Some notes following qus/discussions after lec#1

- "Global" vs "local" solutions: Strong/symmetric/strict hyperbolicity $\Rightarrow$ local existence

global soln is an open-problem (e.g. Clay Prize for Navier Stokes in 3D)

- Constraints, what does a violation mean? E.g. $\vec{\nabla}\vec{B} \neq 0 \to$ monopoles

in GR we have { Hamiltonian $\to$ "energy" ; Momentum $\to$ "flux of energy" } $\to$ theory need not be well-behaved e.g. negative energy?

- Example of Harmonic formulation without constraint transport; it is well posed. Take resolution $\to 0$ the right soln will be obtained. But for practical resolns, the behavior is bad enough.

Coordinate conditions for Harmonic case

Pretorius '05: $\nabla^a \nabla_a H_t = -\xi_1 \frac{\alpha - 1}{\alpha^n} + \xi_2 n^a \partial_a H_t$

if not BBH problem not working.

'07 $\nabla^a \nabla_a x^\mu = 0 \to$ "plain" Harmonic just fine

difference is just resolution! ~ 4x better refined and $\Gamma_a = 0$ ok

Coordinates in general:

- Through GH formulation, from $-\Gamma_a = H_a$, expand and obtain

$$\partial_t \alpha = -\alpha^2 H_a n^a + \dots$$
$$\partial_t \beta^i = \alpha^2 \perp H^i + \dots$$

Thus, one can relate choice of H_a with time derivatives of $\{\alpha, \beta^i\}$

- Also if one has an explicit solution (e.g. Kerr or Ingoing Eddington-Finkelstein coords) one can evaluate explicitly Γ_a and use it to define H_a!

- Last, many coordinate conditions devised to achieve different desired behavior, e.g.

$$\partial_t \alpha - \beta^i \partial_i \alpha = -2\alpha K$$ "drive" $\alpha \to 0$ along n^a

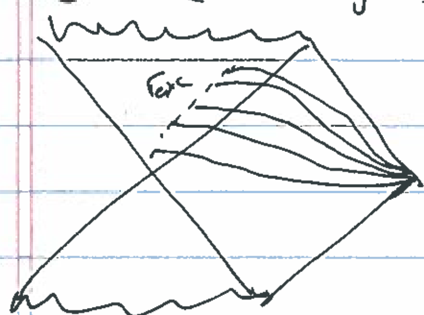
o Many others exist, e.g. in BSSN approaches, it's been useful to adopt

$$(\partial_t - \beta^i \partial_i)\alpha = -\alpha 2 K \quad , \quad \partial_t^2 \beta^i = \frac{3}{4} \partial_t \tilde{\Gamma}^i - \eta \partial_t \beta^i$$

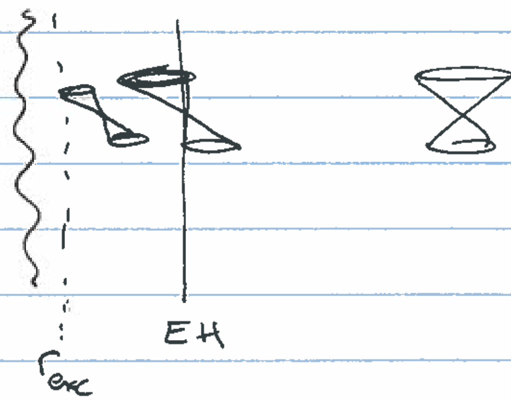
Black hole treatment (inside there is a singularity!)

(A) Black hole excision. Locate "trapped region" (which is contained outside the event horizon) and excise a suitable region

~~Penrose~~ (Penrose diagram) ("poor's man diagram)



Ω



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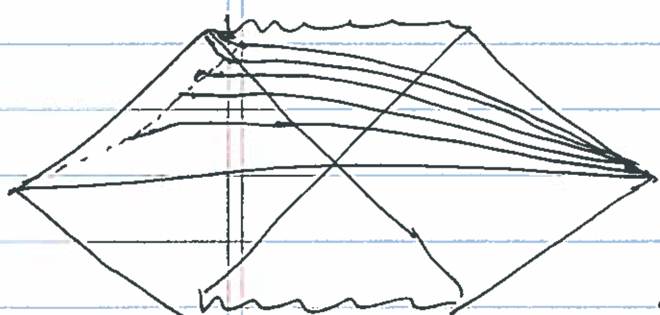
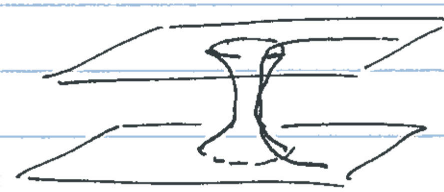
(B) Exploit throat to another region





- Points are "dragged" to the right by the shift condition.

—— o ——

Excision { - requires "sideways" derivatives & total "inflow" of charact.
{ - finding app. horizon

Avoidance requires carefully designed coordinate conditions

Both: ~~Usage~~ Usage of Artificial "Kreiss-Oliger" (FD) or "drop coeffs" (SPECTRAL) to introduce dissipation

Summation by parts operators, why?

Back ~~to~~ analytics. Consider the eqn $u_{,t} = u_{,x}$

We can define an "energy" estimate as $E = \int u^2\,dx$ (a norm of the solution)

$$\Rightarrow E_{,t} = \int u\, u_{,t}\, 2\,dx \underset{\text{eqn}}{=} \int 2 u\, u_{,x}\,dx \underset{\text{IBP}}{=} \int (\partial_x u^2)\,dx \underset{\text{only boundary contribution}}{=} u^2 \Big|_L^R$$

$$\Rightarrow E_{,t} = u_R^2 - u_L^2$$

but

$u_{,t} = u_{,x}$ propagates perturbations to the left boundary. Take boundary condition s.t. $u_R = 0$

$$\Rightarrow \boxed{E_{,t} < 0} \rightarrow \text{Norm remains bounded.}$$

Now, take time being continuous but space discrete. Let's Redo

$$\hat{E}_{,t} = \Sigma' (u_i^2)_{,t}\, h = \Sigma' 2 u_i \dot{u}_i\, h = \Sigma' 2 u_i (Du)_i\, h$$

if $\Sigma' u_i (Du)_i\, 2 = \Sigma' [D(u^2)]_i$ then D satisfies SBP

(summation by parts.)

[Gustaffson-Kreiss-Oliger boo

$$\Rightarrow \hat{E}_{,t} = u_R^2 - u_L^2$$

$\rightarrow$ Semi discrete problem has bounded norm.

But, if using RK3 (or higher) $\Rightarrow$ fully discrete norm as well satisfies it. [TADMOR]

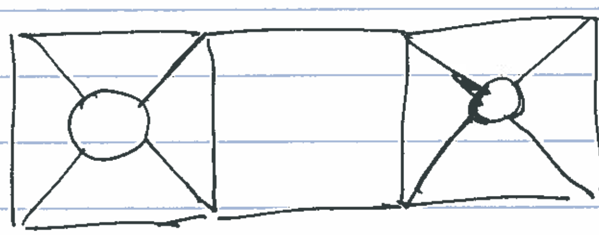
This result is general (strongly/symmetric) for hyperbolic PDEs with (time, space) dependent coefficients! $\Rightarrow$ a theorem that tells us the implementation will be stable.

Summation by parts operators of various orders in: Calabrese, LL, Sarbach, Tiglio CQG V21 (2004)

as well as negative definite dissipation operators for excised regions.

Summary:

1. Strongly/symmetric/strictly hyperbolic a must
2. Constraint damping
3. Operators (derivative & dissipation) with desired properties
4. I.D. & boundary data
5. suitable coord conditions
6. CFL now stricter than in hydro C vs C_s
7. Adaptivity! Most commonly through Cartesian grids and Finite Differences, but also spectral (and FD) through domains. Eg.



9 patches.
(multi-blocks/patches)

8. Where needed, penalty methods (Carpenter) to deal with abutting grids.

CODES/INFRASTRUCTURE: Cactus, PAMR, HAD, CHOMBO, BAM (?), SPEC (?)

Now, what do we want to do / explore?

• Magnetized matter / plasma around dynamical compact objects (BHs, NSs)

• Why? → Possible counterparts (EM) to GWs events
→ Possible impact on dynamics after merger [ang mom. transport!]

e.g. • pre-merger irrelevant for fields $\lesssim 10^{16}$G for NS-NS. But after merger plenty of energy available to crank-up $\vec{B}$ field to equipartition
• impacts the dynamics
• can trigger strong energy outflows
• if MNS (massive neutron star) collapses to a BH, a possible burst of radiation with energy ~ sGRB!

Other relevant systems:
- Jets from BHs e.g. Blandford-Znajek effect [Gammie, McKinney, Tchekovskoy, ...
- Pulsars e.g. Spitkovsky '06

Both are intrinsically related to our systems of interest, so at least them serve as guiding posts.

Disclaimer: From GR point of view we start from a rather naive point, following guidance/intuition from traditional astro-side

Simplest case : Related to BZ effect & Pulsars.

- Ignore inside of stars, only regarded as providing strong enough fields from a surface. For BHs imagine one surrounded by strong enough magnetic fields (anchored by some disk)

Goldreich-Julian ('69) pointed out the region around these compact objects is filled with a low-density plasma where even moderate values of $\vec{B}$ $\Rightarrow$ magnetic field stresses >> pressure gradients

$$\Rightarrow \quad T_{ab} = T_{ab}^{Matter} + T_{ab}^{EM} \simeq T_{ab}^{EM}$$

But $\quad \nabla_a T^{ab} = 0 \Rightarrow \nabla_a \left(T^{ab}_{EM}\right) = 0 \rightarrow F^{ab} J_a = 0$

which implies Lorentz force $= 0$!

check : $(a = k)$ $\quad F^{k0} J_0 + F^{ki} J_i \Rightarrow q E^{j} + (\vec{J} \wedge \vec{B})^k = 0 \quad (*)$

(flat case) $(a = 0)$ $\quad \cancel{F^{00} J_0}^{\,0} + F^{0i} J_i \Rightarrow \vec{J} \cdot \vec{E} = 0$

Some consequences : $\quad \vec{E} \cdot \vec{B} = 0 \qquad$ (From $(*) \cdot \vec{B}$)

(A) $\quad \vec{J} = q \dfrac{\vec{E} \wedge \vec{B}}{B^2} + (\vec{J} \cdot \vec{B}) \dfrac{\vec{B}}{B^2} \qquad$ (From $(*) \wedge \vec{B}$)

(B) $\quad \vec{J} \cdot \vec{B} = \vec{B} (\vec{\nabla} \wedge \vec{B}) - \vec{E} (\vec{\nabla} \wedge \vec{E}) \qquad$ (From $\partial_t (\vec{E} \cdot \vec{B}) = 0$ & eqns)

So, we have Maxwell's equations, with a current $\vec{J}$ ((A) + (B)) ~~and~~ ~~eq~~ [which come from $\vec{E} \cdot \vec{B} = 0$, Lorentz force $= 0$] give a prescription where the behavior of the plasma is implicitly accounted for.

CAVEAT. It must also respect $\quad B^2 > E^2 \quad$ otherwise ill-POSED!

$\rightsquigarrow$ physically this represents new physics!

5c

Implementation. ⊛ Respect $\vec{E}\cdot\vec{B}=0$, $E^2 < B^2$; $\vec{J} = \vec{J}_d + \vec{J}_B$ → bad!

→ Evolve with $\vec{J} = \vec{J}_d$

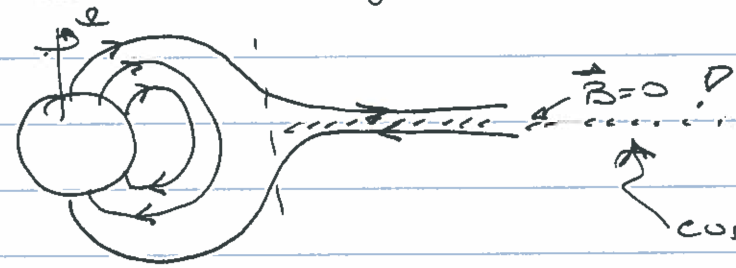
→ substract from $\vec{E}$ any component along $\vec{B}$

→ monitor (and reduce $\vec{E}$) if $|E| > |B|$.

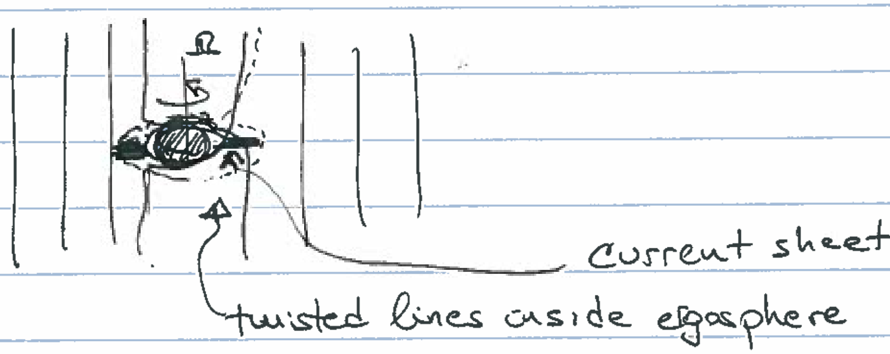
. Where?

- Spitkovsky '06
- Contopoulos '06

NS →

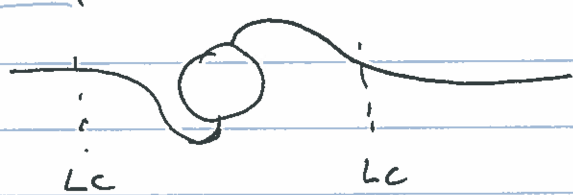


BH →

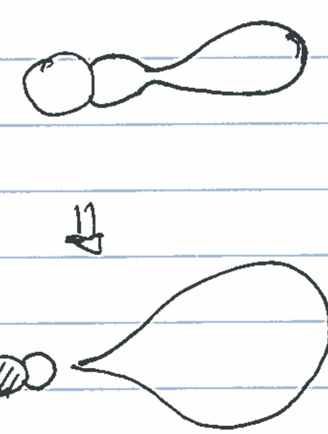


NS → BH ?

From Top



From Side



- Now, let's worry about the sources. For simplicity, let's imagine just Maxwell's equations and since we argued we can analyze things separately, let's start in flat space time.

- Maxwell's equations can be written in covariant form as

$$\nabla_b F^{ab} = I^a \quad ; \quad \nabla_b {}^*F^{ab} = 0$$

where ${}^*F^{ab} = \frac{1}{2} \epsilon^{abcd} F_{cd}$

we can define the Electric & Magnetic field w.r.t. the observer with tangent vector n^a as:

$$E^a = F^{ab} n_b \quad ; \quad B^b = {}^*F^{ab} n_a$$

we can reexpress the Faraday tensor as:

$$F^{ab} = E^b n^a - E^a n^b - \epsilon^{abcd} B_c n_d$$

For a moment consider instead

$$\nabla_b (F^{ab} + g^{ab} \psi) = I^a - \kappa \psi n^a$$

$$\nabla_b ({}^*F^{ab} + g^{ab} \phi) = -\kappa \phi n^a$$

Komisarov MNRAS '07
Palenzuela, LL, Reula + MNRAS 6

$\Rightarrow$ take ∇_a

$$\Rightarrow \quad \nabla^a \nabla_a \psi = -\kappa \nabla_a (n^a \psi)$$
$$\nabla^a \nabla_a \phi = -\kappa \nabla_a (n^a \phi)$$

For simplicity, take $n^a = \frac{\partial}{\partial t}^a$; $(\beta^a = 0, \alpha = 1)$

we get

$$\begin{aligned}
\partial_t \psi + \vec{\nabla}\cdot\vec{E} &= q - \kappa\psi \\
\partial_t \phi + \vec{\nabla}\cdot\vec{B} &= -\kappa\phi \\
\partial_t \vec{E} - \vec{\nabla}\wedge\vec{B} + \vec{\nabla}\psi &= -J \\
\partial_t \vec{B} + \vec{\nabla}\wedge\vec{E} + \vec{\nabla}\phi &= 0
\end{aligned}$$

$\rightarrow$ Hyperbolic divergence cleaning related. [Dedner et al J Comp Phys V175, 2002]

Consider $\partial_t \psi + A = -\lambda\psi$

$$\Rightarrow \partial_t \psi = -(A + \lambda\psi) \xrightarrow{\text{sol}^n} A + \lambda\psi = C_0 e^{-\lambda t}$$

$$\Rightarrow A + \lambda\psi \rightarrow 0$$

$t \rightarrow$ "∞" $\Rightarrow \psi \rightarrow -\frac{A}{\lambda}$

if λ is large $\psi \rightarrow 0$

* Constraints are damped through coupling with damping fields
* Related by constraint damping in GR but addition of fields is needed as we've used up all our freedom.

* We have not used or required any particular algorithm, just modified the equations off the constraint surface to achieve a desired effect.
* We did introduce a new scale in the problem though!

(Δ) On Friday you'll get a code implementing the system above.

Non-flat case $\quad (\gamma^{ij} = h^{ij}$ notation change!)

$$(\partial_t - \mathcal{L}_\beta) E^i - \epsilon^{ijk} D_j (\alpha B_k) + \alpha \gamma^{ij} D_j \psi = \alpha K E^i - 4\pi \alpha J^i$$

$$(\partial_t - \mathcal{L}_\beta) B^i + \epsilon^{ijk} D_j (\alpha E_k) + \alpha \gamma^{ij} D_j \phi = \alpha K B^i$$

$$(\partial_t - \mathcal{L}_\beta) \psi + \alpha D_i E^i = 4\pi \alpha q - \alpha \sigma_2 \psi$$

$$(\partial_t - \mathcal{L}_\beta) \phi + \alpha D_i B^i = - \alpha \sigma_2 \phi$$

writing $\quad I^a = q n^a + J^a \qquad (J^a n_a = 0)$

$$E^a = F^{ab} n_b \quad ; \quad B^a = {}^*F^{ab} n_b$$

Conditions on covariant form: $\quad F_{ab} F^{ab} > 0 \qquad (= 2(B^2 - E^2))$

$\qquad F_{ab} {}^*F^{ab} = 0 \qquad (\propto \vec{E} \cdot \vec{B})$.

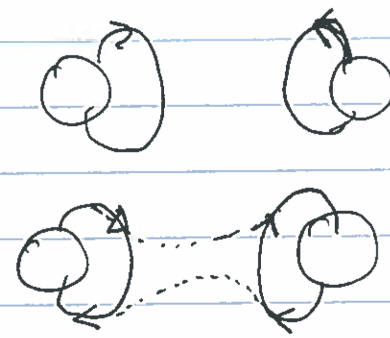
New phenomena due to dynamical gravity or relative motion?

- Jets from non-spinning, boosted BHs wrt $\vec{B}$ (Palenzuela, LL, Liebling'..)
- "gravitation-driven": if background spacetime is dynamic local EM energy density of plasma deviates from equilibrium and induce plasma waves (Fang + Zhang '15)

Basic observations.

- Beat fields into submission

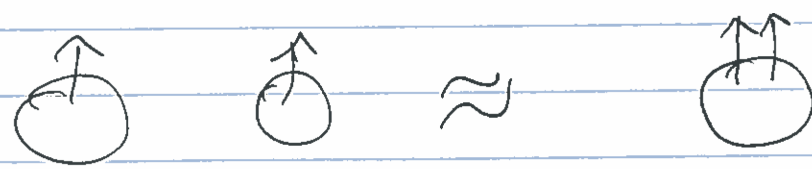
compression!



- BH "extended" B-Z effect

$$L \sim B^2\left[\Omega^2 + \#\left(\frac{v}{c}\right)^2\right]$$

- Binary NS:



$\cong$ From afar aligned NSs $\rightarrow$ single NS $\rightarrow$ similar emission properties

more complex for arbitrary configurations but to zeroth order expect a "pulsar on steroids" near coalescence.

Astro-understanding with "some" GR is quite reasonable for estimating what might happen [though needs to keep an open mind!]

Ideal MHD eqns & some options. $\{\rho, \vec{v}, \vec{B}, p\}$: 8 variables.

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho\vec{v} \\ \vec{B} \\ E\end{pmatrix} + \nabla\cdot\begin{pmatrix}\rho\vec{v} \\ \rho v^i v^j + \delta^{ij}\left(p+\frac{B^2}{2}\right) - B^i B^j \\ v^i B^j - B^i v^j \\ \left[E+p+\frac{B^2}{2}\right]\vec{v} - \vec{B}(\vec{v}\cdot\vec{B})\end{pmatrix} = 0$$

Non-diagonalizable, unless $\vec{\nabla}\cdot\vec{B}=0$ is imposed ("eliminate 1 variable" or d.o.f. through constraint)

Powell:

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho\vec{v} \\ \vec{B} \\ E\end{pmatrix} + \nabla\cdot\underbrace{\begin{pmatrix} \quad \end{pmatrix}}_{\text{as above}} = \begin{pmatrix}0 \\ \vec{B} \\ \vec{v} \\ \vec{v}\cdot\vec{B}\end{pmatrix}\cdot\vec{\nabla}\vec{B}$$

[Powell '94]

Now

1. 8 "independent" characteristics ✓
2. No conservation form → ∃ sources. ✗
3. Potential problems at discontinuities ✗

But

A. straightforwardly implemented
B. in GR sources are present anyways.
C. Constraint can propagate
D. No need to worry about AMR/FMR and what's centered where
E. if $\vec{\nabla}\cdot\vec{B}$ is small enough, we don't change things significantly (or at all for practical purposes)

- What have we really done?

we changed $\vec{B}$ eqns by adding terms proportional to $\vec{\nabla}\vec{B}$.

- Implications for the constraint? [Remember GH equations/Maxwell eqns

(A) For original system, take $\vec{\nabla}.$ of $\partial_t \vec{B}$ eqn.

$$\Rightarrow \frac{\partial}{\partial t}(\partial_i B^i) = 0$$

(B) For Powell's system, take $\vec{\nabla}$ of $\partial_t \vec{B}$ eqn. (call $C \equiv \partial_i B^i$)

$$\Rightarrow \boxed{\partial_t C + (\partial_i u^i) C + u^i \partial_i C = 0}$$

→ (i) C evolves as zero if $C=0$ at $t=0$

(ii) if $C \neq 0$ → C propagates with $\vec{u}$!

Let's work a bit more Dedner's[+] formulation JCP v175, 2002.

They present a few options, we'll stick with the hyperbolic one (well one of them

Consider Powell's version but add to l.h.s of $\partial_t \vec{B}$ eqn the following

and a rhs of $\partial_t E$

$$\partial_t \vec{B} + \dots + \nabla_j (\psi \delta^{ij}) \quad ; \quad \partial_t E + \dots = -(\vec{\nabla}\vec{B})\vec{B} - \vec{B}.\vec{\nabla}\psi$$

and augment the system with an equation for ψ

$$\partial_t \psi + c_h^2 \vec{\nabla}.\vec{B} = -u^e \partial_e \psi - \frac{c_h^2}{c_p^2}\psi$$

What we we gained?

- Before constraint propagated with $\vec{u}$; now with $\vec{u} \pm c_h$ (and ψ).
- Now, the constraint gets damped with timescale $\frac{c_h}{c_p}$.

To see this, take for simplicity $u^i = 0$, then the above equations give:

$$\begin{cases} \partial_t C + \nabla^2 \psi = 0 \\ \partial_t \psi + c_h^2 C = -\frac{c_h^2}{c_p^2}\psi \end{cases} \xrightarrow{\text{take } \partial_t} \partial_t^2 \psi + c_h^2 \partial_t C = -\frac{c_h^2}{c_p^2}\partial_t \psi$$

replace by 1st eqn $\Rightarrow \boxed{\partial_t^2 \psi + \frac{c_h^2}{c_p^2}\partial_t \psi - c_h^2 \nabla^2 \psi = 0}$

$\rightarrow$ "telegraph" equation. [just as before]

Note, this also works, dropping $\vec{\nabla}\vec{B}$ terms from $\vec{B}$ and e equations (leaving only 1 source with $(\vec{\nabla}\vec{B})\vec{B}$ in Mom eqn and $-\vec{B}.\vec{\nabla}\varphi$) in E eqn)

How does this translate to the G.R. case? Liebling, LL, Neilsen, Palenzuela arXiv/1001.0575.

First, define densitized variables

$\tilde{D} = \sqrt{h} D$ with $D = W \rho_0$

$\tilde{S}_i = \sqrt{h} S_i$ $\qquad S_i = (h_e W^2 + B^2) v_i - (B^j v_j) B_i$

$\tilde{\tau} = \sqrt{h}\, \tau$ $\qquad \tau = h_e W^2 + B^2 - P - \frac{1}{2}\left[(B^i v_i)^2 + \frac{B^2}{W^2}\right]$

$\tilde{B}^i = \sqrt{h}\, B^i$

where $W = -n^a u_a$

$v^i = \frac{1}{W} h^i{}_j u^j$

Careful $v_i \neq v^i$ in general!

Eqns

$$\partial_t \tilde{D} + \partial_i \left(\alpha \tilde{D} \left(v^i - \frac{\beta^i}{\alpha}\right)\right) = 0$$

$$\partial_t \tilde{S}_j + \partial_i \left[\alpha \left(\tilde{S}_j \left(v^i - \frac{\beta^i}{\alpha}\right) + \sqrt{h}\, P h^i{}_j\right)\right] = \alpha\, {}^3\Gamma^i_{jk} \left(\tilde{S}_i v^k + \sqrt{h}\, P h_i{}^j\right) + \tilde{S}_\ell \partial_j \beta^\ell - \partial_j \alpha \left(\tilde{\tau} + \tilde{D}\right) - \xi \alpha \left[\frac{\tilde{B}_i}{W^2} + v_i v_j \tilde{B}^j\right] \partial_k \tilde{B}^k$$

$$\partial_t \tilde{\tau} + \partial_i \left[\alpha \left(\tilde{S}^i - \frac{\beta}{\alpha} \tilde{\tau} - v^i \tilde{D}\right)\right] = \alpha \left[K_{ij} \tilde{S}^i v^j + \sqrt{h}\, K P - \frac{1}{\alpha} \tilde{S}^e \partial_e \alpha\right] - \xi \alpha v_j \tilde{B}^j \partial_k \tilde{B}^k$$

$$\partial_t \tilde{B}^\ell + \partial_i \left[\tilde{B}^\ell \left(v^i - \frac{\beta^i}{\alpha}\right) - \tilde{B}^i \left(v^s - \frac{\beta^s}{\alpha}\right)\right] - \alpha \sqrt{h}\, h^{\ell i} \partial_i \psi - \xi \alpha v^\ell \partial_j \tilde{B}^j$$

$$\partial_t \psi = - c_r \alpha \psi - c_a \frac{\alpha}{\sqrt{h}} \partial_i \tilde{B}^i + \left(\beta^i - \alpha v^i\right) \partial_i \psi$$

Typically we use $c_a = 1 = c_r$

→ Implementation:

i) RK3TVD

ii) HLLE

iii) AMR with possible tapered method for dealing with boundaries

iv) 4th order accuracy for GR variables with Summation by parts operators. (SBP)

v) Metric & MHD fields communicated as needed.

OK; we have a prescription for magnetized matter in the ideal MHD case. How about its exterior? (magnetosphere?)

Let's pause and recall $T_{ab} = T_{ab}^{Matter} + T_{ab}^{EM}$

coupled via Ohm's law

Ohm's law written in a Lorentz covariant way reads

$$I_a + (I^b u_b) u_a = \sigma F_{ab} u^b \qquad \sigma : \text{electric conduct.}$$

Expressed in terms of $\vec{E}, \vec{B}$ (switching to flat spacetime for the moment without loss of generality!)

$$\vec{J} = \sigma W [\vec{E} + \vec{v} \wedge \vec{B} - (\vec{E}.\vec{v}) \vec{v}] + q \vec{v}$$

consider the above to $\mathcal{O}\left(\frac{v}{c}\right) \to \vec{J} = \sigma [\vec{E} + \vec{v} \wedge \vec{B}]$

Maxwell's eqns $\Rightarrow \partial_t \vec{E} - \vec{\nabla} \wedge \vec{B} = -\sigma [\vec{E} + \vec{v} \wedge \vec{B}]$

assuming σ = const

$$\hookrightarrow \; -\frac{1}{\sigma}(\partial_{tt} \vec{B} - \nabla^2 \vec{B}) = (\partial_t \vec{B} - \vec{\nabla} \wedge (\vec{v} \wedge \vec{B})) \quad (*)$$

if displacement current $\partial_t E \sim \partial_t^2 B \sim 0$ then $(*)$

$$\Rightarrow \quad \partial_t \vec{B} - \vec{\nabla} \wedge (\vec{v} \wedge \vec{B}) - \frac{1}{\sigma} \nabla^2 B = 0 \quad (\square)$$

$(*)$ and $(\square)$ give the same eqn for $\sigma \to \infty$

but for $\sigma \to 0$

$(*)$ $\square B$ ✓

$(\square)$ $\nabla^2 B \to 0$ ✗

if σ varies different physics & timescales

Generalized Ohm's law

$$J^a = \sigma^{ab} e_b + \lambda b^a$$

anisotropic conductivity (Bekenstein & Oron '78)

generalization of mean field dynamo (Bucciantini & Del Zanna '12)

$\sigma^{ab} \rightarrow$ expressions obtained in collision-time approximation [Bekenstein-Or] or through relativistic, charged multifluids [Andersson '12]

$$\sigma^{ab} = \frac{\sigma}{(1+\xi^2 b^2)} \left(g^{ab} + \xi^2 b^a b^b + \xi \varepsilon^{abcd} u_c b_d \right) \quad \text{with } \xi = \frac{1}{R} = \frac{e\,\tau_r}{m_e}$$

$$\sigma = \frac{R}{(n_e e)}$$

isotropic case — anisotropies due to b^a — related to Hall effect

Simplification 1.

consider ignoring dynamo effects as well as Hall effect

$$\rightarrow J_a = \frac{\sigma}{1+\xi^2 b^2} \left[e_a + \xi^2 (e_b b^b)\, b_a \right] \qquad \text{(Zanotti \& Dumbser '11)}$$

now, write w.r.t "Eulerian" observers

$$J_a n^a = \frac{\sigma}{1+\xi^2 b^2} \left[-W\,(E^k v_k) - W \xi^2 (E^J B_J)(B^k v_k) \right]$$

$$J_i = q\, v_i + \frac{\sigma}{1+\xi^2 b^2} \left[\mathcal{E}_i + \xi^2 (E^k B_k)\, \mathcal{B}^i \right]$$

with $\mathcal{E}_i = W \left[E_i + (v \wedge B)_i - (\vec{E} \cdot \vec{v})\, v_i \right]$

$$\mathcal{B}_i = W \left[B_i - (v \wedge E)_i - (\vec{B} \cdot \vec{v})\, v_i \right]$$

Recall: Force-free current $J^i = q\, v_d^i + (J^k B_k)\dfrac{B^i}{B^2}$

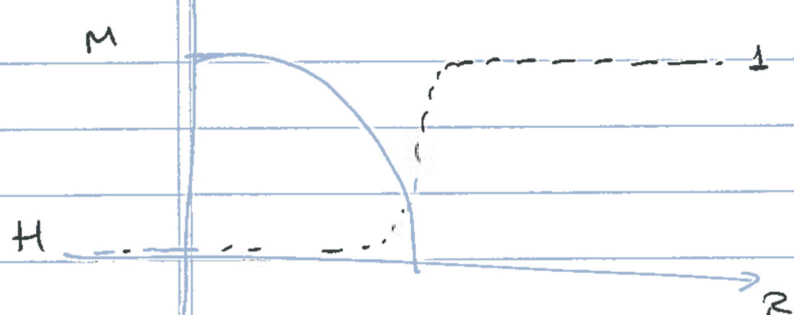
but, a similar behavior can be obtained with

$$J^i = q\, v_d^i + \frac{\sigma_{\parallel}}{B^2}\left[(E^k B_k) B^i + \chi (E^2 - B^2) E^i\right]$$

where $\sigma_{\parallel}$: anisotropic conductivity along magnetic field lines

Simplification 2: The currents and their similar expression suggest defining:

$$J^i = q\left[(1-H)\, v^i + H\, v_d^i\right] + \frac{\sigma}{(1+\xi^2)}\left[E^i + \frac{\xi^2}{B^2}\left\{(\vec{E}\cdot\vec{B}) B^i + \chi (E^2 - B^2) E^i\right\}\right]$$



Now, equation for $\vec{E}$ becomes stiff if

Discussion: Take σ large, then

$H = 0 \longrightarrow J^i \cong q\, v^i + \sigma E^i \longrightarrow$ ideal MHD

$H = 1 \longrightarrow J^i \cong q \dfrac{(\vec{E}\wedge\vec{B})}{B^2} + \sigma F F_{\parallel} \Rightarrow J^i_{FF}$

Now one has an equation which has become stiff but only for $\partial_t E$!

$$\Leftrightarrow \partial_t \left(\sqrt{h}\, E^i\right) = F_E^i + \sqrt{h}\, R_E^i$$

with

$$F_E^i = -\partial_k \left[\sqrt{h}\left(-\beta^k E^i - \alpha\left(\varepsilon^{ikj} B_j - h^{ik}\psi\right)\right)\right]$$
$$- \sqrt{h}\, E^k \left(\partial_k \beta^i\right) + \sqrt{\gamma}\, \psi \left(h^{ij}\partial_j \alpha - \alpha \gamma^{jk} \Gamma^i_{jk}\right)$$
$$- \alpha \sqrt{h}\, J_E^i$$

$$R_E^i = -\alpha J_S^i$$

where:

$$J_E^i = q\left[(1-H)\, v^i + H\, v_d^i\right]$$

$$J_S^i = \frac{\sigma}{1+\xi^2}\left[\mathcal{E}^i + \frac{\xi^2}{B^2}\left\{(E^k \cdot B_k) B^i + \chi\left(E^2 - B^2\right) E^i\right.\right]$$