Einstein eqns & (magneto) hydrodynamics

Lecture #1

-) Introduction & warnings
-) "The importance of being earnest" in Numerical Relativity
-) Einstein equations & basics of differential geometry
  - 3+1 formulation: why & how
  - Generalized Harmonic formulation
  - (a bit of) BSSN formulation.
-) Dealing with constraints (in GR and elsewhere!)

Introduction & warnings

- We are interested in this course on "Computational plasma astrophys
The school has many lectures on different but related topics.
Each will necessarily be incomplete (in a sense) but you'd get an overall exposure to relevant topics and be able to combine them to your primary goals.

- In this set, our goal is to consider scenarios where matter and electromagnetic fields are "beat into submission" by gravity itself. Furthermore, we will consider (or have as target systems) dynamical & strong field regimes.

- Thus, we need to address the impact of a (rapidly) changing strong gravitational field. Since such scenarios typically have characteristic speeds $(v/c) \sim 1$; and fields $(M/r) \sim 1$ $\Rightarrow$ a priori no perturbative treatment of "Φ" can be adopted & we must immerse ourselves into General Relativity.

* Before doing so, we must review some concepts which proved crucial in this task

"Theorems are permanent; tricks are ephemeral"

(Hadamard) Any <u>physical</u> system of equations <u>must</u> give rise to <u>well posed problems</u>

(i) Existence of solutions
(ii) Solution is unique when initial & boundary conditions are defined
(iii) Solution depends continuously on such data: $\|u\| \leq \alpha e^{\beta t} \|u_0\|$

<u>E.g.</u> $u_{,t} = u_{,x}$. Use Fourier representation: $u(t,x) = f(t)\, e^{ikx}$

$$\to \dot{f} = f\, ik \Rightarrow u(t,x) = u_0\, e^{ik(x+t)}$$

(WP)

$$\Rightarrow \|u\|_{t=T} = \|u_0\|$$

$$u_{,t} = u_{,x} + u \to u(t,x) = u_0\, e^{t}\, e^{ik(x+t)}$$

$$\hookrightarrow \|u(T)\| = \|u_0\|\, e^{t}$$

(WP)

$$u_{,t} = -iu \to u(t) = e^{kt}\, e^{ikx}\, u_0 \Rightarrow \|u(T)\| = e^{kt}\|u_0\|$$

(IP)

arbitrary growth!

Sufficient conditions for well posedness of <u>hyperbolic problems</u>

$$\vec{u}_{,t} - \sum_i{}' A^i\, \partial_i \vec{u} + (\text{Rest})$$

$\to$ if A^i : diagonalizable with <u>real</u> eigenvalues!
[strongly, symmetric hyperbolic]

then <u>WP</u> follows in spite of "Rest"

<u>Ref.</u> "Time-dependent problems & difference methods"
Gustafsson - Kreiss - Oliger. Wiley, 2013

Example

$$u_{,t} = \begin{pmatrix} u \\ v \end{pmatrix}_{,t} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_{,x}$$

linear growth.

(A)

$$\Rightarrow \quad u(t,x) = D(F_1(t+x)) \cdot t + F_2(t+x)$$
$$v(t,x) = F_1(t+x)$$

$+ \begin{pmatrix} u \\ v \end{pmatrix}$

$$\rightarrow \quad u(t,x) = [D(F_1) + F_1]\, e^t \cdot t + F_2\, e^t$$
$$v(t,x) = F_1\, e^t$$

(B)

$+ \begin{pmatrix} \cancel{u+v} \\ u+\cancel{v} \end{pmatrix}$

$$\rightarrow \quad v(t,x) = c_1\, e^{c_1(t+x)} \cdot c_2\, e^{t\sqrt{c_1}} + \dots$$
$$u(t,x) = \frac{\sqrt{c_1}\, e^{c_1(t+x)}}{e^{t\sqrt{c_1}}} \dots\dots$$

(C)

Case (C) will have a sdn where the bound growth depends on the initial data !

⇒ ill posed.

WARNING: Even this simple problem shows how bad things can get.

→ In spite of real eigenvalues
→ without worrying about boundaries
→ Just 1 D ; no constraints ;
→ simple problem, simple lower order terms!

Caveat: This is not to say that.

- strong/symmetric hyperbolicity is necessary
- that specific choices can't be adapted to ensure a well posed problems can indeed be obtained even with a "weakly hyperbolic" system [e.g. ideal MHD + CT !!]

However, it might be too much to fight with, especially if underlying system is too complicated to see a way through. So we will as much as possible work analytically first and then at the algorithm level.

Einstein equations

$$G_{ab} = 8\pi\, T_{ab}$$

→ Motivation:
- gravity is one of the fundamental forces of nature,
- field equations are quite complicated & non-linear. Analytic solutions only known in a few special cases.
- perturbative techniques work in weak-field, slow-motion scenarios otherwise numerical solns are required.
- GWs have been detected; sources likely to give strong radn involve bhs & neutron stars
- NSs are magnetized and magnetic effects can play an important role on both dynamics & EM counterparts!

Some (very brief) words about GR and its structure!

- in GR. no gravitational force → we live on a 4-dimensional curved spacetime. It's the curvature of this spacetime what we feel as the gravitational force ($\Rightarrow$ must revise/revisit equation ingredients & interpretation.)

Curved spacetime → describe the geometry by a metric tensor g_{ab}, defined via the line element as:

$$ds^2 = g_{ab}\, dx^a dx^b$$

(Notation! repeated indices sum)

$\Rightarrow$ derivatives must be mindful of change of quantities and change due to curvature.

→ Introduce ∇_a s.that $\nabla_a g_{cd} = 0$.

action on vector V^a ⇒ $\nabla_a V^b = \partial_a V^b + \Gamma^b_{ac} V^c$

covector v_a $\quad \nabla_a v_b = \partial_a v_b - \Gamma^c_{ab} v_c$

function Φ $\quad \nabla_a \Phi = \partial_a \Phi$

(any other higher ranked tensor can be deduced from the above)

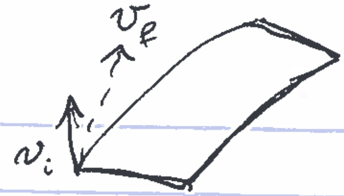
$$\Gamma^a_{bc} = \frac{1}{2} g^{ae}\left(g_{be,c} + g_{ce,b} - g_{bc,e}\right)$$ Christoffel symbol

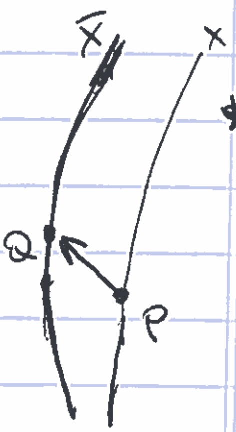
Curvature tensor

$$R^a{}_{bcd} = \partial_c \Gamma^a_{bc} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd} \Gamma^a_{ec} - \Gamma^e_{bc} \Gamma^a_{ed}$$

2 words about the tensor

$$\nabla_b \nabla_c V_a - \nabla_c \nabla_b V_a = R^d{}_{abc} V_d$$





$$\frac{d^2 W^a}{d\lambda^2} + R^a{}_{cbd} \frac{dx^c}{d\lambda} \frac{dx^d}{d\lambda} W^b = 0$$ geodesic deviation.

$\Rightarrow R^a{}_{bcd}$ encodes curvature affects.

<u>Einstein's tensor:</u>

$$G_{ab} = R_{ab} - \frac{1}{2} R\, g_{ab}$$

$R = g^{ab} R_{ab}$

$R_{ab} = R^c{}_{acb}$

We can write:

$$R_{ab} = \Gamma^d{}_{ab,d} - \Gamma^d{}_{db,a} + \Gamma^e{}_{ab} \Gamma^d{}_{ed} - \Gamma^e{}_{db} \Gamma^d{}_{ea}$$

<u>Einstein equations</u>

$$G_{ab} = \frac{8\pi G}{c^4} T_{ab}$$

<u>Note</u>: i) $\nabla^a G_{ab} = 0$ (Bianchi identity $\Rightarrow$ required for $\nabla^a T_{ab} = 0$!)

ii) Often in GR: $G = 1$; $c = 1$; (and 8π absorbed!)

iii) 10 equations, second order, for 10 components of g_{ab}

iv) No definite math character. Covariant theory!

Consequences

⇒ Can arbitrary change g_{ab} via coord. transformations and all representations satisfies the same equations.

⇒ in a sense ill-posed unless we fix coords. (non unig.

Now, we will discuss two decomposition of field eqns (but there are infinitely many!)

* typically we get a couple of 4 elliptic (constraint) equations, 6 hyperbolic (evolution) equations, plus 4 freely specifiable gauge "equations".

* 4 gauge conditions ⇒ 6 "unknowns" tied by 4 constraints ⇒ 2 d.o.f

* Distinguish in between: free evolution, constrained evolution (or in between "partially" constrained)

Further observations

* Either option, one can show (via identities) solve using a particular subset + ID + identities ⇒ solution of full system.

* Numerically though; this need not be the case.

⇒ constraints might grow without bounds

⇒ non linearities, in general, get in the way of developing strategies like Constraint Transport in Maxwell's & ideal MHD.

Now, first a detour. Recall our discussion of principal part

e.g., we only need to care of highest derivatives to analyze the equations. For dynamical gravity with "standard" sources.

$$\underbrace{G_{ab} = 8\pi T_{ab}}_{\{2^{nd}\text{ deriv of } g_{ab},\ \text{none of } T_{ab}.\}} \quad \& \quad \underbrace{\nabla_a T^{ab} = 0}_{\{1^{st}\text{ deriv of } T^{ab},\ 1^{st}\text{ deriv of } g_{ab}\}}$$

$\Rightarrow$ Principal part decouples matter & gravity $\Rightarrow$ we can analyze them independently!

Also: $R_{ab} - \frac{1}{2} g_{ab} R = T_{ab} \Rightarrow g^{ab}(R_{ab} - \frac{1}{2} g_{ab} R) = g^{ab} T_{ab}$

$$\Rightarrow R - 2R = T$$
$$-R = T$$

$$\Rightarrow R_{ab} - T_{ab} + \frac{1}{2} g_{ab} R = T_{ab} - \frac{1}{2} g_{ab} T$$

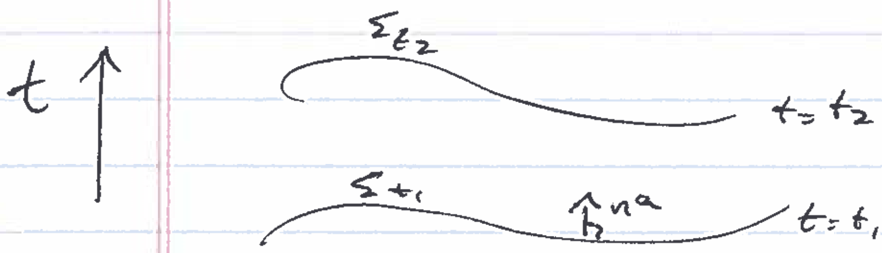
$$\Rightarrow \boxed{R_{ab} = T_{ab} - \frac{1}{2} g_{ab} T}$$ "trace reversed EE".

Suffices to understand what to do with R_{ab}.

$\Rightarrow$ {ADM & Harmonic formulation analysis next.

Setup for dynamics

1) Introduce a foliation. Fix for this course a "spacelike" foliation



- Use signature s.t. $n^a n_a = -1$ i.e. flat metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

if each Σ has $t =$ const $\Rightarrow \nabla_a t$ is $\perp$ to Σ.

$\Rightarrow$ $n_a \propto \partial_a t$ choose $n_a = -\alpha \, \partial_a t$ $\underline{\underline{\alpha > 0}}$ $\to$ lapse function

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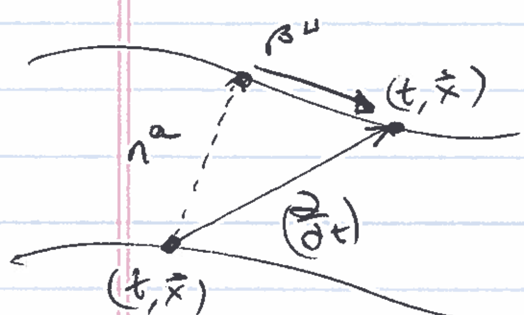

Proper time interval $= \alpha \, dt$

$$\left(\frac{\partial}{\partial t}\right)^a = \alpha n^a + \beta^a$$

$a : 0 \dots 3$
$i, j : 1 \dots 3$,

In terms of this decomposition: $ds^2 = g_{ab} dx^a dx^b$

$$= -\alpha^2 dt^2 + h_{ij} (dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

$\to$ Note: we naturally have a new tensor appearing.

$$\boxed{h_{ab} = g_{ab} + n_a n_b}$$ $\to$ intrinsic metric to Σ_t surfaces.

- h_{ab} also has other properties. (Recall $g_a{}^b = \delta_a{}^b$)

Note:

i) $$h_a{}^b h_b{}^d = (g_a{}^b + n_a n^b)(g_b{}^d + n_b n^d)$$
$$= \delta_a{}^d + n_a n^d + \cancel{n_a n^d} - \cancel{n_a n^d}$$
$$= \delta_a{}^d + n_a n^d$$
$$= h_a{}^d \quad \Rightarrow \boxed{h.h = h}$$

ii) $$h_{ab} n^b = (g_{ab} + n_a n_b)\, n^b$$
$$= n_a - n_a = 0 \quad \Rightarrow h.n = 0$$

iii) Take S^a s.that $S^a n_a = 0$ (i.e. they are orthogonal)

$$\Rightarrow h_{ab} S^b = (g_{ab} + n_a n_b) S^b - g_{ab} S^b + n_a n_b S^b$$
$$= S_a$$
$$\Rightarrow \quad h.S = S \text{ if } S \perp n$$

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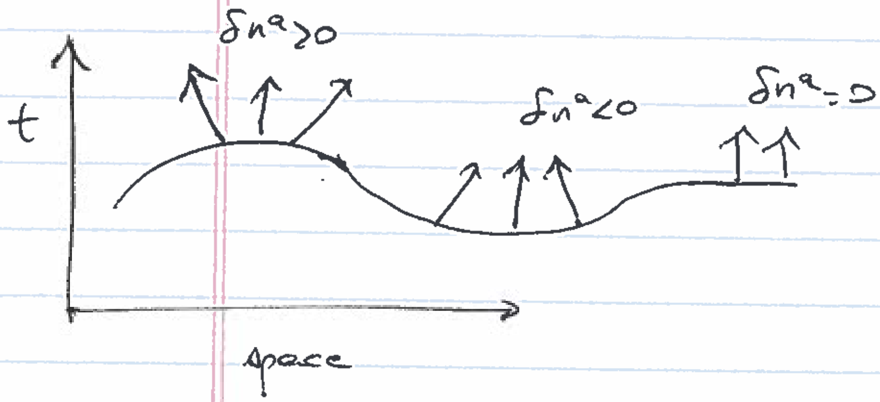
i) + ii) + iii) $\Rightarrow$ h_{ab} is a projection tensor (denoted with $\perp$)

i.e. given a general vector (tensor ...) V_a

$\perp V_i = h_i{}^a V_a$ is its projection onto Σ_t

Why bother? : We are after re-expressing EEs in terms of natural geometric quantities but in a way where we recognize a set of evolution & constraint equations.

So far, we have identified h_{ab} : metric of 3-dimensional surfaces. But these "live" in a higher dimensional spacetime we need a new concept to account for the way Σ_t's are embedded in it.



$$K_{ab} \equiv -\perp \nabla_a n_b$$

[sign chosen so that concave surfaces give positive curvature]

→ Note: by definition K_{ab} is purely spatial

and it can be shown $K_{ab} \equiv$ change of (h_{ab}) "advected" along the normal direction.

it can be expressed as:

$$\boxed{\partial_t h_{ab} = -2\alpha K_{ab} + \beta^e \partial_e h_{ab} + h_{ae}\partial_b \beta^e + h_{be}\partial_a \beta^e}$$

So we have:

h_{ab} : intrinsic metric to Σ_t
K_{ab} : extrinsic curvature of Σ_t
α, β^i : 4 coordinate functions

The derivation of EEs in terms of these quantities is straightforward but lengthy. For references check
MTW (Gravitation); York's 1978 "Sources of Gravitational Radn" article
or e.g. "Introduction to 3+1 Num Rel" Alcubierre

Also, for historical reasons (most often found in derivations) we work with $G_{ab} = 8\pi T_{ab}$.

First define:

$$\rho \equiv T_{ab} n^a n^b$$

$$J^i \equiv -\perp (T^{ib} n_b) \longrightarrow [\perp(T^i) = h^i{}_c T^c]$$

$$S^{ij} \equiv \perp (T^{ij}) \longrightarrow [\perp(T^{ij}) = h^i{}_a h^j{}_b T^{ab}]$$

Now, project EEs wrt:

$n^a n^b \rightarrow$ $$^{(3)}R + K^2 - K_{ij} K^{ij} = 16\pi \rho$$ [$^{(3)}$ quantities related to h_{ab}!]

$n^a h^a{}_b \rightarrow$ $$D_b K^{ab} - D^a K = 8\pi J^a$$ [$D_b h_{ac} = 0$]

$h_a{}^c h_b{}^d \rightarrow$ $$\partial_t K_{ab} = \beta^e \partial_e K_{ab} + K_{ae} \partial_b \beta^e + K_{be} \partial_a \beta^e$$

$$- D_a D_b \alpha$$

$$+ \alpha \left[{}^{(3)}R_{ab} + K K_{ab} - 2 K_{ad} K^d{}_b - 8\pi \left(S_{ab} - \frac{h_{ab}}{2}(S - \rho) \right) \right]$$

and

$$\partial_t h_{ab} = \beta^e \partial_e h_{ab} + h_{ae} \partial_b \beta^e + h_{be} \partial_a \beta^e$$

$$- 2\alpha K_{ab}.$$

What do we have then?

- "Geometrodynamics" : rate of change of geometrical quantities
- Bianchi identities ($\nabla_a G^{ab} = 0$) $\Rightarrow$ ID satisfying constraints initially will continue to do so if evolved with $\{\dot{K}, \dot{g}\}$ eqs.

- system is hyperbolic.

What don't we have?

- Strongly/symmetric hyperbolic system! It's only weakly hyperbolic [Kreiss + Ortiz arxiv: gr-qc/0106085].

- Constraints are "propagated" but not damped [we'll return to this]

- We still need to specify $\{\alpha, \beta^i\}$

∴ e.g. what's wrong with $\alpha = 1$, $\beta^i = 0$?

However: In special cases. This system is fine! so, take a look at what you want & then proceed.

(Generalized) Harmonic system of eqns.:

- Stays "formally" less geometric ("3+1" ish, than ADM)

- We start from $R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2} g_{ab} T\right)$

$$\Rightarrow g^{cd} g_{ab,cd} + 2 g^{cd}{}_{,(a} g_{b)d,c} + 2\Gamma^c_{db}\Gamma^d_{ca} + 8\pi\left(2T_{ab} - g_{ab}T\right) = 0$$
$$+ 2\nabla_{(a}\Gamma_{b)}$$

Principal part of the system? $\Rightarrow g^{cd} g_{ab,cd} + 2\nabla_{(a}\Gamma_{b)}$

"Almost" nicely symmetric hyperbolic! if it weren't for $\nabla_{(a}\Gamma_{b)}$ ....

Harmonic case: choose coordinates s.t. $\nabla_a\nabla^a x^b = 0 \Rightarrow -\Gamma^b = 0$!!

Generalized Harm. Case: " " s.th. $\nabla_a\nabla^a x^b = H^b \Rightarrow -\Gamma^b = H^b$
and replace back

$\Rightarrow$ Principal part is $g^{cd} g_{ab,cd} + \{\partial H\}$

if H^b obeys a wave-equation then $\partial H \Rightarrow$ lower order

We got a symmetric hyperbolic system of equations!

at the expense of using (essentially) all our coord freedom.

Sounds too good to be true right?

- Yes! and this was recognized long ago! Worries where
  - Coords tied to wave eqns, they'd "fly away"
  - Mathematical worries about eqns of the type $\Box \phi_I \sim \sum A \partial \phi_J \partial \phi_J$ admitting solutions with smooth arbitrarily small ID that blow up in finite time. BUT ⇒ [Null Condition: Klainerman '86 also, Lindblad-Rodnianski '2003]

* It's even better actually! [Modulo coord conds]

Define: $C^a \equiv H^a + \nabla^b \nabla_b x^a$ $\qquad$ $C^a = 0$ are the new constraints! as we are evolving all components.

$\Rightarrow H^a = C^a - \Gamma^a$

Qu: How do C^a behave upon evolution?

Consider, for simplicity, $T_{ab} = 0$. $R_{ab} - \nabla_{(a} C_{b)} = R_{ab} - \nabla_{(a} H_{b)} + \nabla_{(a} \Gamma_{b)}$

$$= g^{cd} g_{ab,cd} + \cancel{\nabla_{(a} \Gamma_{b)}} - \nabla_{(a} H_{b)} - \cancel{\nabla_{(a} \Gamma_{b)}}$$

⇒ our system is then "as if we were adding the constraints, which if satisfied, is like we added nothing!"

Let's see this is true

$$\nabla^a G_{ab} = 0 \Rightarrow \nabla^a R_{ab} - \frac{1}{2} \nabla_b R = 0 \quad *$$

but $R_{ab} = \nabla_{(a} C_{b)} \Rightarrow \nabla^a R_{ab} = \nabla^a \nabla_{(a} C_{b)}$

$R = \nabla_a C^a \Rightarrow \nabla^b R = \nabla^b (\nabla_a C^a)$

$* \Rightarrow$ ~~$\nabla^a \ldots$~~

thus:

$$\nabla^a R_{ab} - \frac{1}{2}\nabla_b R = \nabla^a \nabla_{(a} C_{b)} - \frac{1}{2}\nabla_b \nabla_a C^a$$

$$= \frac{1}{2}\nabla^a \nabla_a C_b + \frac{1}{2}\nabla^a \nabla_b C_a - \frac{1}{2}\nabla_b \nabla_a C^a$$

$$= \frac{1}{2}\nabla^a \nabla_a C_b + \frac{1}{2}\left[\nabla_a \nabla_b - \nabla_b \nabla_a\right] C^a$$

$$\boxed{0 = \nabla^a \nabla_a C_b + R_{bc} C^c}$$

which can be written as $\boxed{0 = \nabla^a \nabla_a C_b + C^a \nabla_{(b} C_{a)}}$

This implies that if $\{C_a, \partial_t C_a\} = 0$ initially C^a will stay so!

numerically though this need not be the case [we'll see examples of this!]

Consider instead: $0 = R_{ab} - \nabla_{(a} C_{b)} + \gamma_0 \left[t_{(a} C_{b)} - \frac{1}{2} g_{ab} t^c C_c \right]$

with γ_0 a parameter and t^a a time like vector field pointing towards the future, Proceeding as before:

$$0 = \nabla^a \nabla_a C_b + R_{ab} C^a - 2\gamma_0 \nabla^a (t_{(b} C_{a)})$$

or

$$0 = \nabla^a \nabla_a C_b - 2\gamma_0 \nabla^a [t_{(a} C_{b)}] + C^a \nabla_{(b} C_{a)} - \frac{1}{2}\gamma_0 t_b C^a C_a$$

Now, imagine $C_a \neq 0$ but small (eg $C^2 \sim 0$)

$$\Rightarrow \quad 0 \simeq \nabla^a \nabla_a C_b - 2\gamma_0 \nabla^a [t_{(a} C_{b)}]$$

$\Rightarrow$ decaying wave equation along t^a!

So far this is quite promising; but what happens with the coordinates? and the I.D?

Coordinates?, Freedom is now encoded in H^a, notice that

$$H_a n^a = -n^a \partial_a \ln\alpha - K$$

$$\perp H^i = H_a h^{ai} = \frac{1}{\alpha} n^a \partial_a \beta^i + h^{iJ} \partial_J \ln\alpha - {}^{(3)}\Gamma^i_{Jk} h^{Jk}$$

thus,

$$\partial_t \alpha = -\alpha^2 H_a n^a + \ldots$$

$$\partial_t \beta^i = \alpha^2 \perp H^i + \ldots$$

thus H^a implicitly determine the evolution of our $\{\alpha, \beta^i\}$

to not spoil well-posedness, can use $\Box H^a = \ldots$!

[Pretorius '05]

(though there is an interesting story here!)

$H^a = 0$ also works! but with better resolution!

Initial data?

- in ADM, solve Hamiltonian & Momentum constraint.

- in GH? ⇒ Evolving g_{ab} in full! → We need to provide I.D. such that $g_{ij}\{g_{ab}; \partial_t g_{ab}\}$ is consistent with $\{C_a=0, \partial_t C_a=0\}$

Procedure: ① Solve ADM constraints [Useful step as there is much known/done here] $\to \{h_{ij}, \partial_t h_{ij}\}$ if α, β^i given

② Choose H^a at $t=0$. from $C^a=0$ $\alpha_{,t}$ & $\beta^i_{,t}$ are obtained

① + ② $\Rightarrow \{ g_{ab}, \partial_t g_{ab} \}$ are known.

③ Identity $R_{ab} = \nabla_{(a} C_{b)}$ and $M_a \equiv (R_{ab} - \frac{1}{2} g_{ab} R) n^b$

$$\Rightarrow n^b \nabla_b C_a = 2 M_a + (g^{bc} n_a - n^c g^b{}_a) \nabla_b C_c$$

but $C_a = 0$ & $\partial_i C_a = 0$ & $M_a = 0$ because of $H=0$ Mom Const.

$$\Rightarrow \partial_t C_a = 0.$$

Last, a few words w.r.t BSSN.

- widely used formulation of GR.
- takes from both ADM & GH.

Ingredients: * Introduce conformal decomposition. $\tilde{h}_{ij} = \varphi^{-4} h_{ij}$

* Introduce a trace-free representation $A_{ij} = K_{ij} - \frac{1}{3} h_{ij} K$

and rescale conformally

$$\tilde{A}_{ij} = \varphi^{-4} A_{ij}$$

* Introduce the conformal $\tilde{\Gamma}^i_{jk}$

and the variables $\tilde{\Gamma}^i \equiv \tilde{h}^{jk} \tilde{\Gamma}^i_{jk}$

$\to$ for the same reason we introduced H^a!

Equations (replace directly / obtain from ADM eqns)

$$\frac{d}{dt}\tilde{h}_{ij} = -2\alpha\tilde{A}_{ij}$$

$$\frac{d}{dt}\phi = -\frac{1}{6}\alpha K \qquad (\phi \equiv \ln\varphi)$$

$$\frac{d}{dt}\tilde{A}_{ij} = e^{-4\phi}\left\{-D_i D_j\alpha + \alpha R_{ij} + 4\pi\alpha\left[h_{ij}(S-\rho) - 2S_{ij}\right]\right\}^{TF} + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k{}_j\right)$$

$$\frac{d}{dt}K = D_i D^i\alpha + \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) + 4\pi\alpha(\rho + S)$$

with
$$\frac{d}{dt}F_{ij} = \frac{\partial}{\partial t}F_{ij} + \beta^\ell\partial_\ell F_{ij} + F_{i\ell}\partial_j\beta^\ell + F_{j\ell}\partial_i\beta^\ell \cdot \#$$
$$\frac{d}{dt}S = \frac{\partial}{\partial t}S + \beta^\ell\partial_\ell S \cdot \#$$

with
$\# = \frac{1}{6}$ for ϕ
$\# = -\frac{2}{3}$ for $\tilde{h}_{ij}$ and $\tilde{A}_{ij}$
$\# = 1$ for K.

Notice: $R_{ij} = h^{\ell m}\partial_\ell\partial_m h_{ij} + D_{(i}\Gamma_{j)} + \dots$

so we have the same issue as in GH. To fix this the variables $\tilde{\Gamma}^i$ are introduced.
$$\rightarrow \frac{d}{dt}\tilde{\Gamma}^i = \dots \quad \partial_j\tilde{A}^{ij}$$

Also, replace $\partial_j\tilde{A}^{ij}$ by employing the momentum constraint which replaces

$$\partial_j\tilde{A}^{ij} = \left\{\Gamma.A\ ,\ \tilde{A}\,\partial_j\phi\ ,\ \tilde{h}.\partial_j K\right\},$$

[see e.g. Alcubierre's book for details]

What else is "new"?

① Gravity does not develop shocks! [linearly degenerate systems]

unless they are put by hand or a coordinate effect

② ~~At~~ worst it gives a singularity! what to do?

Ⓐ Excise the black hole: Find apparent horizon and cut region inside of it.

(better be the case that all fluxes is ingoing there!)

Ⓑ Choose to go beyond horizon and "slow" things down there by suitable gauge condition.

③ Do you write your own code? Arguably things are less mature than in the Astro community. However, there are codes and publicly available infrastructure you can get your hands onto.

- CACTUS
- PAMR
- HAD
- GRCHOMBO
- BAM (?)

- Now, let's worry about the sources. For simplicity, let's imagine just Maxwell's equations and since we argued we can analyze things separately, let's start on flat space-time.

- Maxwell's equations can be written in covariant form as

$$\nabla_b F^{ab} = I^a \quad ; \quad \nabla_b {}^*F^{ab} = 0$$

where ${}^*F^{ab} = \frac{1}{2} \epsilon^{abcd} F_{cd}$

we can define the Electric & Magnetic field w.r.t. the observer with tangent vector n^a as:

$$E^a = F^{ab} n_b \quad ; \quad B^b = {}^*F^{ab} n_a$$

we can reexpress the Faraday tensor as:

$$F^{ab} = E^b n^a - E^a n^b - \epsilon^{abcd} B_c n_d$$

For a moment consider instead

Komisarov MNRAS '07

Palenzuela, LL, Reula + MNRAS 0?

$$\nabla_b (F^{ab} + g^{ab} \psi) = I^a - \kappa \psi n^a$$

$$\nabla_b ({}^*F^{ab} + g^{ab} \phi) = - \kappa \phi n^a$$

$\Rightarrow$ take ∇_a

$$\Rightarrow \quad \nabla^a \nabla_a \psi = - \kappa \nabla_a (n^a \psi)$$

$$\nabla^a \nabla_a \phi = - \kappa \nabla_a (n^a \phi)$$

For simplicity, take $n^a = \frac{\partial}{\partial t}^a$; $(\beta^a = 0,\ \alpha = 1)$

we get

$$\left.\begin{aligned} \partial_t \psi + \vec{\nabla}\cdot\vec{E} &= q - \kappa\psi \\ \partial_t \phi + \vec{\nabla}\cdot\vec{B} &= -\kappa\phi \\ \partial_t \vec{E} - \vec{\nabla}\wedge\vec{B} + \vec{\nabla}\psi &= -J \\ \partial_t \vec{B} + \vec{\nabla}\wedge\vec{E} + \vec{\nabla}\phi &= 0 \end{aligned}\right]$$

$\Rightarrow$ Hyperbolic divergence cleaning related. [Dedner et al J Comp Phys V175, 2002]

Consider $\partial_t \psi + A = -\lambda\psi$

$$\Rightarrow \partial_t \psi = -(A + \lambda\psi) \xrightarrow{\text{soln}} A + \lambda\psi = C_0 e^{-\lambda t}$$

$$\Rightarrow A + \lambda\psi \to 0$$

$$t \to \text{"}\infty\text{"} \quad \Rightarrow \quad \psi \to -\frac{A}{\lambda}$$

if λ is large $\psi \to 0$

* Constraints are damped through coupling with damping fields
* Related to constraint damping in GR but addition of fields is needed as we've used up all our freedom.

* We *have not* used or required any particular algorithm, just modified the equations off the constraint surface to achieve a desired effect.
* We did introduce a new scale in the problem though!

Ⓐ On Friday you'll get a code implementing the system above.