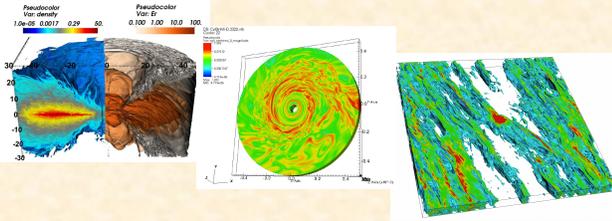


Numerical Methods for Radiation Magnetohydrodynamics

<http://www.astro.princeton.edu/~jstone/downloads/papers/>



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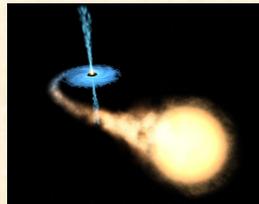
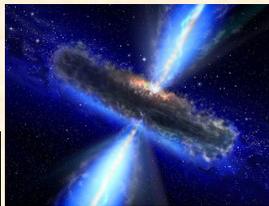
Lecture 2:

1. Radiation hydrodynamics.
2. Numerical methods for radiation hydrodynamics.
 - Flux-limited diffusion
 - Full transport methods
3. Radiation hydrodynamics in Godunov schemes.
4. Athena++.

Why Radiation Hydrodynamics?

Example: Black hole accretion flows
Accretion powers the most luminous sources in the universe

Quasars
Active Galactic Nuclei

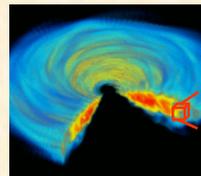


X-ray binaries

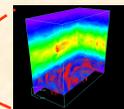
MHD is essential

Angular momentum transport in black hole accretion disks is driven by MHD turbulence produced by the magneto-rotational instability (MRI): Balbus & Hawley (1991)

Nonlinear saturation of MRI widely studied with both *local* ("shearing-box") and *global* simulations.



Global simulation



Local simulation

Radiation is essential

In black hole accretion disks, radiation pressure exceeds gas pressure inside

$$r/R_G < 170(L/L_{\text{Edd}})^{16/21}(M/M_\odot)^{2/21}$$

(Shakura & Sunyaev 1973). Radiation needs to be included in dynamical models.

If stress $\tau_{r\phi} = \alpha P$ then radiation dominated disks are subject to both

- Viscous instability (Lightman & Eardley 1974)
- Thermal instability (Shakura & Sunyaev 1976)

Still not clear if such instabilities really exist with MRI.



Foundations of Radiation Hydrodynamics

Numerical MHD is easy compared to radiation hydrodynamics.

Some of the reasons why radiation hydrodynamics is hard:

- Which equations (transfer equation or its moments)?
- Which frame (co-moving, Eulerian, mixed-frame)?
- Proper closure of moment equations.
- Mathematical problem changes in different regimes: *hyperbolic* in streaming limit, mixed *hyperbolic-parabolic* in diffusion limit.
- Wide range of timescales requires implicit methods.
- Frequency dependence adds another dimension to solution
- Non-LTE effects requires modeling level populations.

This complexity means that radiation hydrodynamics means different things to different people.

In some cases, only need to include energy transport via material-radiation energy exchange term:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{v}] = 0$$

$$\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v} \mathbf{v} + P] = 0$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P)\mathbf{v}] = -g^0$$

$$g^0 = \int d\nu \int_{4\pi} d\Omega (j_\nu - \kappa_\nu k_\nu) \quad \begin{array}{l} \text{Optically thin cooling.} \\ \text{Heating by (ionizing) radiation.} \end{array}$$

Examples: diffuse ISM, HII regions.

In general (non-LTE with scattering), the emission and scattering terms may be complicated to evaluate.

Aside: Adding source terms.

Simple source terms usually added via *operator splitting*.

1. Update flux divergence terms ignoring source terms
2. Update source term.

For Godunov methods, simple operator splitting:

1. formally makes scheme first-order in time
2. can lead to stability problems

Second-order can be achieved using multi-step methods (easy using van Leer unsplit integrator, or RK time stepping).

Stability issues can be addressed using implicit methods, e.g. IMEX

Simple source term: Optically-thin cooling

Adds source terms to energy equation: $\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F}_E = -\rho^2 \Lambda(T) + \rho H$

Where $\Lambda(T)$ is per-particle cooling rate, H is per particle heating rate.

Depending on cooling function, terms are usually nonlinear in T , and very stiff. *Forward Euler* differencing requires very small Δt

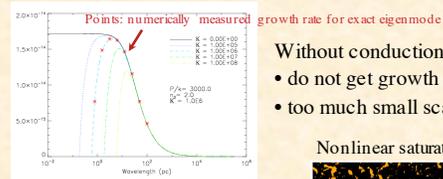
Better to use *Crank-Nicholson* (semi-implicit) differencing, where source terms are calculated at both current and advanced time (using E^n and E^{n+1}).

Not difficult to add cooling directly to integrator in Godunov methods by adding cooling term to calculation of every partial update.

Warning: easy to add cooling, but makes physics of MHD much more complex. For example, need to add thermal conduction to be able to resolve field length to get correct dynamics with cooling instability.

Moral: It takes work to really understand what is going on in both the physics and numerics.

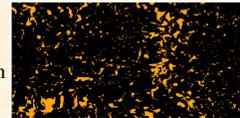
Example: thermal instability. Adding heat conduction is crucial.



- Without conduction:
- do not get growth rate correct;
 - too much small scale structure

Nonlinear saturation at 200 Myr

No conduction



With conduction



Ionizing radiation transport

Application: growth of HII regions in ISM. Solve MHD equations for 2-fluid (ions + neutrals) medium, including heating, cooling, photoionization, and recombination.

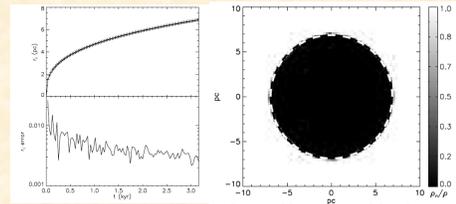
$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0, \\ \frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v} - \mathbf{B} \mathbf{B}) + \nabla P^* &= 0, \\ \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v} \mathbf{B} - \mathbf{B} \mathbf{v}) &= 0, \\ \frac{\partial E}{\partial t} + \nabla \cdot [(E + P^*) \mathbf{v} - \mathbf{B} (\mathbf{B} \cdot \mathbf{v})] &= G - \mathcal{L}, \\ \frac{\partial \rho_n}{\partial t} + \nabla \cdot (\rho_n \mathbf{v}) &= \mathcal{R} - \mathcal{I}, \\ \mathcal{I}_{\text{ph}} &= \sigma \rho_n \sum_n \frac{s_n}{4\pi |x - x_n|^2} e^{-\tau(x, x_n)}, \\ \tau(x, x_n) &\approx \int_{x_n}^x (\sigma_{\text{HI}} + \sigma_{\text{dH}}) dl \end{aligned}$$

Challenge: compute optical depth from every point source to every grid cell.

Algorithm

- Use adaptive ray-tracing method of Abel & Wandelt (2002) and Whalen & Norman (2006) using HEALPix to compute ionization rate in each cell
- Limit cooling in mixed cells
- Tests: propagation of R- and D-type I-fronts.

Krumholz, Stone, & Gardiner (2007)



Test of growth of R-type front with no recombinations

In some cases, may “only” need to include momentum exchange terms.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{v}] = 0$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v} \mathbf{v} + P] = -\mathbf{g}$$

$$\mathbf{g} = \frac{1}{c} \int d\nu \int_{4\pi} d\Omega \mathbf{n} (j_\nu - \kappa_\nu k_\nu)$$

e.g. line-driven winds (assuming gas is isothermal).

Of course, computing \mathbf{g} can be extremely difficult!

In some cases, need to include *both* energy *and* momentum exchange terms.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{v}] = 0$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v} \mathbf{v} + P] = -\mathbf{g}$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P) \mathbf{v}] = -g^0$$

$$\frac{\partial E_r}{\partial t} + \nabla \cdot \mathbf{F}_r = g^0$$

$$\frac{1}{c^2} \frac{\partial \mathbf{F}_r}{\partial t} + \nabla \cdot \mathbf{P}_r = \mathbf{g}$$

$$g^0 = \int d\nu \int_{4\pi} d\Omega (j_\nu - \kappa_\nu k_\nu)$$

$$\mathbf{g} = \frac{1}{c} \int d\nu \int_{4\pi} d\Omega \mathbf{n} (j_\nu - \kappa_\nu k_\nu)$$

Examples:
radiation-dominated disks
core-collapse SN

All of these problems could be called “radiation hydrodynamics”.

Obviously, the numerical methods required in each regime are very different.

Transfer equation.

Fundamental description of the radiation field is the frequency-dependent transfer equation

$$\frac{1}{c} \frac{\partial I_\nu}{\partial t} + \nabla \cdot (\mathbf{n} I_\nu) = j_\nu - \kappa_\nu I_\nu$$

Can be thought of as the “collisionless Boltzmann equation for photons”, so that I is the “photon distribution function”.

Only in LTE are emission coefficient j_ν and scattering term $\kappa_\nu I_\nu$ simple.

Just like the fluid equations, can take moments over phase space (angles) and frequency to derive a set of moment equations.

Why? Reduces dimensions of problem, making it easier to solve.

Grid-based method versus particles for radiation transfer

Even though we use a grid for the MHD, we could still choose to use either a grid or particles (Monte Carlo) to solve the transfer equation.

Grid:

More accurate and less noise

Difficult to extend to include scattering, and line-transport

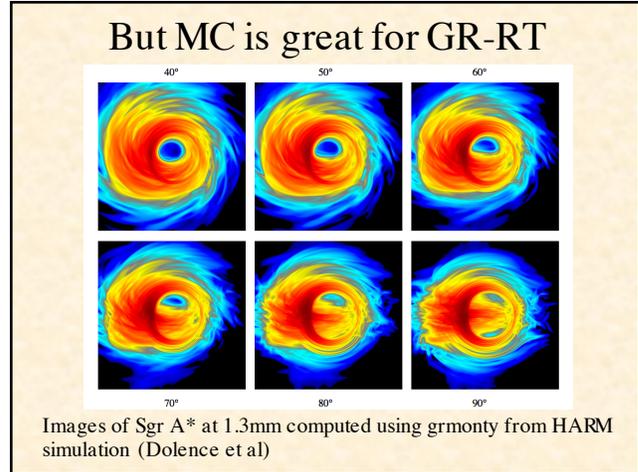
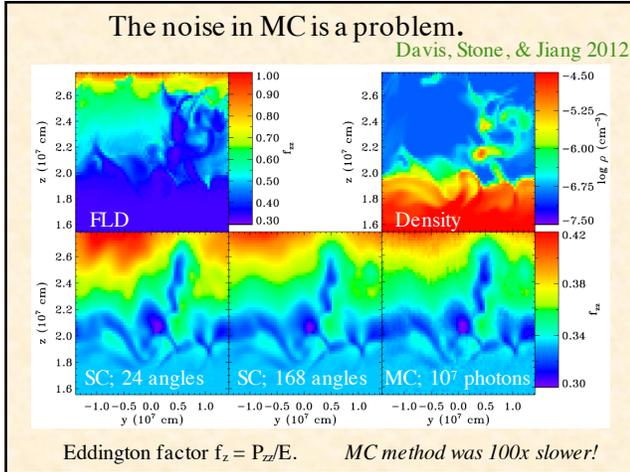
Very expensive

Particles (Monte Carlo):

Very flexible, easy to extend to frequency-dependent transport, etc.

Embarrassingly parallel

Noisy, especially in optically thick regions



Radiation Moment equations

Euler equations + Maxwell's equations + moment equations.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + P + B^2/2 - \mathbf{B}\mathbf{B}) = -\mathbf{F}\mathbf{S}_M$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P)\mathbf{v} + (B^2/2)\mathbf{v} - \mathbf{B}(\mathbf{B} \cdot \mathbf{v})] = -\mathbf{P}\mathbf{C}\mathbf{S}_E$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$$

$$\frac{\partial E_r}{\partial t} + \mathbf{C}\nabla \cdot \mathbf{F}_r = \mathbf{C}\mathbf{S}_E$$

$$\frac{\partial \mathbf{F}_r}{\partial t} + \mathbf{C}\nabla \cdot \mathbf{P}_r = \mathbf{C}\mathbf{S}_M$$

$E_r, \mathbf{F}_r, \mathbf{P}_r$ are radiation energy density, flux, pressure in Eulerian frame.
Source terms are $O(v/c)$ expansion of material-radiation interaction terms in fluid frame (Lowrie et al 1999). Or can simply use Lorentz boost to transfer Eulerian frame quantities to fluid frame to compute source terms.

Numerical Methods: radiation transport (RT)

- Crucial issue: need a closure relation $\mathbf{P} = \mathbf{f} E$
- Various approximations commonly used

(1) Flux-limited diffusion.
Assume radiation flux given by Ficks's Law

$$\mathbf{F} = -\frac{c\lambda}{\chi} \nabla E$$
 $\lambda = \lambda(E)$ is limiter that prevents super-luminal transport in optically thin regions
 Reduces equations to two-temperature diffusion approximation.

(2) M1 closure.
Assume Eddington tensor given by:

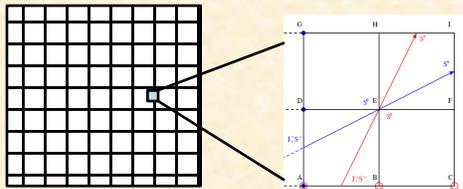
$$\mathbf{f} = \frac{1-\xi}{2} \mathbf{I} + \frac{3\xi-1}{2} \mathbf{n}_F \otimes \mathbf{n}_F,$$

$$\xi = \frac{3 + 4 \|\mathbf{F}_r^n / (c\mathbf{E}_r^n)\|^2}{5 + 2\sqrt{4 - 3 \|\mathbf{F}_r^n / (c\mathbf{E}_r^n)\|^2}}$$

(3) Variable Eddington Tensor (VET)

For non-relativistic flows, the light-crossing time is much shorter than an MHD time step. In this case, solve the *time-independent* transfer equation using the *method of short characteristics* along N_r rays per cell. $\partial I / \partial s = \kappa(S - I)$

Include scattering and non-LTE effects using *accelerated lambda iteration* (ALI). Olson & Kunasz 1987; Stone, Mihalas, & Norman 1992; Trujillo Bueno & Fabiani Bendicho 1995; Davis, Stone, & Jiang 2012



Then compute \mathbf{f} directly from moments of I .
$$\mathbf{f} = \frac{P_r}{E_r} = \frac{\int \hat{\mathbf{n}} \hat{\mathbf{n}} I d\omega}{\int I d\omega}$$

Solving the closure problem: Flux-limited diffusion

Adopt the diffusion approximation everywhere $\mathbf{F} = -D\nabla E$
 Superluminal transport in optically thin regions, unless flux is limited: $D = \frac{c\lambda}{\chi}$
 $\lambda = \lambda(E)$ is limiter
 These reduces the RMHD equations to a two-temperature diffusion problem. Turner & Stone 2001

$$\begin{aligned} \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} &= 0 \\ \rho \frac{D\mathbf{v}}{Dt} &= -\nabla P + \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} + \frac{1}{c} \chi_F \mathbf{F} \\ \rho \frac{D}{Dt} \left(\frac{E}{\rho} \right) &= -\nabla \cdot \mathbf{F} - \nabla \mathbf{v} : \mathbf{P} + 4\pi \kappa_F B - c\kappa_E E \\ \rho \frac{D}{Dt} \left(\frac{e + E}{\rho} \right) &= -\nabla \cdot \mathbf{P} - P \nabla \cdot \mathbf{v} - \nabla \cdot \mathbf{F} \\ \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}) \end{aligned}$$

- Pros:** easy to solve
- Cons:** lost information about direction of flux
 magnitude of flux in optically thin regions is ad-hoc
 no radiation inertia (superluminal wave speeds)
 no radiation shear viscosity

Flux limiter

Arbitrary function of E . Most popular form is due to Levermore & Pomraning (1981)

$$\lambda(R) = \frac{2 + R}{6 + 3R + R^2} \quad R = |\nabla E|/E$$

Main purpose of limiter is to give correct flux in optically thin and thick limits

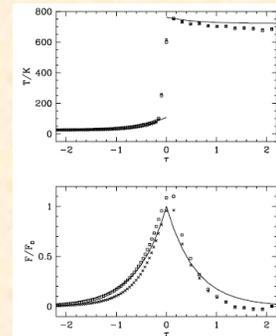
$$\lim_{R \rightarrow \infty} \lambda(R) = \frac{1}{R} \quad \text{Optically thin limit, } F \sim cE$$

$$\lim_{R \rightarrow 0} \lambda(R) = \frac{1}{3} \quad \text{Optically thick limit, } F \sim \text{Grad}(E)$$

Test of FLD: subcritical shock

X = Minerbo limiter
 □ = Levermore & Pomraning limiter

Solid line = approx analytic solution from Zel'dovich & Raizer (1967)

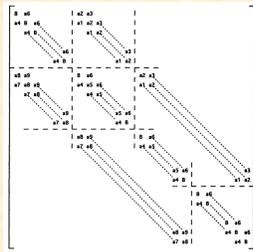


Parameters same as in Sincell, Gehmeyr & Mihalas (1999)

Implicit differencing.

Material-radiation interaction and radiation transport terms have a very restrictive time step limit, and must be solved implicitly.

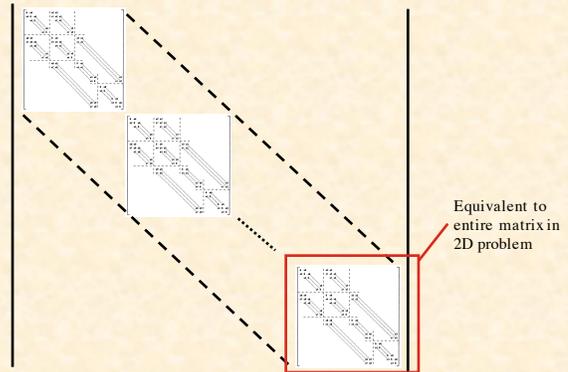
Equations are nonlinear in unknowns, so must use Newton-Raphson iteration. Requires solving large sparse-banded matrix for every NR iteration.



Matrix solved for each NR iteration is very sparse, so use iterative methods like GMRES or ICCG.

Schematic of matrix in 2D

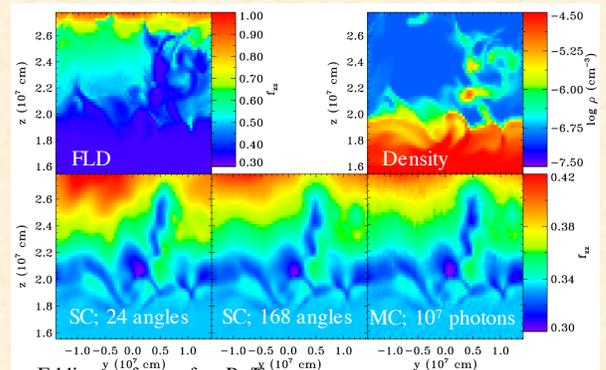
In 3D, matrix to be solved in each NR step is $N^3 \times N^3$ where N is number of grid points along each dimension.



Reduced speed of light methods

- To avoid implicit differencing, simply assume c is larger than sound speed, but smaller than the true value.
- Can derive constraints on slowest speed allowed that gives correct dynamics, e.g. Skinner & Ostriker (2013)

FLD doesn't give correct values even in optically thin regions



Eddington factor $f_z = P_{zz}/E$

Davis, Stone, & Jiang 2012

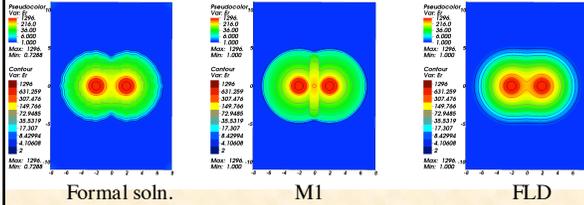
M1 closure

To avoid problems with FLD, new local closures have been tried
Most popular currently is M1 (Gonzalez et al 2007)

- Keeps flux as separate variable
- Uses local information to construct direction of flux

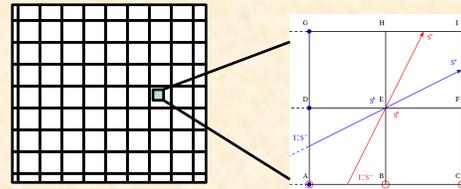
M1 fixes one problem (lack of shadows with FLD), but replaces it with another (photons collide and merge with M1)

Radiation energy density from two radiating spheres:



Variable Eddington Tensor

Compute using short characteristics to solve time-independent transfer equation along N_r rays per cell. $\partial I / \partial s = \kappa(S - I)$ Olson & Kunasz 1987; Stone, Mihalas, & Norman 1992; Davis, Stone, & Jiang 2012

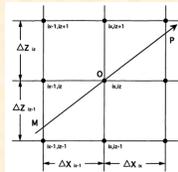


Method includes scattering and non-LTE effects using accelerated lambda iteration (ALI). Trujillo Bueno & Fabiani Bendicho 1995

Then compute f directly from moments of I . $f = \frac{P_r}{E_r} = \frac{\int \hat{n} \hat{n} I d\omega}{\int I d\omega}$

Short versus long characteristics

Short characteristics (Kunasz & Auer 1988): solve along ray segments that cross a single zone, and interpolate I to start of next ray segment, $O(N^3)$ in 3D

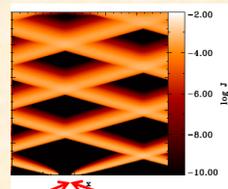


Long characteristics: for each cell, solve along rays that cross entire grid, $O(N^4)$ in 3D.

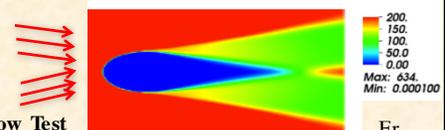
Short characteristics are much faster, but can have problems in treating point sources.

Tests of Transfer Solver

Davis, Stone, & Jiang 2012



Beam Test
Two "flashlight" beams in optically thin, periodic domain.



Shadow Test
Optically thick, spherical cloud irradiated by two beams.

Modified Godunov method

Miniati & Colella 2007
Sekora & Stone 2010

Stable, 2nd order accurate scheme for handling stiff source terms.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial(\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v} + P + B^2/2 - \mathbf{B}\mathbf{B}) = -\mathbb{P}\mathbf{S}_M$$

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + P + B^2/2)\mathbf{v} - \mathbf{B}(\mathbf{B} \cdot \mathbf{v})] = -\mathbb{P}\mathbf{C}S_E$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$$

Uses modified wave speeds and eigenvectors to compute fluxes.

Semi-implicit (Picard iteration) scheme ensures stability.

Implicit solution of moment equations.

Method must be implicit to allow $\delta t > dx/c$.
Solving entire system of equations implicitly is expensive and inaccurate.

Instead, split fully-implicit solution of radiation moment equations from modified Godunov method for MHD equations.

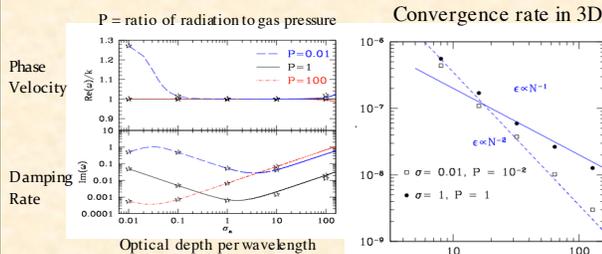
$$\frac{\partial E_r}{\partial t} + \mathbb{C}\nabla \cdot \mathbf{F}_r = \mathbb{C}S_E$$

$$\frac{\partial \mathbf{F}_r}{\partial t} + \mathbb{C}\nabla \cdot \mathbf{P}_r = \mathbb{C}\mathbf{S}_M$$

Requires inverting large sparse matrix every time step. *This is usually the slowest step in the entire algorithm.*

Test of Full Code: Linear Waves

Quantitative measure of error and convergence rate.



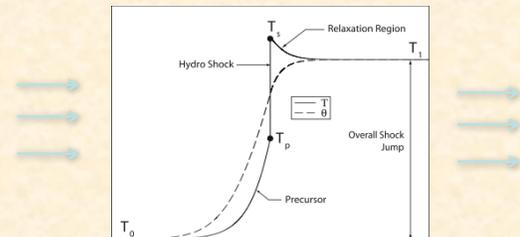
Stars are measured phase velocity and damping rate from 1D code.

Jiang et al. 2012

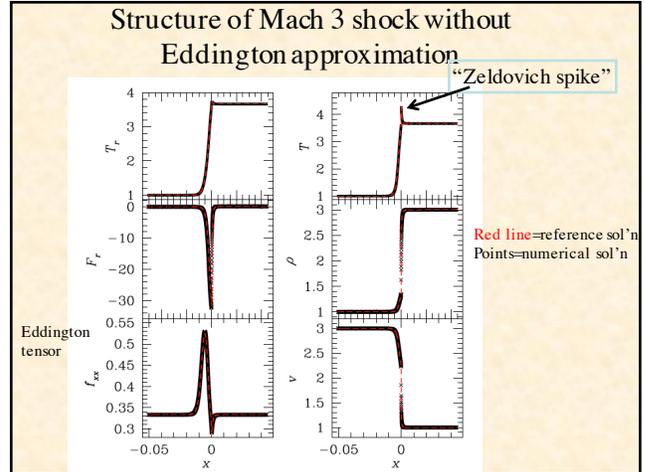
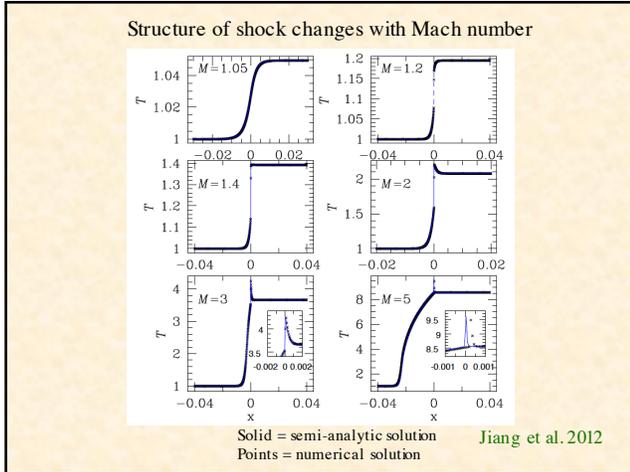
Radiation Shock Tests

- 1D steady shock with pure absorption opacity
- Semi-analytic solution possible in nonequilibrium diffusion limit (Eddington approximation, $f=1/3$)

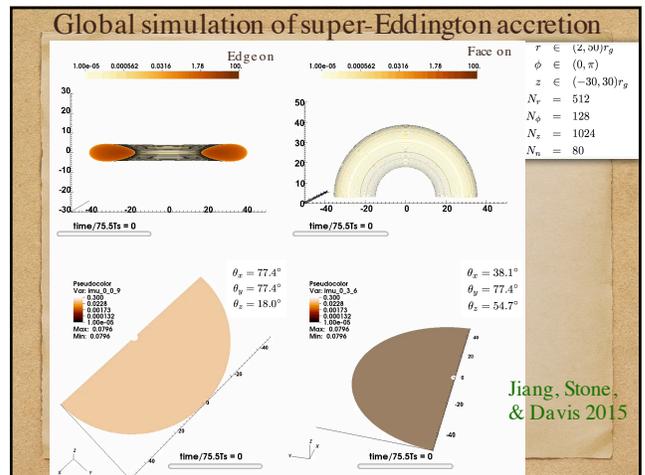
Lowrie & Edwards (2008)

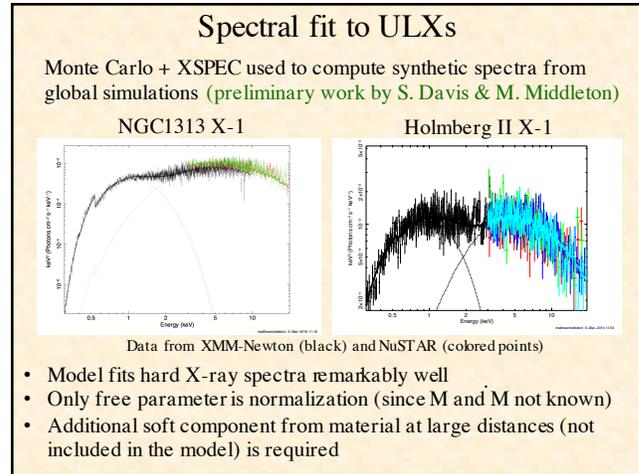
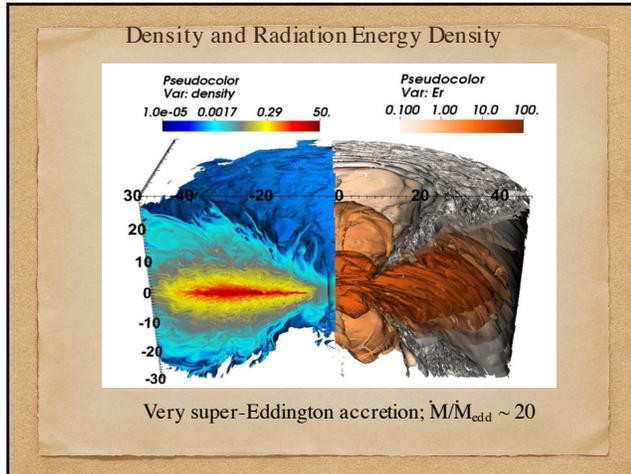


Shock structure changes with different Mach numbers.



- ### Computational challenges
- Cost of 3D radiation MHD simulations using explicit differencing scale as: $N_x N_y N_z N_m N_n$
 - Number of angles
 - Number of frequencies
 - Efficient mixed parallelization possible.
 - Adaptive angles and frequencies could prove extremely powerful
 - Implicit differencing also requires inversion of $4N_x N_y N_z$ matrix every time step, parallelization is more difficult.
 - Either way, access to petascale resources crucial





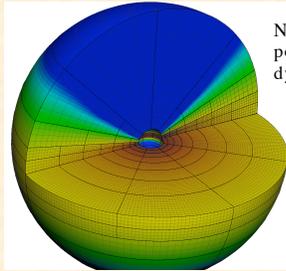
- ### Athena++: A new framework
- Project goals:
 - Rewrite in C++ to make it more modular
 - Implement new capabilities (non-uniform mesh, AMR, GRMHD)
 - Try to improve performance on vector (SIMD) processors
 - Implement mixed parallelization (OpenMP and MPI) with overlapping computation/communication
 - Use common framework so that same code can implement hydro, MHD, hybrid PIC, RT, etc.

K. Tomida (PU)

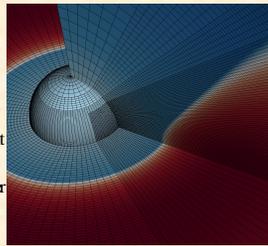
AMR Design: Grid Structure

	(A) Block (Patch) Based	(B) Oct-Tree-Block Based	(C) Cell(Tree)-Based
Pros	High adaptivity Uniform within block Use of existing scheme	Simple relation btw blocks Uniform within block Use of existing scheme Parallelization by space-filling curve	Highest adaptivity Logically beautiful Parallelization by space-filling curve
Cons	Grids are not unique Non-trivial grid generation Complex parallelization	Lower adaptivity (depending on patch size)	Performance Issue Complicated grids (non-trivial neighbor cell) Hard to write, read, analyze
Examples	Orion, PLUTO(Chombo), Enzo, Athena SMR, ...	FLASH(PARAMESH), Nirvana, SFUMATO, ...	RAMSES, ART

Athena++: non-uniform and curvilinear meshes



Non-uniform grid in spherical polar coordinates allows large dynamic range in radius.



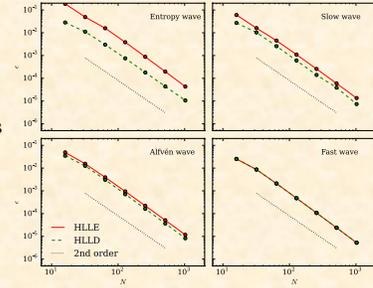
Nested grid in polar angle allows de-refinement towards pole, avoiding very small cells there. Polar boundary condition allows free-flow over poles

GRMHD

C. White (PU)

- Both SR and GR hydro and MHD have been implemented
- GRMHD algorithm significantly different than, e.g. HARM
 - Uses more advanced Riemann solvers (HLLC, HLLD)
 - Staggered-grid CT

Linear wave test in GRMHD shows less diffusion with more advanced solvers



Single Core Performance

Recent results for full code on 2.6Ghz Intel Haswell

Hydro, HLLC, 2nd order PLM, with intel v15:

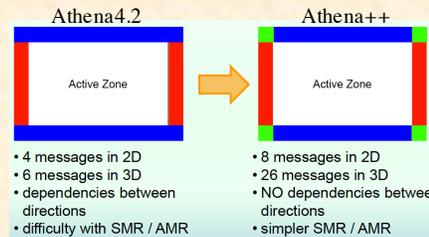
- 2.5M zone-cycles/sec per core
- 3.2 Gflops in 2.5GHz Intel IvyBridge [15% of theoretical peak]
- 25% of FP operations are vectorized, 75% are SIMD (0.7% are scalar)

MHD, HLLD, 2nd order PLM, with intel v15:

- 1.3M zone-cycles/sec per core

Mixed Parallelization

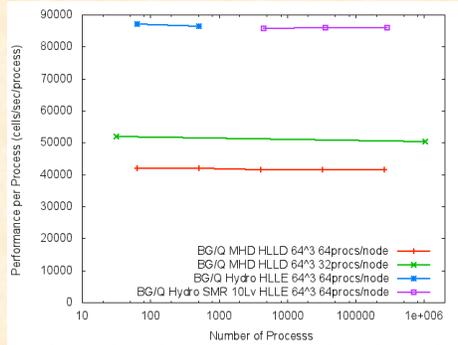
1. **Distributed memory parallelization** using domain decomposition and MPI. Athena++ uses new MPI communication patterns, and Z-ordering for load balancing.



2. **Shared memory parallelization** within a mesh block using OpenMP. Allows use of multi-core architectures (Intel Xeon Phi)

Weak scaling.

Athena++ shows excellent efficiency (better than Athena4.2), achieving 98.6% efficiency on up to 256,000 physical cores.



IBM Blue Gene Q
(Mira) at ALCF

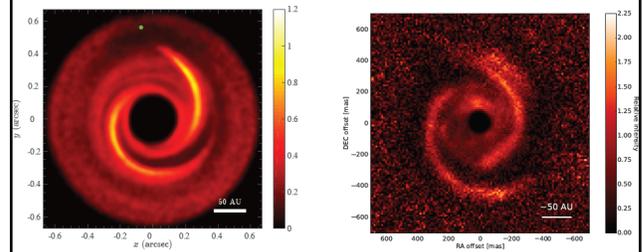
First application using Athena++

3D structure of spiral density waves launched by
disk-planet interaction

Dong+ 2015
Zhu+ 2015

Synthetic scattered-light image
computed using MC

MWC 758



Summary

- Finite volume methods for MHD are now mature.
- They are workhorse methods for many problems in astrophysics
- Such methods can scale extremely well to 10^5 - 10^6 cores, even with mesh refinement.
- Higher-order methods are becoming increasingly important
 - High-order FV methods on compact stencils
 - DG methods
- Methods for radiation hydrodynamics are still under active development