s0}$,

$$\frac{Df_{s0}}{Dt_3} + w_\parallel\underbrace{\cancel{\hat{b}\cdot\frac{D\hat{b}}{Dt_3}}}_{0}\frac{\partial f_{s0}}{\partial w_\parallel} + w_\parallel\hat{b}\cdot\vec{\nabla} f_{s0} + w_\parallel^2\underbrace{\cancel{\hat{b}\hat{b}:\vec{\nabla}\hat{b}}}_{0}\frac{\partial f_{s0}}{\partial w_\parallel}$$
$$+ \frac{w_\perp^2}{2}(\vec{\nabla}\cdot\hat{b})\frac{\partial f_{s0}}{\partial w_\parallel} - \frac{w_\perp^2}{2}\underbrace{\cancel{\hat{b}\hat{b}:\vec{\nabla}\hat{b}}}_{0}\frac{\partial f_{s0}}{\partial w_\parallel} + \frac{w_\parallel}{w}\frac{\partial f_{s0}}{\partial w}\hat{b}\cdot\left(\frac{q_s\vec{E}'}{m_s} + \vec{g} - \frac{D\vec{u}_s}{Dt_3}\right)$$
$$+ \left(\frac{q_s\vec{E}'}{m_s} + \vec{g} - \frac{D\vec{u}_s}{Dt_3}\right)\cdot\hat{b}\,\frac{\partial f_{s0}}{\partial w_\parallel} - \frac{w_\parallel^2}{w}\frac{\partial f_{s0}}{\partial w}\hat{b}\hat{b}:\vec{\nabla}\vec{u}_s - \frac{w_\perp^2}{2w}\vec{\nabla}\cdot\vec{u}_s\frac{\partial f_{s0}}{\partial w}$$
$$+ \frac{w_\perp^2}{2}\frac{\hat{b}\hat{b}:\vec{\nabla}\vec{u}_s}{w}\frac{\partial f_{s0}}{\partial w} - w_\parallel\hat{b}\hat{b}:\vec{\nabla}\vec{u}_s\frac{\partial f_{s0}}{\partial w_\parallel} + \phi = C[f_{s0}]$$

$$\boxed{\begin{aligned}&\left(\frac{D}{Dt_3} + w_\parallel\hat{b}\cdot\vec{\nabla}\right)f_{s0} + \frac{w_\perp^2}{2}(\vec{\nabla}\cdot\hat{b})\frac{\partial f_{s0}}{\partial w_\parallel} + \left(\frac{q_sE_\parallel}{m_s} + g_\parallel - \hat{b}\cdot\frac{D\vec{u}_s}{Dt_3}\right)\left(\frac{w_\parallel}{w}\frac{\partial f_{s0}}{\partial w} + \frac{\partial f_{s0}}{\partial w_\parallel}\right)\\ &- \frac{w_\perp^2}{2w}\frac{\partial f_{s0}}{\partial w}(\vec{\nabla}\cdot\vec{u}_s) + \hat{b}\hat{b}:\vec{\nabla}\vec{u}_s\left(\frac{w_\perp^2}{2w}\frac{\partial f_{s0}}{\partial w} - \frac{w_\parallel^2}{w}\frac{\partial f_{s0}}{\partial w} - w_\parallel\frac{\partial f_{s0}}{\partial w_\parallel}\right) = C[f_{s0}]\end{aligned}} \tag{7}$$

This is it! Our drift-kinetic equation.

If you look this up in Kulsrud's notes, you won't find it in this form. He works instead in $(v_\parallel, w_\perp)$ variables, a mix of full and peculiar velocity coordinates. To understand why, let's go back to that thing about $\vec{E}'$ that I post-poned...

I said that $|\vec{E}'| \sim \epsilon \frac{v_{ths}|\vec{B}|}{c}$, which means that $\vec{E}_\perp$ and $-\frac{\vec{u}_s \times \vec{B}}{c}$ differ by an asymptotically small value, i.e. to leading order, $\boxed{\vec{E}_\perp = -\frac{\vec{u}_s \times \vec{B}}{c}}$. This implies that

$$\boxed{\vec{u}_{s\perp} = c\frac{\vec{E}\times\vec{B}}{B^2}},$$ which is <u>species-independent</u>!

Thus, by working in $w_\perp$ variables, we've separated out the species-independent $E\times B$ drift from the particle motion ... thus, "<u>Drift</u>" <u>Kinetics</u>. It's up to you whether to use $v_\parallel$ or $w_\parallel$ ... each has different advantages. For the record, here is our DKEqn. in $(v_\parallel, w_\perp)$ variables; with $\frac{D}{Dt_s} \equiv \frac{\partial}{\partial t} + \vec{u}_s \cdot \vec{\nabla} + v_\parallel \hat{b}\cdot\vec{\nabla}$,

$$(8) \quad \frac{Df_s}{Dt_s} + \frac{D\ln B}{Dt_s}\frac{w_\perp}{2}\frac{\partial f_s}{\partial w_\perp} + \left(\frac{q_s}{m_s}E_\parallel - \hat{b}\cdot\frac{D\vec{u}_{s\perp}}{Dt} - \frac{w_\perp^2}{2}\hat{b}\cdot\nabla\ln B\right)\frac{\partial f_s}{\partial v_\parallel} = C[f_s]$$

~~where~~ where $\frac{D\ln B}{Dt_s} = \hat{b}\hat{b} : \vec{\nabla}\vec{u}_s - \vec{\nabla}\cdot\vec{u}_s + v_\parallel \hat{b}\cdot\nabla\ln B$

and sometimes, the velocity-space variable $\mu_s \equiv \frac{1}{2} m_s \frac{w_\perp^2}{B}$ is used, in place of $w_\perp$. Then the DKEqn. becomes

$$(9) \qquad \frac{Df_s}{Dt_s} + \left(\frac{q_s}{m_s} E_\parallel - \frac{D\vec{u}_{s\perp}}{Dt_s} \cdot \hat{b} - \mu_s \nabla_\parallel B \right) \frac{\partial f_s}{\partial v_\parallel} = C[f_s]$$

Note that there are no μ_s-derivatives! This means that the DKEqn. conserves μ_s. In other words, if $f_s = f_s(t, \vec{r}, \mu_s, v_\parallel)$, then

$$\frac{df_s}{dt} = \frac{\partial f_s}{\partial t} + \frac{d\vec{r}}{dt} \cdot \frac{\partial f_s}{\partial \vec{r}} + \underbrace{\frac{d\mu_s}{dt} \cdot \frac{\partial f_s}{\partial \mu_s}}_{=0 \text{ since } \frac{d\mu_s}{dt} = 0.} + \frac{dv_\parallel}{dt} \cdot \frac{\partial f_s}{\partial v_\parallel} = C[f_s]$$

(adiabatic invariant is exactly constant in our ordering)

Now, one last thing. Why do $\vec{E}$ and $-\frac{\vec{u}_s \times \vec{B}}{c}$ differ by a small amount? If you take the first moment of our DKEqn. (7), you can show that

$$\vec{E} = -\frac{\vec{u}_s \times \vec{B}}{c} + \frac{\vec{\nabla} \cdot \overleftrightarrow{P}_s}{q_s n_s} - \frac{\vec{R}_s}{q_s n_s} + \frac{m_s}{q_s} \frac{D\vec{u}_s}{Dt_s}$$

where $\overleftrightarrow{P}_s \equiv \int m_s \vec{w}\vec{w}\, d^3\vec{w}\, f_s$ is the pressure tensor, $\vec{R}_s \equiv \int C[f_s]\, \vec{w}\, d^3\vec{w}$ is the frictional force, and

$n_s \equiv \int f_s\, d^3\vec{w}$. Let's estimate the sizes of each term...

$$\frac{|\vec{\nabla}\cdot\overleftrightarrow{P}_s|}{q_s n_s}\frac{1}{|\vec{u}_s\times\vec{B}/c|} \sim k\rho_s \frac{v_{ths}}{u_s} \sim \frac{k\rho_s}{M_{as}} \ll 1$$

$$\frac{|\vec{R}_s/q_s n_s|}{|\vec{u}_s\times\vec{B}/c|} \sim \frac{\nu_{coll,s}}{\Omega_s} \ll 1$$

$$\frac{|(m_s/q_s)D\vec{u}_s/Dt_s|}{|\vec{u}_s\times\vec{B}/c|} \sim \frac{\omega}{\Omega_s} \ll 1$$

So, everything is small compared to $\vec{u}_s\times\vec{B}/c$! Then

$$\vec{E} + \frac{\vec{u}_s\times\vec{B}}{c} = O(\epsilon)\frac{v_{ths}B}{c},$$

which means that $\vec{u}_{\perp s} \equiv \vec{u}_\perp$ is species-independent.

Let us now return to this form of the DKE:

$$\left(\frac{D}{Dt_s} + w_\parallel\hat{b}\cdot\nabla\right)f_s + \frac{w_\perp^2}{2}(\nabla\cdot\hat{b})\frac{\partial f_s}{\partial w_\parallel} + \left(\frac{q_s E_\parallel}{m_s} + g_\parallel - \hat{b}\cdot\frac{D\vec{u}_s}{Dt_s}\right)\left(\frac{w_\parallel}{w}\frac{\partial f_s}{\partial w} + \frac{\partial f_s}{\partial w_\parallel}\right)$$
$$-\frac{w_\perp^2}{2w}\frac{\partial f_s}{\partial w}(\vec{\nabla}\cdot\vec{u}_s) + \hat{b}\hat{b}:\nabla\vec{u}_s\left(\frac{w_\perp^2}{2w}\frac{\partial f_s}{\partial w} - \frac{w_\parallel^2}{w}\frac{\partial f_s}{\partial w} - w_\parallel\frac{\partial f_s}{\partial w_\parallel}\right) = C[f_s]$$

I know... this form is a mess. BUT! It's really easy to take moments of. Let's do that now.

(remember that $w_\perp^2 = w^2 - w_\parallel^2$)

0th moment: $$\frac{D}{Dt_s}\cancelto{n_s}{\int f_s} + \cancelto{0}{\hat{b}\cdot\nabla\int w_\parallel f_s} + \cancelto{0}{(\nabla\cdot\hat{b})\int\frac{w_\perp^2}{2}\frac{\partial f_s}{\partial w_\parallel}}$$

$$+\left(\frac{q_s E_\parallel}{m_s} + g_\parallel - \hat{b}\cdot\frac{D\vec{u}_s}{Dt_s}\right)\left[\cancelto{0}{\int\frac{w_\parallel}{w}\frac{\partial f_s}{\partial w}} + \cancelto{0}{\int\frac{\partial f_s}{\partial w_\parallel}}\right]$$

$$-\int\frac{(\overset{-n_s}{w^2} - \overset{0}{w_\parallel^2})}{2w}\frac{\partial f_s}{\partial w}\,(\nabla\cdot\vec{u}_s) + \hat{b}\hat{b}:\nabla\vec{u}_s\left[\cancelto{-n_s}{\int\frac{w}{2}\frac{\partial f_s}{\partial w}} - \cancelto{0}{\int\frac{3}{2}\frac{w_\parallel^2}{w}\frac{\partial f_s}{\partial w}} - \cancelto{-n_s}{\int w_\parallel\frac{\partial f_s}{\partial w_\parallel}}\right]$$

$$= \cancelto{0}{\int C[f_s]}$$

$$\Rightarrow \frac{\partial}{\partial t}n_s + \vec{u}_s\cdot\vec{\nabla}n_s + n_s(\vec{\nabla}\cdot\vec{u}_s) = 0. \qquad \text{continuity eqn!}$$

Do the 1st moment yourself. You'll get the parallel component of

$$m_s n_s\left(\frac{\partial}{\partial t} + \vec{u}_s\cdot\vec{\nabla}\right)\vec{u}_s = -\vec{\nabla}\cdot\overleftrightarrow{P}_s + q_s n_s\left(\vec{E} + \frac{\vec{u}_s\times\vec{B}}{c}\right) + m_s n_s\vec{g},$$

where $$\overleftrightarrow{P}_s = p_{\perp s}(\overleftrightarrow{I} - \hat{b}\hat{b}) + p_{\parallel s}\hat{b}\hat{b}$$

$$= \int\frac{1}{2}m w_\perp^2 f_s\, d^3v\,(\overleftrightarrow{I} - \hat{b}\hat{b}) + \int m w_\parallel^2 f_s\, d^3v\,\hat{b}\hat{b}$$

I'll do the 2nd moments with you (next page).

$\int \frac{m_s w_\perp^2}{2}$ of DKE:

$$\frac{D}{Dt_s}\underbrace{\int \frac{m_s w_\perp^2}{2} f_s}_{\equiv p_{\perp s}} + \hat b\cdot\nabla \underbrace{\int w_\parallel \frac{m_s w_\perp^2}{2} f_s}_{\equiv Q_{\perp s}} + (\nabla\cdot\hat b)\overbrace{\int \frac{w_\perp^4}{4} m_s \frac{\partial f_s}{\partial w_\parallel}}^{=\int \frac{m_s}{4}(w^2-w_\parallel^2)^2 \frac{\partial f_s}{\partial w_\parallel} = \int m_s (w^2-w_\parallel^2) w_\parallel f_s \,\equiv\, 2Q_{\perp s}}$$

$$+\left(\frac{q_s E_\parallel}{m_s} + g_\parallel - \hat b\cdot\frac{D\vec u_s}{Dt_s}\right)\underbrace{\left[\overbrace{\int \frac{m_s w_\perp^2 w_\parallel}{2w}\frac{\partial f_s}{\partial w}}^{-\int m_s w_\parallel f_s} + \overbrace{\int \frac{m_s w_\perp^2}{2}\frac{\partial f_s}{\partial w_\parallel}}^{\int m_s w_\parallel f_s}\right]}_{\to 0}$$

$$-(\nabla\cdot\vec u_s)\underbrace{\int \frac{m_s w_\perp^2}{2}\frac{w_\perp^2}{2w}\frac{\partial f_s}{\partial w}}_{-2p_{\perp s}} + \hat b\hat b:\nabla\vec u_s \overbrace{\int \frac{m_s w_\perp^2}{2}\frac{w_\perp^2}{2w}\frac{\partial f_s}{\partial w}}^{= \frac{m_s}{2}\int (w^2-w_\parallel^2)^2 \frac{\partial f_s}{\partial w^2} = -m_s\int w_\perp^2 f_s = -2p_{\perp s}}$$

$$+\hat b\hat b:\nabla\vec u_s \underbrace{\int \frac{m_s w_\perp^2}{2}\left(-\frac{w_\parallel^2}{w}\right)\frac{\partial f_s}{\partial w}}_{\substack{=-\int m_s (w^2-w_\parallel^2) w_\parallel^2 \frac{\partial f_s}{\partial w^2} \\ = \int m_s \underset{\uparrow w_\parallel^2}{f_s} = p_{\parallel s}}} - \hat b\hat b:\nabla\vec u_s \underbrace{\int \frac{m_s w_\perp^2}{2} w_\parallel \frac{\partial f_s}{\partial w_\parallel}}_{\substack{=\int \frac{m_s}{2}(w^2-w_\parallel^2) w_\parallel \frac{\partial f_s}{\partial w_\parallel} \\ = -\int \frac{m_s}{2} w^2 f_s \\ +\int \frac{m_s}{2} 3 w_\parallel^2 f_s \\ = -p_{\perp s} - \frac{p_{\parallel s}}{2} + \frac{3}{2} p_{\parallel s}}} = \int \frac{m_s w_\perp^2}{2} C[f_s]$$

$$\longrightarrow \quad \frac{Dp_{\perp s}}{Dt_s} + \hat b\cdot\nabla Q_{\perp s} + (\nabla\cdot\hat b)\, 2Q_{\perp s} + 2p_{\perp s}(\nabla\cdot\vec u_s) - p_{\perp s}\,\hat b\hat b:\nabla\vec u_s$$

$$= \int \frac{m_s w_\perp^2}{2} C[f_s]$$

Use Krook operator for simplicity $\longrightarrow -\int \frac{m_s w_\perp^2}{2}(f_s - \underset{\text{Maxwellian}}{f_{Ms}})\nu_s$

$$= -\nu_s(p_{\perp s} - p_s) = -\frac{\nu_s}{3}(p_{\perp s} - p_{\parallel s})$$

Using $\frac{D\ln n_s}{Dt_s} = \nabla\cdot\vec u_s$ and $\frac{D\ln B}{Dt_s} = -\nabla\cdot\vec u_s + \hat b\hat b:\nabla\vec u_s$, this becomes

$$\boxed{\frac{Dp_{\perp s}}{Dt_s} + \hat{b}\cdot\nabla Q_{\perp s} + 2Q_{\perp s}\vec{\nabla}\cdot\hat{b} - p_{\perp s}\frac{D\ln B n_s^{-1}}{Dt_s} = -\frac{\nu_s}{3}\left(p_{\perp s} - p_{\parallel s}\right)}$$

note the adiabatic terms!

collisions relax anisotropy

heat flows redistribute $p_{\perp s}$ along $\hat{b}$

$\int m_s w_\parallel^2$ of DKE:

$$\frac{D}{Dt_s}\cancelto{= p_{\parallel s}}{\int m_s w_\parallel^2 f_s} + \hat{b}\cdot\nabla\cancelto{\equiv Q_{\parallel s}}{\int m_s w_\parallel^3 f_s} + (\vec{\nabla}\cdot\hat{b})\underbrace{\int m_s w_\parallel^2\frac{w_\perp^2}{2}\frac{\partial f_s}{\partial w_\parallel}}_{\substack{= -\int m_s w_\parallel w_\perp^2 f_s \\ + \int m_s 2 w_\parallel^3 f_s \\ = Q_{\parallel s} - 2Q_{\perp s}}}$$

$$+\left(\frac{q_s E_\parallel}{m_s} + g_\parallel - \hat{b}\cdot\frac{D\vec{u}_s}{Dt_s}\right)\cancelto{0}{\left[\int m_s\frac{w_\parallel^3}{w}\frac{\partial f_s}{\partial w} + \int m_s w_\parallel^2\frac{\partial f_s}{\partial w_\parallel}\right]}$$

$$-(\vec{\nabla}\cdot\vec{u}_s)\underbrace{\int m_s w_\parallel^2 w_\perp^2\frac{\partial f_s}{\partial w^2}}_{\substack{= -\int m_s w_\parallel^2 f_s \\ = -p_{\parallel s}}} + \hat{b}\hat{b}:\vec{\nabla}\vec{u}_s\left[\overbrace{\int m_s w_\parallel^2\, w_\perp^2\frac{\partial f_s}{\partial w^2}}^{-p_{\parallel s}} - \cancelto{0}{\int m_s\frac{w_\parallel^4}{w}\frac{\partial f_s}{\partial w}} - \overbrace{\int m_s w_\parallel^3\frac{\partial f_s}{\partial w_\parallel}}^{-3p_{\parallel s}}\right]$$

$$= \int m_s w_\parallel^2 C[f_s] = \int m_s w_\parallel^2(-\nu_s)(f_s - f_{Ms}) = -\nu_s(p_{\parallel s} - p_s)$$
$$= -\frac{2}{3}\nu_s\left(p_{\parallel s} - p_{\perp s}\right)$$

$$\Rightarrow \frac{Dp_{\parallel s}}{Dt_s} + \hat{b}\cdot\vec{\nabla}Q_{\parallel s} + (\vec{\nabla}\cdot\hat{b})\left(Q_{\parallel s} - 2Q_{\perp s}\right) + p_{\parallel s}\vec{\nabla}\cdot\vec{u}_s + 2p_{\parallel s}\hat{b}\hat{b}:\vec{\nabla}\vec{u}_s$$
$$= -\frac{2}{3}\nu_s\left(p_{\parallel s} - p_{\perp s}\right)$$

or

$$\boxed{\frac{Dp_{\parallel s}}{Dt_s} + \overbrace{\hat{b}\cdot\nabla Q_{\parallel s} + (\vec{\nabla}\cdot\hat{b})(Q_{\parallel s} - 2Q_{\perp s})}^{\text{heat flows redistribute } p_{\parallel s}} + \overbrace{p_{\parallel s}\,(2)\,\frac{D \ln B n_s^{-3/2}}{Dt_s}}^{\text{adiabatic invariance}} = \underbrace{-\frac{2}{3}\nu_s\,(p_{\parallel s} - p_{\perp s})}_{\text{collisions remove anisotropy}}}$$

Can rearrange the two boxed equs. to give

$$\text{Ⓐ}\quad p_{\perp s}\frac{D}{Dt_s}\ln\left(\frac{p_{\perp s}}{n_s B}\right) = -\vec{\nabla}\cdot\vec{Q}_{\perp s} - (\vec{\nabla}\cdot\hat{b})\,Q_{\perp s} - \frac{\nu_s}{3}\,(p_{\perp s} - p_{\parallel s})$$

$$\text{Ⓑ}\quad p_{\parallel s}\frac{D}{Dt_s}\ln\left(\frac{p_{\parallel s}B^2}{n_s^3}\right) = -\vec{\nabla}\cdot\vec{Q}_{\parallel s} + 2(\vec{\nabla}\cdot\hat{b})\,Q_{\perp s} - \frac{2}{3}\nu_s\,(p_{\parallel s} - p_{\perp s})$$

These are the full CGL equs., with collisions added and heat fluxes retained.

One can keep going, to derive evolutionary equations for $\vec{Q}_{\perp s}$ and $\vec{Q}_{\parallel s}$, but let's stop here and see if we can recover Braginskii MHD. Take $l \gg \lambda_{mfp}$ $(\omega \ll \nu)$. Then Ⓐ − Ⓑ gives

$$\frac{p_{\perp s} - p_{\parallel s}}{p_s} \simeq \frac{3}{\nu_s}\frac{D}{Dt_s}\ln\frac{B}{n_s^{2/3}} + \frac{1}{p_s\nu_s}\left[-\vec{\nabla}\cdot(\vec{Q}_{\perp s} - \vec{Q}_{\parallel s}) - 3(\vec{\nabla}\cdot\hat{b})\,Q_{\perp s}\right]$$

Because $\nu_i \ll \nu_e$ (mass ratio!), the ions dominate the pressure anisotropy:

$$p_\perp - p_\parallel \simeq \underbrace{\frac{3p_i}{\nu_i}\frac{D}{Dt}\ln\frac{B}{n^{2/3}}}_{\text{"Braginskii"}} \; \underbrace{\frac{1}{\nu_i}\left[3(\vec{\nabla}\cdot\hat{b})Q_{\perp i} + \vec{\nabla}\cdot\left(\vec{Q}_{\perp i} - \vec{Q}_{\parallel i}\right)\right]}_{\text{"Mikhailovskii-Tsypin"}}$$

We've already discussed the Braginskii term — deviations from a Maxwellian are produced by adiabatic invariance, shaped by the magnetic field, and relaxed by collisions. The M-T terms also have a simple interpretation. Consider the 2nd one: $\vec{\nabla}\cdot(\vec{Q}_{\perp i} - \vec{Q}_{\parallel i})$. This is a total divergence, and so it can only redistribute the pressure anisotropy. Indeed, if the flows of $p_{\perp i}$ and $p_{\parallel i}$ are different, then the local anisotropy can change. The first term, $3(\vec{\nabla}\cdot\hat{b})Q_{\perp i}$, is related to the pinching of field lines on which the (perpendicular) heat flows. If the lines are perturbed $\left[i\vec{k}\cdot\delta b = i\vec{k}_\perp\cdot\frac{\delta\vec{B}_\perp}{B_0} = -ik_\parallel\frac{\delta B_\parallel}{B_0}\right]$, then there are regions where the field is compressed ($\delta B_\parallel > 0$) and the flux of heat through that region increases. This also generates pressure anisotropy.

Barnes Damping and linear KMHD

Our KMHD equations are:

$$\Sigma_s q_s \int f_s \, d^3v = 0$$

$$\Sigma_s q_s \int \vec{v} f_s \, d^3v = \frac{c}{4\pi} \vec{\nabla} \times \vec{B}$$

$$m_i n_i \left(\frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla} \right) \vec{u} = -\vec{\nabla} \cdot \left(\overleftrightarrow{P} + \overleftrightarrow{I} \frac{B^2}{8\pi} - \frac{\vec{B}\vec{B}}{4\pi} \right)$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{u} \times \vec{B})$$

$$\frac{Df_s}{Dt} + \frac{D \ln B}{Dt} \frac{w_\perp}{2} \frac{\partial f_s}{\partial w_\perp} + \left(\frac{q_s}{m_s} E_\parallel - \hat{b} \cdot \frac{D\vec{u}_\perp}{Dt} - \frac{w_\perp^2}{2} \hat{b} \cdot \vec{\nabla} \ln B \right) \frac{\partial f_s}{\partial v_\parallel} = C[f_s]$$

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{u}_\perp \cdot \vec{\nabla}_\perp + v_\parallel \hat{b} \cdot \vec{\nabla}$ and we've taken $m_i \vec{u}_i + \vec{u}_e m_e \simeq m_i \vec{u}_i = m_i \vec{u}$.

The pressure tensor $\overleftrightarrow{P} = p_\perp (\overleftrightarrow{I} - \hat{b}\hat{b}) + p_\parallel \hat{b}\hat{b}$ with $p_\perp \equiv \Sigma_s p_{\perp s}$ and $p_\parallel \equiv \Sigma_s p_{\parallel s}$; the pressures are determined from $p_{\perp s} = \int \frac{1}{2} m w_\perp^2 f_s \, d^3\vec{v}$ & $p_{\parallel s} = \int m_s w_\parallel^2 f_s \, d^3v$.

Let us do linear theory about a homogeneous, Maxwellian, stationary equilibrium: $f_s = f_{Ms} + \delta f_s$, $\vec{B} = B_0 \hat{z} + \delta\vec{B}$, etc.

w/ $\delta \sim \exp(-i\omega t + i\vec{k} \cdot \vec{r})$

linearized induction: $-i\omega \frac{\delta B_\parallel}{B_0} = -i\vec{k}_\perp \cdot \delta\vec{u}_\perp \qquad -i\omega \frac{\delta \vec{B}_\perp}{B_0} = i k_\parallel \delta\vec{u}_\perp$

linearized momentum: $$-i\omega \delta\vec{u}_\perp = -i\left[\frac{(\vec{k}-k_\parallel\hat{b})\delta p_\perp + k_\parallel \hat{b}\,\delta p_\parallel}{m_i n_{0i}}\right]_\perp - i\vec{k}_\perp v_A^2 \frac{\delta B_\parallel}{B_0} + i k_\parallel v_A^2 \frac{\delta\vec{B}_\perp}{B_0}$$

$$\Rightarrow -i\omega \frac{\delta B_\parallel}{B_0} = -i\vec{k}_\perp \cdot \left(\frac{1}{-i\omega}\right)\left\{-i\vec{k}_\perp v_A^2 \frac{\delta B_\parallel}{B_0} + i k_\parallel v_A^2 \frac{\delta \vec{B}_\perp}{B_0} - \frac{i\vec{k}_\perp \delta p_\perp}{m_i n_{0i}}\right\}$$

$$+\cancel{i}\left(\omega^2 - k^2 v_A^2\right)\frac{\delta B_\parallel}{B} = +\cancel{i}\,\frac{k_\perp^2 \delta p_\perp}{m_i n_{0i}}$$

we just need $\delta p_\perp = \Sigma_s \int \frac{1}{2} m_s w_\perp^2 \delta f_s \, d^3v$ from linearized DKE:

$$-i\omega \delta f_s - i\omega \frac{\delta B_\parallel}{B_0}\frac{w_\perp}{2}\frac{\partial f_{0s}}{\partial w_\perp} + i k_\parallel v_\parallel \delta f_s + i k_\parallel v_\parallel \frac{\delta B_\parallel}{B_0}\frac{w_\perp}{2}\frac{\partial f_{0s}}{\partial w_\perp}$$

$$+\left(\frac{q_s}{m_s}E_\parallel - \frac{w_\perp^2}{2} i k_\parallel \frac{\delta B_\parallel}{B_0}\right)\frac{\partial f_{0s}}{\partial v_\parallel} = C[f_s] = 0$$

Drop collisions (let $k\lambda_{mfp} \gg 1$)

$$\Rightarrow \delta f_s + \frac{w_\perp}{2}\frac{\partial f_{0s}}{\partial w_\perp}\frac{\delta B_\parallel}{B_0} + \frac{\left(\frac{q_s}{m_s}E_\parallel\right) - \left(\frac{w_\perp^2}{2} i k_\parallel \frac{\delta B_\parallel}{B_0}\right)}{-i(\omega - k_\parallel v_\parallel)}\frac{\partial f_{0s}}{\partial v_\parallel} = 0.$$

~~[illegible]~~ will give Landau damping when $\omega \sim k_\parallel v_{th}$

mirror force also gives collisionless ("Barnes") damping when $\omega \sim k_\parallel v_{th}$.

$$\delta p_{\perp s} = \int \frac{1}{2} m_s w_\perp^2 \, \delta f_s \, d^3v \quad \leftarrow 2\pi w_\perp dw_\perp dv_\parallel$$

$$= \int \frac{1}{2} m_s w_\perp^2 \left(-\frac{w_\perp}{2} \frac{\partial f_0}{\partial w_\perp} \right) \frac{\delta B_\parallel}{B_0} + \int \frac{1}{2} m_s w_\perp^2 \frac{\partial f_{0s}}{\partial v_\parallel} \frac{\left(\frac{q_s}{m_s} E_\parallel - \frac{w_\perp^2}{2} i k_\parallel \frac{\delta B_\parallel}{B_0} \right)}{i(\omega - k_\parallel v_\parallel)}$$

$$= \underbrace{\int m_s w_\perp^2 f_{0s} \frac{\delta B_\parallel}{B_0}}_{2p_{0s}} + \frac{1}{(-ik_\parallel)} \int \underbrace{\frac{\partial f_{0s}/\partial v_\parallel}{v_\parallel - \omega/k_\parallel}}_{\left(\frac{-\frac{2v_\parallel}{v_{ths}^2} f_{0s}}{v_\parallel - \omega/k_\parallel} \right)} \frac{1}{2} m_s w_\perp^2 \left(\frac{q_s}{m_s} E_\parallel - \frac{i k_\parallel w_\perp^2}{2} \frac{\delta B_\parallel}{B_0} \right)$$

$$= 2p_{0s} \frac{\delta B_\parallel}{B_0} + \frac{iT_s}{k_\parallel} \frac{(-2)}{v_{ths}^2} \int \frac{v_\parallel}{v_{ths}} \frac{f_{0s}}{\frac{v_\parallel}{v_{ths}} - \frac{\omega}{k_\parallel v_{ths}}} \frac{w_\perp^2}{v_{ths}^2} \left[\frac{q_s}{m_s} E_\parallel - \frac{i k_\parallel w_\perp^2}{v_{ths}^2} \frac{T_s}{m_s} \frac{\delta B_\parallel}{B_0} \right]$$

$$= 2p_{0s} \frac{\delta B_\parallel}{B_0} - \frac{2i}{k_\parallel} \frac{m_s}{2T_s} \frac{n_{0s}}{T_s} \left[\frac{q_s}{m_s} E_\parallel \left(1 + \xi_s Z(\xi_s) \right) - i k_\parallel \frac{T_s}{m_s} \frac{\delta B_\parallel}{B_0} 2 \left(1 + \xi_s Z(\xi_s) \right) \right]$$

where $\xi_s \equiv \frac{\omega}{|k_\parallel| v_{ths}}$ and Z is plasma dispersion function

$$\Rightarrow \delta p_\perp = 2p_0 \frac{\delta B_\parallel}{B_0} - \frac{i}{k_\parallel} \sum_s p_{0s} \left[1 + \xi_s Z(\xi_s) \right] \left(\frac{q_s E_\parallel}{T_{0s}} - 2 i k_\parallel \frac{\delta B_\parallel}{B_0} \right)$$

This goes into our momentum equ., but first we need to get $E_\parallel$. This is from quasi-neutrality: $\sum_s q_s \int \delta f_s \, d^3v = 0$

$$0 = \sum_s q_s \int d^3v \left[-\frac{w_\perp}{2}\frac{\partial f_{0s}}{\partial w_\perp}\frac{\delta B_\parallel}{B_0} + \frac{\left(\frac{q_s}{m_s}E_\parallel - \frac{w_\perp^2}{2}\, i k_\parallel \frac{\delta B_\parallel}{B_0}\right)\frac{\partial f_{0s}}{\partial v_\parallel}}{-i(\omega - k_\parallel v_\parallel)}\right]$$

$$= \cancelto{0}{\sum_s q_s n_{0s}\frac{\delta B_\parallel}{B_0}} + \sum_s \frac{q_s^2}{m_s}\frac{(-2)E_\parallel}{v_{ths}}\int \frac{v_\parallel}{v_{ths}} f_{0s}\left(\frac{-i}{k_\parallel}\right)\frac{1}{\frac{v_\parallel}{v_{ths}} - \frac{\omega}{k_\parallel v_{ths}}}\left(\frac{1}{v_{ths}}\right)$$

$$+ \sum_s q_s \frac{\cancel{i k_\parallel}}{\cancel{2}}\frac{\delta B_\parallel}{B_0}\frac{2T_{0s}}{m_s}\frac{(+\cancelto{1}{2})}{v_{ths}}\int \frac{v_\parallel}{v_{ths}} f_{0s}\frac{w_\perp^2}{v_{ths}^2}\left(\frac{+\cancel{i}}{\cancel{k_\parallel} v_{ths}}\right)\frac{1}{\frac{v_\parallel}{v_{ths}} - \frac{\omega}{k_\parallel v_{ths}}}$$

$$= \frac{2i}{k_\parallel}\sum_s \frac{q_s^2}{\cancel{m_s}}\frac{\cancel{m_s}}{2T_{0s}} n_{0s}\left[1+\zeta_s Z(\zeta_s)\right]E_\parallel$$

$$+ \sum_s q_s \frac{\delta B_\parallel}{B_0}\frac{\cancel{2T_{0s}}}{\cancel{m_s}}\frac{\cancel{m_s}}{\cancel{2T_{0s}}} n_{0s}\left[\underbrace{\cancel{1}}_{0 \text{ by } \sum_s q_s n_s = 0} + \zeta_s Z(\zeta_s)\right]$$

$$\Rightarrow \frac{iE_\parallel}{k_\parallel} = \left[\frac{-\sum_s q_s n_{0s}\zeta_s Z(\zeta_s)}{\sum_s \frac{q_s^2 n_{0s}}{T_{0s}}\left[1+\zeta_s Z(\zeta_s)\right]}\right]\frac{\delta B_\parallel}{B_0} \equiv -\chi\frac{\delta B_\parallel}{B_0}$$

$$\Rightarrow \delta p_\perp = 2p_0\frac{\delta B_\parallel}{B_0} + \frac{\cancel{i}}{\cancel{k_\parallel}}\sum_s p_{0s}\left[1+\zeta_s Z(\zeta_s)\right]\frac{\delta B_\parallel}{B_0}\left[\frac{q_s\chi}{T_{0s}} - 2\right]\left(\frac{\cancel{k_\parallel}}{\cancel{i}}\right)$$

$$= \left\{2p_0 - 2\sum_s p_{0s}\left[1+\zeta_s Z(\zeta_s)\right] + \frac{\left(\sum_s q_s n_{0s}\zeta_s Z(\zeta_s)\right)^2}{\sum_s \frac{q_s^2 n_{0s}}{T_{0s}}\left(1+\zeta_s Z(\zeta_s)\right)}\right\}\frac{\delta B_\parallel}{B_0}$$

$$= \left[-2\sum_s p_{0s}\zeta_s Z(\zeta_s) + \frac{\left(\sum_s q_s n_{0s}\zeta_s Z(\zeta_s)\right)^2}{\sum_s \frac{q_s^2 n_{0s}}{T_{0s}}\left(1+\zeta_s Z(\zeta_s)\right)}\right]\frac{\delta B_\parallel}{B_0}$$

$$\Rightarrow \quad \omega^2 - k_\parallel^2 v_A^2 - k_\perp^2 v_A^2 = \frac{k_\perp^2}{m_i n_{0i}} \left[-2 \sum_s p_{0s} \xi_s Z(\xi_s) + \frac{\left(\sum_s q_s n_{0s} \xi_s Z(\xi_s) \right)^2}{\sum_s \frac{q_s^2 n_{0s}}{T_{0s}} \left(1 + \xi_s Z(\xi_s) \right)} \right]$$

This is the dispersion relation governing compressive fluctuations. The simplest case is for cold electrons or $\beta_i \gg 1$, for which the $E_\parallel$ term (the last in brackets) can be dropped. Then, with $\xi_e/\xi_i \sim (m_e/m_i)^{1/2} \ll 1$ and $Z(\xi_i) \simeq i\sqrt{\pi}$ (can check a posteriori), we have

$$+k^2 v_A^2 \simeq +2 k_\perp^2 \left(\frac{T_{0i}}{m_i} \right) \frac{\omega}{|k_\parallel| v_{thi}} \, i\sqrt{\pi} = \frac{k_\perp^2 v_{thi}}{|k_\parallel|} \, \omega \, i\sqrt{\pi}$$

$$\Rightarrow \quad \omega \simeq \frac{-i}{\sqrt{\pi}} \frac{|k_\parallel| v_{thi}}{\beta_i} \left(\frac{k^2}{k_\perp^2} \right) \qquad \text{DAMPING!}$$

weakest when $k^2 \simeq k_\perp^2$ (i.e., $k_\parallel / k_\perp \ll 1$): $\boxed{\omega \simeq \frac{-i}{\sqrt{\pi}} \frac{|k_\parallel| v_A}{\sqrt{\beta_i}}}$

This is Barnes damping. (Note that $\xi_i \sim \frac{1}{\beta_i} \ll 1$.)

Particles that are almost at rest w/ respect to the slow wave (i.e., "Landau resonant" with $\omega \sim k_\parallel v_\parallel$) are subject to the action of the mirror force associated with the magnetic compression in the wave. Since, for $\partial f_{0i}/\partial v_\parallel < 0$, there are more particles with $v_\parallel < \omega/k_\parallel$ than with $v_\parallel > \omega/k_\parallel$, the energy exchange between resonant particles and the wave leads to a net gain (loss) of energy by the particles (wave). Put differently, the only way to maintain perpendicular pressure balance for a slow wave is to increase the energy of the resonant particles at the expense of the wave energy. The result is wave damping.

Firehose & Mirror Instabilities

Given that many astrophysical plasmas are weakly collisional (see Eliot's table), there is little reason to believe they are always Maxwellian. Because $\frac{\omega}{\Omega_i} \ll 1$ often, we expect any non-Maxwellianities to know about the field direction and be biased with respect to it. A common way of describing this is by the bi-Maxwellian

$$f_{bi\text{-}M_s}(v_\parallel, w_\perp) = \frac{n_s}{\pi^{3/2} v_{th\perp s}^2 v_{th\parallel s}} \exp\left(\frac{-v_\parallel^2}{v_{th\parallel s}^2}\right) \exp\left(\frac{-w_\perp^2}{v_{th\perp s}^2}\right)$$

where $v_{th\perp s}^2 = \frac{2T_{\perp s}}{m_s}$ and $v_{th\parallel s}^2 = \frac{2T_{\parallel s}}{m_s}$. Let us consider a uniform, magnetized, stationary plasma with $f_{0s} = f_{bi\text{-}M}(v_\parallel, w_\perp)$, and do linear theory on it.

As before, we have $-i\omega \frac{\delta B_\parallel}{B_0} = -i\vec{k}_\perp \cdot \delta \vec{u}_\perp \qquad -i\omega \frac{\delta \vec{B}_\perp}{B_0} = i k_\parallel \delta \vec{u}_\perp$

The momentum eqn. is now more complicated; look at $\vec{\nabla} \cdot \overleftrightarrow{P}_s$:

$$-i\vec{k} \cdot \delta \overleftrightarrow{P}_s = -i\vec{k} \cdot \Big[\delta p_{\perp s} (\overleftrightarrow{I} - \hat{b}_0\hat{b}_0) + \delta p_{\parallel s} \hat{b}_0\hat{b}_0 + (p_{\parallel 0s} - p_{\perp 0s})(\hat{b}\,\delta\hat{b} + \delta\hat{b}\,\hat{b}) \Big]$$

$$= -i\vec{k}_\perp \delta p_{\perp s} + (-ik_\parallel \hat{b}_0)\delta p_{\parallel s} + (p_{\parallel 0} - p_{\perp 0})(-i)\Big(k_\parallel \overset{\delta \vec{B}_\perp / B_0}{\delta\hat{b}} + \underbrace{\vec{k} \cdot \delta\hat{b}}_{-k_\parallel \frac{\delta B_\parallel}{B_0}}\, \hat{b}_0 \Big)$$

$$= -i\vec{k}_\perp \delta p_{\perp s} - i k_\parallel \frac{\delta \vec{B}_\perp}{B_0}(p_{\parallel 0} - p_{\perp 0}) - i k_\parallel \hat{b}_0 \delta p_{\parallel s} + i k_\parallel \frac{\delta B_\parallel}{B_0} \hat{b}_0 (p_{\parallel 0} - p_{\perp 0})$$

With $k_\perp = 0$, we have $-i\omega \frac{\delta \vec{B}_\perp}{B_0} = i k_\parallel \delta \vec{u}_\perp = i k_\parallel \left(-\frac{1}{i\omega}\right)\Big[i k_\parallel v_A^2 \frac{\delta \vec{B}_\perp}{B_0} - i k_\parallel \frac{\delta \vec{B}_\perp}{B_0} \big(\;\big) \Big]$

$\big(\;\big) = \frac{\sum_s (p_{\parallel 0s} - p_{\perp 0s})}{m_i n_{0i}}$

$$\Rightarrow \boxed{\omega^2 = k_\parallel^2 v_A^2 + k_\parallel^2 \frac{(p_{\perp 0} - p_{\parallel 0})}{m_i n_{0i}}}$$

Don't need to know δp for these fluctuations!

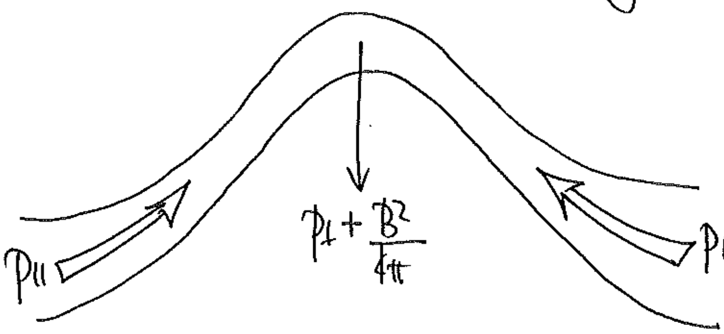
These are Alfvén waves ($\delta B_\parallel = 0$) with a modified propagation speed:

$$v_{A\,eff}^2 = v_A^2 + \frac{p_{\perp 0} - p_{\parallel 0}}{m_i n_{0i}} = \left(\frac{B_0^2}{4\pi p_0} + \frac{p_{\perp 0} - p_{\parallel 0}}{p_0}\right)\frac{v_{thi}^2}{2}$$

$$= v_A^2 \left[1 + \frac{p_{\perp 0} - p_{\parallel 0}}{B_0^2/4\pi} \right]$$

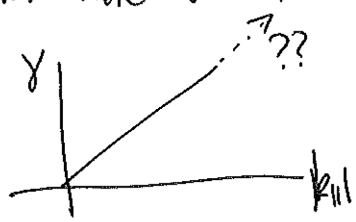
Note that, if the term in brackets is < 0, then ω is imaginary!

This is the <u>firehose instability</u>.



Parallel pressure overwhelms restoring forces and drives Alfvén wave unstable.

problem: growth rate $\gamma \equiv -i\omega = |k_\parallel| v_A \sqrt{\frac{p_{\parallel 0} - p_{\perp 0}}{B_0^2/4\pi} - 1} \propto |k_\parallel|$



UV catastrophe. This is only resolved by including finite-Larmor-radius effects.

What about $k_\perp \neq 0$?

$$-i\omega \frac{\delta B_\parallel}{B_0} = -i\vec{k}_\perp \cdot \delta\vec{u}_\perp = \frac{-i\vec{k}_\perp}{(-i\omega)} \cdot \left[i k_\parallel v_A^2 \frac{\delta \vec{B}_\perp}{B_0} - i\vec{k}_\perp v_A^2 \frac{\delta B_\parallel}{B_0} - i\vec{k}_\perp \sum_s \frac{\delta p_{\perp s}}{m_i n_{0i}} - i k_\parallel \frac{\delta \vec{B}_\perp}{B_0} \frac{(p_{\parallel 0} - p_{\perp 0})}{m_i n_{0i}} \right]$$

$$\left[-\omega^2 + k_\perp^2 v_A^2 + k_\parallel^2 v_A^2 + k_\parallel^2 (p_{\perp 0} - p_{\parallel 0}) \right] \frac{\delta B_\parallel}{B_0} = -k_\perp^2 \sum_s \frac{\delta p_{\perp s}}{m_i n_{0i}}$$

$$= +\frac{k_\perp^2}{m_i n_{0i}} \sum_s \int \frac{1}{2} m_s w_\perp^2 \left[+\frac{w_\perp}{2} \frac{\partial f_{0s}}{\partial w_\perp} \frac{\delta B_\parallel}{B_0} + i \frac{\partial f_{0s}}{\partial w_\parallel} \frac{\left(\frac{q_s}{m_s} E_\parallel - \frac{w_\perp^2}{2} i k_\parallel \frac{\delta B_\parallel}{B_0} \right)}{\omega - k_\parallel v_\parallel} \right]$$

Set $\frac{E_\parallel i}{k_\parallel} = -\chi \frac{\delta B_\parallel}{B_0}$ like before.

$$\left(-\omega^2 + k_\perp^2 v_A^2 + k_\parallel^2 v_{Aeff}^2\right)\frac{\delta B_\parallel}{B_0}$$

$$= -2k_\perp^2 \sum_s \frac{p_{\perp 0s}}{m_i n_{0i}}\frac{\delta B_\parallel}{B_0}$$

$$+ k_\perp^2 \sum_s \frac{T_{\perp 0s}}{m_i n_{0i}} \int \frac{w_\perp^2}{v_{ths}^2} \frac{2 v_\parallel}{v_{ths}^2} \frac{\left[\frac{q_s}{m_s}\frac{k_\parallel}{}(-\chi) + \frac{w_\perp^2}{v_{ths}^2}\frac{k_\parallel}{}\frac{T_{\perp 0s}}{m_s}\right]\frac{\delta B_\parallel}{B_0}}{k_\parallel v_{th\parallel s}\left(\frac{v_\parallel}{v_{th\parallel s}} - \frac{\omega}{k_\parallel}\right)} f_{0s}$$

$$= -2k_\perp^2 \frac{p_{\perp 0}}{m_i n_{0i}}\frac{\delta B_\parallel}{B_0} + k_\perp^2 \sum_s \frac{p_{\perp 0s}}{m_i n_{0i}}\frac{\delta B_\parallel}{B_0}\frac{m_s}{T_{\parallel 0s}}\left[1 + \xi_s Z(\xi_s)\right] \cdot \left(-\frac{q_s}{m_s}\chi + 2\frac{T_{\perp 0s}}{m_s}\right)$$

$$\Rightarrow \omega^2 - k_\perp^2 v_A^2 - k_\parallel^2 v_{Aeff}^2 = +2k_\perp^2\frac{p_{\perp 0}}{m_i n_{0i}} - k_\perp^2 \sum_s \frac{p_{\perp 0s}^2}{m_i n_{0i}\, p_{\parallel 0s}}\left(2 - \frac{q_s \chi}{T_{\perp 0s}}\right)\left[1 + \xi_s Z(\xi_s)\right]$$

Once again, you can get χ from $\sum_s q_s \int \delta f_s d^3v = 0$; I'll leave that as an exercise. In the high-β limit, it doesn't matter anyhow. With $Z(\xi_i) \simeq i\sqrt{\pi}$, $\xi_e/\xi_i \ll 1$, and $|\xi_i| \ll 1$, we find

$$m_i n_{0i}\left(-k_\perp^2 v_A^2 - k_\parallel^2 v_{Aeff}^2\right) = 2k_\perp^2 p_{\perp 0} - 2k_\perp^2 \frac{p_{\perp 0i}^2}{p_{\parallel 0i}}\left[1 + i\sqrt{\pi}\frac{\omega}{|k_\parallel| v_{th\parallel i}}\right]$$

$$- 2k_\perp^2 \frac{p_{\perp 0e}^2}{p_{\parallel 0e}}\left[1\right]$$

$$+k_\perp^2 \frac{B_0^2}{4\pi} + k_\parallel^2\left[\frac{B_0^2}{4\pi}\right]\left[1+\frac{p_{\perp 0}-p_{\parallel 0}}{B_0^2/4\pi}\right] + 2k_\perp^2 p_{\perp 0} - 2k_\perp^2\left(\frac{p_{\perp 0i}^2}{p_{\parallel 0i}} + \frac{p_{\perp 0e}^2}{p_{\parallel 0e}}\right)$$

$$= +2k_\perp^2 \frac{p_{\perp 0i}^2}{p_{\parallel 0i}} \; \frac{i\sqrt{\pi}\;\omega}{|k_\parallel| v_{th\parallel i}}$$

Set $\omega = +i\gamma$ and divide through by $2k_\perp^2 \frac{p_{\perp 0i}^2}{p_{\parallel 0i}}$:

$$\frac{\gamma\sqrt{\pi}}{|k_\parallel| v_{th\parallel i}} = -\frac{p_{\parallel 0i}}{p_{\perp 0i}^2}\left[\frac{B_0^2}{8\pi} + p_{\perp 0i} + p_{\perp 0e} - \frac{p_{\perp 0i}^2}{p_{\parallel 0i}} - \frac{p_{\perp 0e}^2}{p_{\parallel 0e}} + \frac{k_\parallel^2}{k_\perp^2}\frac{B_0^2}{8\pi}\left(1+\frac{p_{\perp 0}-p_{\parallel 0}}{B_0^2/4\pi}\right)\right]$$

$$\boxed{\Rightarrow \gamma = \frac{|k_\parallel| v_{th\parallel i}}{\sqrt{\pi}}\left(\frac{p_{\parallel 0i}}{p_{\perp 0i}}\right)\left[\frac{p_{\perp 0i}-p_{\parallel 0i}}{p_{\parallel 0i}} + \frac{p_{\perp 0e}}{p_{\perp 0i}}\left(\frac{p_{\perp 0e}-p_{\parallel 0e}}{p_{\parallel 0e}}\right) - \frac{1}{\beta_{\perp i}} - \frac{k_\parallel^2}{k_\perp^2}\frac{1}{\beta_{\perp i}}\left(1+\frac{p_{\perp 0i}-p_{\parallel 0i}+p_{\perp 0e}-p_{\parallel 0e}}{B_0^2/4\pi}\right)\right]}$$

Mirror instability when term in brackets > 0. Note that this formula generalizes the Barnes Damping result for anisotropic distribution functions. Max. growth when

$$\left(\frac{\partial\gamma}{\partial k_\parallel}\right)_{k_\perp} = 0 = \frac{p_{\perp 0i}-p_{\parallel 0i}}{p_{\parallel 0i}} + \frac{p_{\perp 0e}}{p_{\perp 0i}}\left(\frac{p_{\perp 0e}}{p_{\parallel 0e}}-1\right) - \frac{1}{\beta_{\perp i}} - 3\frac{k_\parallel^2}{k_\perp^2}\frac{1}{\beta_{\perp i}}\left(\ldots\right)$$

$$\Rightarrow \boxed{\gamma_{max} = \frac{|k_\parallel| v_{th\parallel i}}{\sqrt{\pi}} \left[\frac{2}{3}\frac{p_{\parallel 0i}}{p_{\perp 0i}}\right]\left[\frac{p_{\perp 0i}-p_{\parallel 0i}}{p_{\parallel 0i}} + \frac{p_{\perp 0e}}{p_{\perp 0i}}\left(\frac{p_{\perp 0e}-p_{\parallel 0e}}{p_{\parallel 0e}}\right) - \frac{1}{\beta_{\perp i}}\right]}$$

Note that this also has a UV catastrophe ($\gamma \propto |k_\parallel|$), which is resolved by finite-Larmor-radius effects.

Note also that, if we are near marginal instability (i.e. term in brackets is small), then $\gamma \ll |k_\parallel| v_{thi}$ (as promised) and

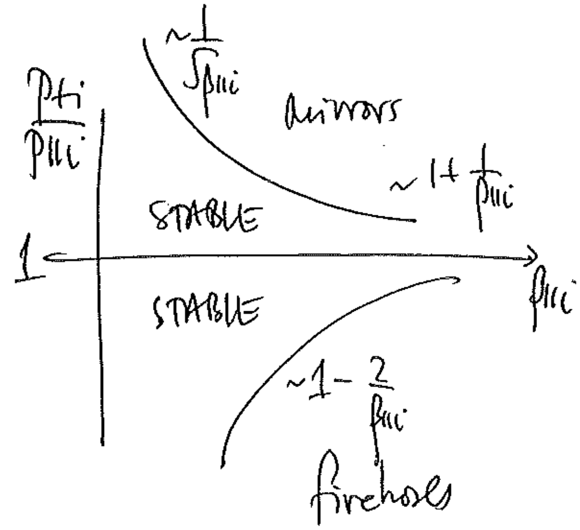
$\frac{k_\parallel}{k_\perp} \sim \sqrt{\Lambda} \ll 1$ (where $\Lambda \equiv \frac{p_{\perp 0i}-p_{\parallel 0i}}{p_{\parallel 0i}} + \frac{p_{\perp 0e}}{p_{\perp 0i}}\left(\frac{p_{\perp 0e}-p_{\parallel 0e}}{p_{\parallel 0e}}\right) - \frac{1}{\beta_{\perp i}}$)

Thus, mirror modes are <u>highly oblique</u>.

with cold electrons (simple):

Firehose: $\frac{p_{\perp i}}{p_{\parallel i}} - 1 < -\frac{2}{\beta_{\parallel i}}$

Mirror: $\frac{p_{\perp i}}{p_{\parallel i}}\left(\frac{p_{\perp i}}{p_{\parallel i}} - 1\right) > \frac{1}{\beta_{\parallel i}}$



(can show that $k_\parallel{}_{peak}\,\rho_i \sim \Lambda$, $k_\perp{}_{peak}\,\rho_i \sim \sqrt{\Lambda}$, $\gamma_{max} \sim \Omega_i \Lambda^2$ for mirror when including FLR effects.)

"Astrophysical" (Slab) Gyrokinetics

(Howes et al 2006, Schekochihin et al 2009, Frieman & Chen 1982, Kunz et al. 2015)

not the most elegant derivation of GK, but perhaps the most physically intuitive and readily understandable.

Start w/ V-M: (Landau too)

$$\frac{\partial f_s}{\partial t} + \vec{v}\cdot\vec{\nabla} f_s + \frac{q_s}{m_s}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \vec{v}} = C[f_s]$$

$$\frac{\partial \vec{B}}{\partial t} = -c\vec{\nabla}\times\vec{E} \qquad 0 = \sum_s q_s n_s = \sum_s q_s \int f_s\, d^3v$$

$$\vec{J} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B} = \sum_s q_s n_s \vec{u}_s = \sum_s q_s \int \vec{v} f_s\, d^3v$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

Introduce potentials: $\vec{E} = -\vec{\nabla}\varphi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ $\quad \vec{B} = B_0\hat{z} + \vec{\nabla}\times\vec{A}$

with $\vec{\nabla}\cdot\vec{A} = 0$

Ordering:

$$\frac{\omega}{\Omega_s} \sim \frac{\rho_s}{L} \sim \frac{k_\parallel}{k_\perp} \sim \frac{u_\perp}{v_A} \sim \frac{\delta B_\perp}{B_0} \sim \frac{u_\parallel}{v_A} \sim \frac{\delta B_\parallel}{B_0} \sim \frac{\delta f_{1s}}{f_0} \sim \epsilon \ll 1$$

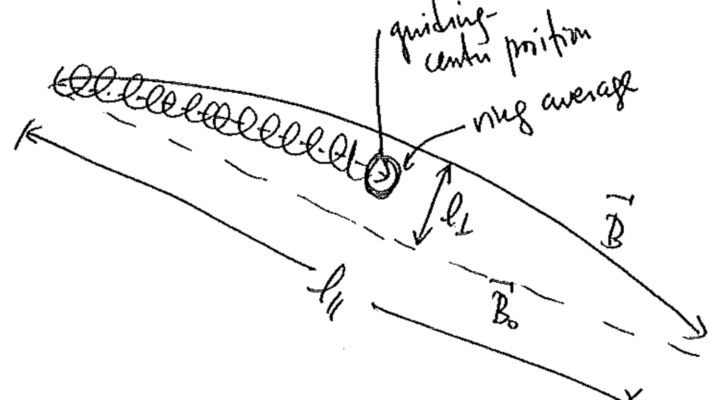
$$f_s = f_{0s} + \delta f_{1s} + \delta f_{2s} + \ldots = f_0 + \delta f_s$$

$$k_\perp \rho_s \sim \beta_s \sim \tau \sim 1 \qquad \text{and} \qquad \omega \sim \nu_s$$

> Small fluctuations
> slow eq. evolution
> intermed. evol. of fluc.
> intermed. collis. times
> $k_\perp \rho_s \sim 1$
> $k_\parallel L \sim 1$

Take f_0 homogeneous and $\frac{\partial f_0}{\partial t} \sim \epsilon^2 \frac{f_0 v_{thi}}{L}$.

Picture:



Order rel. to ωf_{0s}:

$$\underset{(2)}{\frac{\partial f_{0s}}{\partial t}} + \underset{(1)}{\frac{\partial \delta f_s}{\partial t}} + \underset{(0)}{v_\perp \cdot \nabla_\perp \delta f_s} + \underset{(1)}{v_\parallel \hat{z}\cdot\nabla \delta f_s}$$

$$+ \frac{q_s}{m_s}\Big[\underset{(0)}{-\vec{\nabla}_\perp \varphi} - \underset{(1)}{\vec{\nabla}_z \varphi} + \underset{(-1)}{\frac{\vec{v}\times\hat{z} B_0}{c}} - \underset{(1)}{\frac{1}{c}\frac{\partial \vec{A}}{\partial t}} + \underset{(0)}{\frac{\vec{v}\times\delta\vec{B}}{c}}\Big]\cdot\frac{\partial f_{0s}}{\partial \vec{v}}$$

$$+ \frac{q_s}{m_s}\Big[\underset{(1)}{-\vec{\nabla}_\perp \varphi} - \underset{(2)}{\vec{\nabla}_z \varphi} + \underset{(0)}{\frac{\vec{v}\times\hat{z} B_0}{c}} - \underset{(2)}{\frac{1}{c}\frac{\partial \vec{A}}{\partial t}} + \underset{(1)}{\frac{\vec{v}\times\delta\vec{B}}{c}}\Big]\cdot\frac{\partial \delta f_s}{\partial \vec{v}}$$

$$= \underset{(0)}{C[f_{0s}, f_{0s}]} + \underset{(1)}{C[f_{0s}, \delta f_s]} + \underset{(1)}{C[\delta f_s, f_{0s}]} + \underset{(2)}{C[\delta f_s, \delta f_s]}$$

(-1): $\frac{q_s B_0}{m_s c}(\vec{v}\times\hat{z})\cdot\frac{\partial f_{0s}}{\partial \vec{v}} = 0$ write $\vec{v} = v_\parallel \hat{z} + v_\perp(\cos\vartheta\,\hat{x} + \sin\vartheta\,\hat{y})$

$$\Rightarrow \boxed{-\Omega_s \left.\frac{\partial f_s}{\partial \vartheta}\right|_r = 0}$$

$$\Rightarrow \boxed{f_{0s} = f_{0s}(v_\parallel, v_\perp, t) \text{ independent of gyrophase}}$$

(0): $v_\perp\cdot\nabla_\perp \delta f_s - \frac{q_s}{m_s}\nabla_\perp\varphi\cdot\frac{\partial f_s}{\partial \vec{v}}^{\text{"1"}} + \frac{q_s}{m_s}\frac{\vec{v}\times\delta\vec{B}}{c}\cdot\frac{\partial f_{0s}}{\partial \vec{v}} + \frac{q_s B_0}{m_s c}(\vec{v}\times\hat{z})\cdot\frac{\partial \delta f_s}{\partial \vec{v}}^{\text{"1"}}$

$$= C[f_{0s}, f_{0s}]$$

First, multiply by $(1+\ln f_0)$ and integrate over phase-space:

$$\int d^3r \int d^3v \, \ln f_{0s}\, C[f_{0s}, f_{0s}] = 0.$$

$$\Rightarrow \boxed{f_{0s} = \frac{n_{0s}}{\pi\sqrt{\pi}\, v_{ths}^3}\exp\left(\frac{-v^2}{v_{ths}^2}\right) \; ; \; v_{ths}^2 \equiv \frac{2T_{0s}}{m_s}.}$$

NB: in most derivations of GK, f_0 is not determined but is often assumed to be Maxwellian.

Subbing f_{0s} back in yields: (w/ $\frac{\partial f_s}{\partial \vec{v}} = -\frac{2\vec{v}}{v_{ths}^2} = -\frac{m_s \vec{v}}{T_s}$)

$$v_\perp \cdot \nabla_\perp \delta f_1 + \frac{q_s}{m_s} \vec{v}_\perp \cdot \vec{\nabla}_\perp f_{0s} \frac{m_s \varphi}{T_{0s}} - \Omega_s \frac{\partial \delta f_1}{\partial \vartheta}\Big|_r = 0.$$

$$\left(\vec{v}_\perp \cdot \vec{\nabla}_\perp - \Omega_s \frac{\partial}{\partial \vartheta}\Big|_r \right) \delta f_{1s} = -\frac{q_s}{T_{0s}} (\vec{v}_\perp \cdot \vec{\nabla}_\perp \varphi) f_0$$

Solution: δf_{1s} = particular solution and homogeneous solution

$$\boxed{\delta f_{1s,p} = -\frac{q_s f_0}{T_{0s}} \varphi}$$

$$\Big(\underbrace{\vec{v}_\perp \cdot \vec{\nabla}_\perp - \Omega_s \frac{\partial}{\partial \vartheta}\Big|_r}_{= -\Omega_s \frac{\partial}{\partial \vartheta}\big|_{R_s}} \Big) \delta f_{1s,homo} = 0.$$

where $\vec{R}_s = \vec{r} + \frac{\vec{v} \times \hat{z}}{\Omega_s}$.

$$\Rightarrow \boxed{\delta f_{1s,homo} = h_s(t, \vec{R}_s, v_\parallel, v_\perp)}$$

$$\delta f_{1s} = \underbrace{-\frac{q_s f_0}{T_{0s}} \varphi}_{\text{"Boltzmann response"}} + \underbrace{h_s(t, \vec{R}_s, v_\parallel, v_\perp)}_{\text{"gyrokinetic response"}}$$

"Boltzmann response"

"gyrokinetic response"
response of charged rings to perturbed fields

Note: $f_0 + \delta f_{Boltz} = f_{0s}(v, \epsilon^2 t)\left[1 - \frac{q_s \varphi}{T_{0s}}\right] \simeq f_{0s}(v, \epsilon^2 t)\, e^{-q_s\varphi/T_{0s}}$

$$= \frac{n_{0s}}{\pi^{3/2} v_{ths}^3} \exp\left[-\frac{\mathcal{E}_s}{T_{0s}}\right] \quad \text{w/} \quad \mathcal{E}_s = \frac{1}{2}mv^2 + q_s\varphi$$

Boltzmann response is here because we're not working w/ $\mathcal{E}_s$ as a velocity-space variable — arises from evolution of f_{0s} under influence of perturbed fields.

$$f\left(\tfrac{1}{2}mv^2, t\right) + \delta f_{Boltz} = f_{0s} + q_s\varphi \frac{\partial f_{0s}}{\partial \mathcal{E}_{0s}} \qquad \mathcal{E}_{0s} \equiv \frac{1}{2}mv^2$$

$$= f_{0s} + \Delta\mathcal{E}_s \frac{\partial f_{0s}}{\partial \mathcal{E}_{0s}}$$

$$\simeq f_{0s}(\mathcal{E}_{0s} + \Delta\mathcal{E}_s, t) \approx f_0(\mathcal{E}_s, t)$$

$$\Rightarrow \boxed{f_s = f_0(v, \epsilon^2 t)\exp\left(-\frac{q_s\varphi(r,t)}{T_{0s}}\right) + h_s(\vec{R}_s, t, v, v_\parallel) + \delta f_{2s} + \ldots}$$

(1) $$\cancel{\frac{\partial \delta f_{1s}}{\partial t} + v_\parallel \hat{z}\cdot\nabla\delta f_{1s} + \vec{v}_\perp\cdot\nabla_\perp \delta f_{2s} - \frac{q_s}{m_s}\frac{\partial\varphi}{\partial z}\frac{\partial f_{0s}}{\partial v_z} - \frac{q_s}{m_s}\frac{1}{c}\frac{\partial \vec{A}}{\partial t}\cdot\frac{\partial f_{0s}}{\partial \vec{v}}}$$

$$\cancel{-\frac{q_s}{m_s}\nabla_\perp\varphi\cdot\frac{\partial \delta f_{1s}}{\partial \vec{v}} - \Omega_s \frac{\partial \delta f_{2s}}{\partial \theta}\Big|_r + \frac{q_s}{m_s}\frac{\vec{v}\times\delta\vec{B}}{c}\cdot\frac{\partial \delta f_{1s}}{\partial \vec{v}} = C[f_{0s}, \delta f_{2s}] + C[\delta f_{1s}, f_{0s}]}$$

$$\cancel{= C[f_{0s}, h_s] + C[h_s, f_{0s}]}$$

(1): $$\frac{\partial}{\partial t}\left(h_s(t,\vec{R}_s,v,v_\parallel) - \frac{q_s\varphi}{T_0} f_0\right) + \vec{v}_\perp\cdot\vec{\nabla}_\perp \delta f_{2s} + v_\parallel \frac{\partial}{\partial z}\left[h_s - \frac{q_s\varphi}{T_0} f_0\right]$$

$$+ \frac{q_s}{m_s}\left[+\frac{\partial\varphi}{\partial z}\hat{z} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right]\cdot\left(\frac{+m_s\vec{v}\, f_{0s}}{T_0}\right) - \Omega_s \frac{\partial \delta f_{2s}}{\partial\vartheta}\Big|_{\vec{r}}$$

$$+ \frac{q_s}{m_s}\left[-\vec{\nabla}_\perp\varphi + \frac{\vec{v}\times\delta\vec{B}}{c}\right]\cdot\frac{\partial}{\partial\vec{v}}\left[h_s - \frac{q_s\varphi}{T_0} f_0\right] = C[f_0, h_s] + C[h_s, f_0]$$

$$\left[\frac{\partial}{\partial t} + v_\parallel\frac{\partial}{\partial z} + \frac{q_s}{m_s}\left(-\vec{\nabla}_\perp\varphi + \frac{\vec{v}\times\delta\vec{B}}{c}\right)\cdot\frac{\partial}{\partial\vec{v}}\Big|_{r}\right] h_s$$

$$= C[f_0, h_s] + C[h_s, f_0] + \Omega_s \frac{\partial \delta f_{2s}}{\partial\vartheta}\Big|_{\vec{R}_s}$$

$$+ \frac{q_s f_{0s}}{T_{0s}}\frac{\partial\varphi}{\partial t} + \cancel{v_\parallel \frac{\partial\varphi}{\partial z}\frac{q_s f_0}{T_0}} - \frac{q_s}{T_0} f_0\left[\cancel{v_\parallel\frac{\partial\varphi}{\partial z}} + \frac{\partial\,\vec{v}\cdot\vec{A}}{c\,\partial t}\right]$$

$$+ \frac{q_s\varphi}{T_0}\left(\frac{+\cancel{m_s}}{T_0}\right) f_0 \left(+\frac{q_s}{\cancel{m_s}}\right)\vec{v}_\perp\cdot\vec{\nabla}_\perp\varphi$$

this is just $$\dot{h}_s = \frac{\partial h_s}{\partial t} + \dot{\vec{R}}_s\cdot\frac{\partial h_s}{\partial \vec{R}_s} + \frac{q_s}{m_s}\left(-\vec{\nabla}_\perp\varphi + \frac{\vec{v}\times\delta\vec{B}}{c}\right)\cdot\frac{\partial h_s}{\partial\vec{v}}\Big|_{\vec{R}_s}$$

To eliminate δf_{2s} from this, we ring average over ϑ ~~and~~ at fixed $\vec{R}_s$:
$$\langle a(t,\vec{r},\vec{v})\rangle_{\vec{R}_s} \equiv \frac{1}{2\pi}\oint d\vartheta\, a\left(t, \vec{R}_s - \frac{\vec{v}\times\hat{z}}{\Omega_s}, \vec{v}\right).$$

$$\Rightarrow \quad \frac{\partial h_s}{\partial t} + \langle \dot{\vec{R}}_s \rangle \cdot \frac{\partial h_s}{\partial \vec{R}_s} + \cancel{\frac{q_s}{m_s}\left\langle \left(-\vec{\nabla}_\perp \varphi + \frac{\vec{v}\times\delta\vec{B}}{c}\right)\cdot \frac{\partial h_s}{\partial \vec{v}}\Big|_{R_s} \right\rangle_{\vec{R}_s}} \rightarrow 0 \text{ prove this}$$

$$= \langle C[h_s] \rangle_{\vec{R}_s} + \frac{q_s f_{0s}}{T_{0s}} \frac{\partial}{\partial t} \left\langle \varphi - \frac{\vec{v}\cdot\vec{A}}{c} \right\rangle_{\vec{R}_s} + \cancel{\left\langle \Omega_s \frac{\partial \delta f_s}{\partial \vartheta}\Big|_{R_s} \right\rangle_{R_s}}^{\,0}$$

$$+ \cancel{\frac{q_s^2}{T_0^2} f_{0s} \left\langle \frac{1}{2} \vec{v}_\perp \cdot \vec{\nabla}_\perp \varphi^2 \right\rangle_{\vec{R}_s}}$$

0 (prove this)

Define $\chi \equiv \varphi - \frac{\vec{v}\cdot\vec{A}}{c}$ as the gyrokinetic potential. Then our GK equation is

$$\boxed{\frac{\partial h_s}{\partial t} + \langle \dot{\vec{R}}_s \rangle \cdot \frac{\partial h_s}{\partial \vec{R}_s} = \frac{q_s f_{0s}}{T_{0s}} \frac{\partial \langle \chi \rangle_{R_s}}{\partial t} + \langle C[h_s] \rangle_{R_s}}$$

Note that
$$\left\langle \frac{d\vec{R}_s}{dt} \right\rangle_{\vec{R}_s} = \underbrace{v_\parallel \hat{z}}_{\text{(streaming)}} - \underbrace{\frac{c}{B_0} \langle \vec{\nabla}_\perp \varphi \rangle_{\vec{R}_s} \times \hat{z}}_{\text{((E×B) drift)}} + \underbrace{\frac{v_\parallel}{B_0} \langle \vec{\nabla}_\perp A_\parallel \rangle_{\vec{R}_s} \times \hat{z}}_{\text{(streaming along } \delta\vec{B}_\perp = \vec{\nabla} A_\parallel \times \hat{z})}$$

$$- \underbrace{\frac{1}{B_0} \langle \vec{v}_\perp \delta B_\parallel \rangle_{\vec{R}_s}}_{(\nabla B\text{-drift})} = v_\parallel \hat{z} - \frac{c}{B_0} \frac{\partial \langle \chi \rangle_{R_s}}{\partial \vec{R}_s} \times \hat{z}$$

(used $\langle \vec{v}_\perp \delta B_\parallel \rangle_{\vec{R}_s} = -\langle \vec{\nabla}_\perp (\vec{v}_\perp \cdot \vec{A}_\perp) \rangle_{R_s}$)

Define $\{a, b\} = \hat{z} \cdot \left(\frac{\partial a}{\partial \vec{R}_s} \times \frac{\partial b}{\partial \vec{R}_s} \right)$. Then

$$\boxed{\frac{\partial h_s}{\partial t} + v_\parallel \frac{\partial h_s}{\partial z} + \{ \langle \chi \rangle_{R_s}, h_s \} = \underbrace{\frac{q_s f_{0s}}{T_{0s}} \frac{\partial \langle \chi \rangle_{R_s}}{\partial t}}_{\text{(wave-ring interaction)}} + \langle C[h_s] \rangle_{R_s}}$$

GK Field Equations: (Key is that these are equations at $\vec{r}$, not $\vec{R}_s$!)

quasi-neutrality: $\Sigma_s q_s \int f_s \, d^3v = 0$

$$\Rightarrow \underbrace{\cancel{\Sigma_s q_s \int f_{0s} d^3v}}_{0} + \Sigma_s q_s \left(\frac{-q_s \varphi}{T_{0s}}\right) \int f_{0s} d^3v + \Sigma_s q_s \int h_s\left(t, \vec{r} + \frac{\vec{v} \times \hat{z}}{\Omega_s}, v_\parallel, v_\perp\right) d^3v = 0.$$

$$\Rightarrow \boxed{\Sigma_s \frac{q_s^2 n_{0s} \varphi}{T_{0s}} = \Sigma_s q_s \int d^3v \, \langle h_s \rangle_r}$$

where $\langle \ldots \rangle_r$ denotes gyroaverage

Ampère's law (parallel): $\frac{4\pi}{c} j_\parallel = \boxed{-\nabla_\perp^2 A_\parallel} = \frac{4\pi}{c} \Sigma_s q_s \int d^3v \, v_\parallel f_s$

$$\boxed{= \Sigma_s q_s \frac{4\pi}{c} \int d^3v \, v_\parallel \langle h_s \rangle_r}$$

Ampère's law (perp.): $\cancel{\frac{4\pi}{c} \hat{z} \times \vec{j} = \nabla_\perp \delta B_\parallel = \Sigma_s \frac{4\pi}{c} q_s \int d^3v \, \hat{z} \times \vec{v}_\perp f_s}$

$\cancel{= \Sigma_s \frac{4\pi}{c} q_s \int d^3v \, \langle \hat{z} \times \vec{v}_\perp h_s \rangle_r}$

$$\frac{4\pi}{c} \vec{j}_\perp = \frac{4\pi}{c} \Sigma_s q_s \int d^3v \, \langle \vec{v}_\perp h_s \rangle_r$$

$$\Rightarrow \nabla_\perp^2 \delta B_\parallel = -\frac{4\pi}{c} \hat{z} \cdot \left(\nabla_\perp \times \vec{j}_\perp\right) = -\frac{4\pi}{c} \hat{z} \cdot \left[\nabla_\perp \times \Sigma_s q_s \int d^3v \langle \vec{v}_\perp h_s \rangle_r\right]$$

(int. by parts wrt. $\vartheta \rightarrow \nabla_\perp \nabla_\perp : \left(\overleftrightarrow{\delta p}_\perp + \overleftrightarrow{I} \frac{B_0 \delta B_\parallel}{4\pi}\right) = 0.$

with $\overleftrightarrow{\delta p}_\perp = \Sigma_s \int d^3v \, m_s \langle \vec{v}_\perp \vec{v}_\perp h_s \rangle_r$)

(pressure balance)

Some notes:

(1) Gyro/ring-averaging is best done in Fourier space:

$$\chi(t,\vec{r},\vec{v}) = \Sigma_k \chi(t,\vec{v}) \exp(i\vec{k}\cdot\vec{r})$$

$$= \Sigma_k \chi(t,\vec{v}) \exp\left[i\vec{k}\cdot\left(\vec{R}_s - \frac{\vec{v}\times\hat{z}}{\Omega_s}\right)\right]$$

$$\Rightarrow \langle \chi_k(t,\vec{v})\rangle_{R_s} = \frac{1}{2\pi}\oint d\vartheta \left(\varphi_k - \frac{v_\parallel A_{\parallel k}}{c} - \frac{\vec{v}_\perp\cdot\vec{A}_{\perp k}}{c}\right)\exp\left(-i\vec{k}\cdot\frac{\vec{v}_\perp\times\hat{z}}{\Omega_s}\right)$$

$$= J_0(a_s)\left(\varphi_k - \frac{v_\parallel A_{\parallel k}}{c}\right) + \frac{T_s}{q_s}\frac{2v_\perp^2}{v_{th s}^2}\frac{J_1(a_s)}{a_s}\frac{\delta B_{\parallel k}}{B_0}$$

w/ $a_s \equiv \frac{k_\perp v_\perp}{\Omega_s}$ and $\delta B_{\parallel k} = \hat{z}\cdot(i\vec{k}\times\vec{A}_{\perp k})$

Also, $h_s(t,\vec{R}_s,v_\parallel,v_\perp) = \Sigma_k h_{sk}(t,v_\parallel,v_\perp)\exp(i\vec{k}\cdot\vec{R}_s)$

$$\Rightarrow \langle h_{sk}\rangle_r = \frac{1}{2\pi}\oint d\vartheta\, h_{sk}\exp\left(i\vec{k}\cdot\frac{\vec{v}_\perp\times\hat{z}}{\Omega_s}\right) = J_0(a_s)\, h_{sk}$$

and $$\langle \vec{v}_\perp h_{sk}\rangle_r = -i\vec{k}\times\hat{z}\,\frac{v_\perp^2}{\Omega_s}\frac{J_1(a_s)}{a_s}\, h_{sk}$$

$$\langle \vec{v}_\perp\vec{v}_\perp h_{sk}\rangle_r = v_\perp^2\left[\frac{\vec{k}_\perp\vec{k}_\perp}{k_\perp^2}\frac{J_1(a_s)}{a_s} + \frac{(\vec{k}\times\hat{z})(\vec{k}\times\hat{z})}{k_\perp^2}\frac{dJ_1(a_s)}{da_s}\right] h_{sk}.$$

(2) h_{sk} is NOT! the best quantity to work with for numerical work. Physically, this is because Alfvénic fluctuations have a gyrokinetic response that is largely cancelled at long wavelengths by the Boltzmann response. Let's see that:

$$\vec{u}_\perp = -\frac{\omega}{k_\parallel}\frac{\delta\vec{B}_\perp}{B_0} \quad \text{(Alfvén waves)}$$

$$\frac{c}{B}\hat{z}\times\vec{\nabla}_\perp\varphi(\vec{r}) = \frac{-\omega}{k_\parallel V_A}\left[-\frac{V_A}{B_0}\hat{z}\times\vec{\nabla}_\perp A_\parallel(\vec{r})\right]$$

$$\rightarrow \varphi = \frac{\omega A_{\parallel k}}{k_\parallel c}$$

Can show that linear h_{sk} for this situation is

$$h_s = \frac{q_s f_0}{T_s}\langle\varphi\rangle_{R_s} = -\langle\delta f_{Boltz}\rangle$$

So that $\delta f_{1s} = h_s + \delta f_B = \delta f_B - \langle\delta f_B\rangle$,

which has ~~zero~~ zero long-wavelength limit.

Mathematically, the problem is that Alfvén waves don't change the form of the distribution function, but rather define the moving frame in which any changes to it are to be measured:

$$\delta f_B - \langle \delta f_B \rangle = \frac{2\vec{v}_\perp \cdot \vec{u}_\perp}{v_{ths}^2} f_s$$

$$\Rightarrow f_s\left(\frac{1}{2}m_s v^2, \frac{m_s v_\perp^2}{2B_0}\right) \to f_s\left(\frac{1}{2}m_s v_\parallel^2 + \frac{1}{2}m_s \underbrace{|\vec{v}_\perp - \vec{u}_\perp|^2}_{w_\perp^2}, \frac{m_s \overbrace{|\vec{v}_\perp - \vec{u}_\perp|^2}^{w_\perp^2}}{2B_0}\right)$$

Physically, this is b/c particles in a magnetized plasma adjust on a cyclotron timescale to take on the $\vec{E}\times\vec{B}$ velocity. (This is what underlies Kulsrud's formulation of KMHD — compressive component is passively advected by Alfvénic fluctuations in the inertial range.)

Instead,
Usually work with

$$g_s = h_s - \frac{q_s}{T_{0s}} \langle \chi \rangle_{Rs} f_{0s}$$

$$\left(= \langle \delta f_{1s} \rangle_{Rs} + \frac{q_s}{T_{0s}} \left\langle \frac{v_\perp \cdot A_\perp}{c} \right\rangle_{Rs} f_{0s} \right)$$

$$= h_s + \langle \delta f_{Boltz} \rangle + \frac{q_s}{T_{0s}} \left\langle \frac{v_\perp \cdot A_\perp}{c} \right\rangle_{Rs} f_{0s}$$

$$\left(\text{or} \quad \widetilde{\delta f}_s = h_s + \langle \delta f_{Boltz} \rangle - \frac{q_s}{B_0} \left\langle \frac{v_\perp \cdot A_\perp}{c} \right\rangle_{Rs} \frac{\partial f_{0s}}{\partial \mu_{0s}} \quad \text{if } f_{0s} \text{ is anisotropic} \right)$$

Amounts to shifting from $f_{0s}(\frac{1}{2}mv^2)$ to $f_{0s}(\underbrace{\frac{1}{2}mv^2 + q_s\varphi - q_s\langle\varphi\rangle_{Rs}}_{= \mathcal{E}_s})$

Many standard treatments of GK use g_s instead of h_s:

(a) in electrostatic limit, $g_s = \langle \delta f_{1s} \rangle_{R_s}$ aids in interpretation of polarization effects (Krommes 2012), places GK eqn. in a numerically convenient characteristic form (Lee 1983), and arises naturally from Hamiltonian formulation of GK (Dubin et al. 1983; Brizard & Hahm 2007).

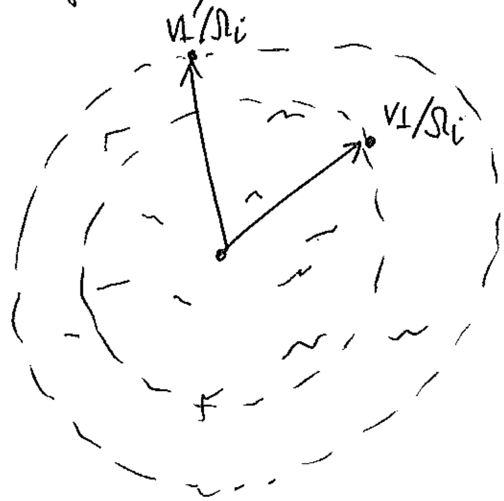
(b) h_s: physically extended rings of charge moving through vacuum

g_s: gas of point-particle-like gyrocenters moving in a polarizable medium (Krommes 93, 12; Abel et al. 13)

"gyrocenter distribution function"

(3) See Kunz et al. 2015 for extension to non-Maxwellian plasmas, Abel et al. 2013 for extension to transport timescales in tokamak geometry.

(4) Entropy cascade due to nonlinear phase mixing:



Two particles with different $v_\perp$ but same gyrocenter, different ring average. If $\frac{k_\perp v_\perp}{\Omega}$ and $\frac{k_\perp v_\perp'}{\Omega}$ differ by order unity, i.e.

$$\frac{\delta v_\perp}{v_{thi}} = \frac{|v_\perp - v_\perp'|}{v_{thi}} \sim \frac{1}{k_\perp \rho_i},$$

ring-averaged fluctuations will come from spatially uncorrelated ~~eddies~~ fluctuations.

Analyzed in Schekochihin et al. 2009, but appeared in earlier gyrofluid models by Hammett, Dorland, et al.