Calculating a Cosmic Ray Spectrum

This argument is based on the original argument by Fermi (1949) and summarized in Kulsrud's book.

Suppose particles are injected at a constant rate and suffer a constant probability of being lost. Then, the number of particles with ages between t and $t+dt$ will be

$$\tilde{n}(t)\,dt = \tilde{n}_0 e^{-t/\tau_L}\,dt \qquad (1)$$

Where τ_L is the loss time and n_0 is a normalization. If we want $\int \tilde{n}\,dt$ to have units of particles/volume then $\tilde{n}$ and $\tilde{n}_0$ have units of particles/(volume × time).

Now suppose particles are injected at some energy E_0 and gain energy at a rate $\frac{dE}{dt}$ such that at time t they have energy $E(t)$. Then the number of particles with energies between E and $E+dE$ is

$$n(E)\,dE = \tilde{n}(t(E))\,dt\,; \qquad (2)$$

$$n(E) = \frac{\tilde{n}(t(E))}{dE/dt} \qquad (3)$$

This is a very general result. Now we make it more specific by supposing that

$$\frac{dE}{dt} = \frac{E}{\tau_{Acc}} \tag{4}$$

for some constant acceleration time τ_{Acc} so that

$$E(t) = E_0 e^{t/\tau_{Acc}} \tag{5}$$

Inverting this

$$\frac{t(E)}{\tau_{Acc}} = \ln\left(\frac{E}{E_0}\right) \tag{6}$$

Substituting eqns. (4) and (6) into eqn. (3) gives

$$\frac{\tilde{n}_0 \tau_{acc}}{E_0} \left(\frac{E}{E_0}\right)^{-\left(1+\frac{\tau_{Acc}}{\tau_L}\right)} \tag{7}$$

This sort of distributed acceleration process can probably be ruled out for Galactic cosmic rays and suffers a defect that must be addressed for all cosmic rays. The first objection is that the model predicts the most energetic cosmic rays are also the oldest, so they should be most enriched in light elements. However, the reverse is true.

The second objection is that the cosmic ray spectrum seems to be quite universal over time & from galaxy to galaxy. This means τ_{acc}/τ_L should have a universal value.

Fermi's model

Fermi envisioned colliding from "magnetic clouds" with speed V and mean separation L. A particle of momentum $\vec{p}$ is changed in energy by $-2\vec{p}\cdot\vec{V}$ upon reflection (head on collisions increase energy & overtaking collision decrease energy).

The frequency of head on collision is $\frac{V+c}{L}$ and the frequency of overtaking collision is $\frac{c-V}{L}$. Hence, summing over many collisions,

$$\frac{dE}{dt} = 2pV\left(\frac{c+V}{L} - \frac{c-V}{L}\right) = 4\frac{E}{c}\frac{V^2}{L} \qquad (8)$$

We should do a proper angular average, so the 4 is replaced by another number of order unity.

This acceleration law is of the same form as eqn (4), with

$$\tau_{acc} \sim \frac{Lc}{V^2} = \frac{L}{c}\left(\frac{c}{V}\right)^2 \qquad (9)$$

We know that in the Milky Way $\frac{c}{V} \sim 10^9$ and $\tau_L \sim 2\text{-}3\times10^7$ yr–

Say $2.5\ 10^7$ yr $= 8\times10^{14}$ s. So, for $\frac{\tau_{acc}}{\tau_L} \sim 1$,

$$L < .01 \text{ pc}$$

Impossible! More generally, matching τ_{acc} to τ_L requires "Fine tuning". This is called 2nd order Fermi acceleration because it's 2nd order in V/c.

Derivation of a Fokker-Planck equations for pitch angle scattering

Define $\mu \equiv \frac{P \cdot B}{PB}$

We want to see the cumulative effect of small random changes in μ, e.g. due to pitch angle scattering. Let $p(\mu, \Delta\mu)$ be the probability that in time Δt a particle originally at μ is scattered by $\Delta\mu$, and let $f(\mu, t)$ be the pitch angle distribution at time t (suppressing the dependence on particle energy E to save writing). ~~Assume~~ p is normalized to ξ. Then

$$f(\mu, t+\Delta t) = \frac{1}{\xi} \int f(\mu - \Delta\mu, t)\, p(\mu - \Delta\mu, \Delta\mu)\, d\Delta\mu \quad (10)$$

Expand the LHS w.r.t Δt and the RHS w.r.t. $\Delta\mu$:

$$f(\mu, t) + \Delta t \frac{\partial f(\mu,t)}{\partial t} = \frac{1}{\xi} \int \Big\{ f(\mu, t)\, p(\mu, \Delta\mu) - \Delta\mu \frac{\partial}{\partial\mu}\big(f(\mu,t)\, p(\mu,\Delta\mu)\big) + \frac{(\Delta\mu)^2}{2} \frac{\partial^2}{\partial\mu^2}\big(f(\mu,t)\, p(\mu,\Delta\mu)\big) \Big\}\, d\Delta\mu \quad (11)$$

Using the normalization and dividing by Δt, we obtain

$$\frac{\partial f(\mu,t)}{\partial t} = -\frac{\partial}{\partial\mu} \frac{\langle \Delta\mu \rangle}{\Delta t} f(\mu,t) + \frac{1}{2} \frac{\partial^2}{\partial\mu^2} \frac{\langle \Delta\mu\Delta\mu \rangle}{\Delta t} f(\mu,t) \quad (12)$$

Where the Fokker-Planck coefficients $\frac{\langle \Delta\mu \rangle}{\Delta t}$ and $\frac{\langle \Delta\mu\Delta\mu \rangle}{2\Delta t}$

are defined by

$$\frac{\langle \Delta\mu \rangle}{\Delta t} \equiv \frac{1}{\xi \Delta t} \int \Delta\mu \, p(\mu, \Delta\mu) \, d\Delta\mu \qquad (13)$$

$$\frac{\langle \Delta\mu \Delta\mu \rangle}{\Delta t} \equiv \frac{1}{2\xi \Delta t} \int \Delta\mu \Delta\mu \, p(\mu, \Delta\mu) \, d\Delta\mu \qquad (14)$$

So far we have left the properties of $p(\mu, \Delta\mu)$ unspecified. However, let's assume (going back to Jokipii 1966) that it drives f toward isotropy and that if f is isotropic scattering produces no further evolution. Unpacking the derivatives on the RHS of eqn (12) and grouping them gives

$$\frac{\partial f(\mu,t)}{\partial t} = f(\mu,t) \frac{\partial}{\partial \mu}\left[-\frac{\langle \Delta\mu \rangle}{\Delta t} + \frac{\partial}{\partial \mu} \frac{\langle \Delta\mu \Delta\mu \rangle}{2\Delta t}\right] + \frac{\partial f}{\partial \mu}\left[-\frac{\langle \Delta\mu \rangle}{\Delta t} + \frac{\partial}{\partial \mu} \frac{\langle \Delta\mu \Delta\mu \rangle}{\Delta t}\right] \qquad (15)$$

$$+ \frac{\partial^2 f}{\partial \mu^2} \frac{\langle \Delta\mu \Delta\mu \rangle}{2\Delta t}$$

We see that $p(\mu, \Delta\mu)$ preserves isotropy only if

$$-\frac{\langle \Delta\mu \rangle}{\Delta t} + \frac{\partial}{\partial \mu} \frac{\langle \Delta\mu \Delta\mu \rangle}{2\Delta t} \equiv 0 \qquad (16)$$

Imposing (16), we can rewrite (15) as

$$\frac{\partial f(\mu,t)}{\partial t} = \frac{\partial}{\partial \mu} \frac{\langle \Delta\mu \Delta\mu \rangle}{2\Delta t} \frac{\partial f}{\partial \mu} \qquad (17)$$

i.e. diffusion in μ space. Notice however that diffusion is always accompanied by drag.

Derivation of a <u>spatial</u> diffusion equation:

Let's assume $\vec{B}$ is along $\hat{z}$ and make a 1D model of evolution that includes pitch angle scattering.

$$\frac{\partial f}{\partial t} + \mu v \frac{\partial f}{\partial z} + q\frac{(\vec{v}\times\vec{B})}{c}\cdot\frac{\partial f}{\partial \vec{p}} = \frac{\partial}{\partial \mu}\frac{\langle \Delta\mu\Delta\mu\rangle}{2\Delta t}\frac{\partial f}{\partial \mu} \qquad (18)$$

The $\frac{\partial f}{\partial \vec{p}}$ term vanishes for a <u>gyrotropic</u> distribution, i.e.

$$\frac{\partial f}{\partial p_x} = \frac{p_x}{p_\perp}\frac{\partial f}{\partial p_\perp}, \quad \frac{\partial f}{\partial p_y} = \frac{p_y}{p_\perp}\frac{\partial f}{\partial p_\perp}$$

so we can omit the Lorentz force. Let's also save writing by defining

$$D_{\mu\mu} = \frac{\langle \Delta\mu\Delta\mu\rangle}{2\Delta t} \equiv \frac{\nu(1-\mu^2)}{2}$$

$$\frac{\partial f}{\partial t} + \mu v \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\nu\frac{(1-\mu^2)}{2}\frac{\partial f}{\partial \mu} \qquad (19)$$

(it can be shown that $\nu \equiv \frac{\langle(\Delta\theta)^2\rangle}{\Delta t}$ where $\theta \equiv \cos^{-1}\mu$.)

There are 3 timescales in ~~eqn~~ (19)

ν^{-1} (scattering time)

$\frac{L}{v}$ (dynamical time; L is a typical gradient)

t (evolution time).

We know cosmic rays are well scattered & confined for much longer than ~~their~~ light travel time, so let's assume

$$t \gg \frac{L}{v} \gg \frac{1}{\nu}$$

Then, to lowest order in the scattering time, (19) becomes

$$0 \simeq \frac{\partial}{\partial \mu} \nu \frac{(1-\mu^2)}{2} \frac{\partial f_0}{\partial \mu} \qquad (20)$$

i.e. f is isotropic. To next order

$$\mu v \frac{\partial f_0}{\partial z} = \frac{\partial}{\partial \mu} \nu \frac{(1-\mu^2)}{2} \frac{\partial f_1}{\partial \mu} \qquad (21)$$

Integrate (21) over μ

$$\int_\mu^1 \mu' v \frac{\partial f_0}{\partial z} d\mu' = \int_\mu^1 \frac{\partial}{\partial \mu'} \nu \frac{(1-\mu'^2)}{2} \frac{\partial f_1}{\partial \mu'} d\mu' \qquad (22)$$

$$\frac{(1-\mu^2)}{2} v \frac{\partial f_0}{\partial z} = -(1-\mu^2)\frac{\nu}{2} \frac{\partial f_1}{\partial \mu} \qquad (23)$$

$$\frac{\partial f_1}{\partial \mu} = -\frac{v}{\nu} \frac{\partial f_0}{\partial z} \qquad (24)$$

This lets us write the diffusion eqn. as

$$\frac{\partial f_0}{\partial t} - v\mu \frac{\partial}{\partial z}\left(-\frac{v\mu}{\nu} \frac{\partial f_0}{\partial z}\right) = \frac{\partial}{\partial \mu} \frac{(1-\mu^2)}{2} \frac{\partial f}{\partial \mu} \qquad (25)$$

Averaging over μ,

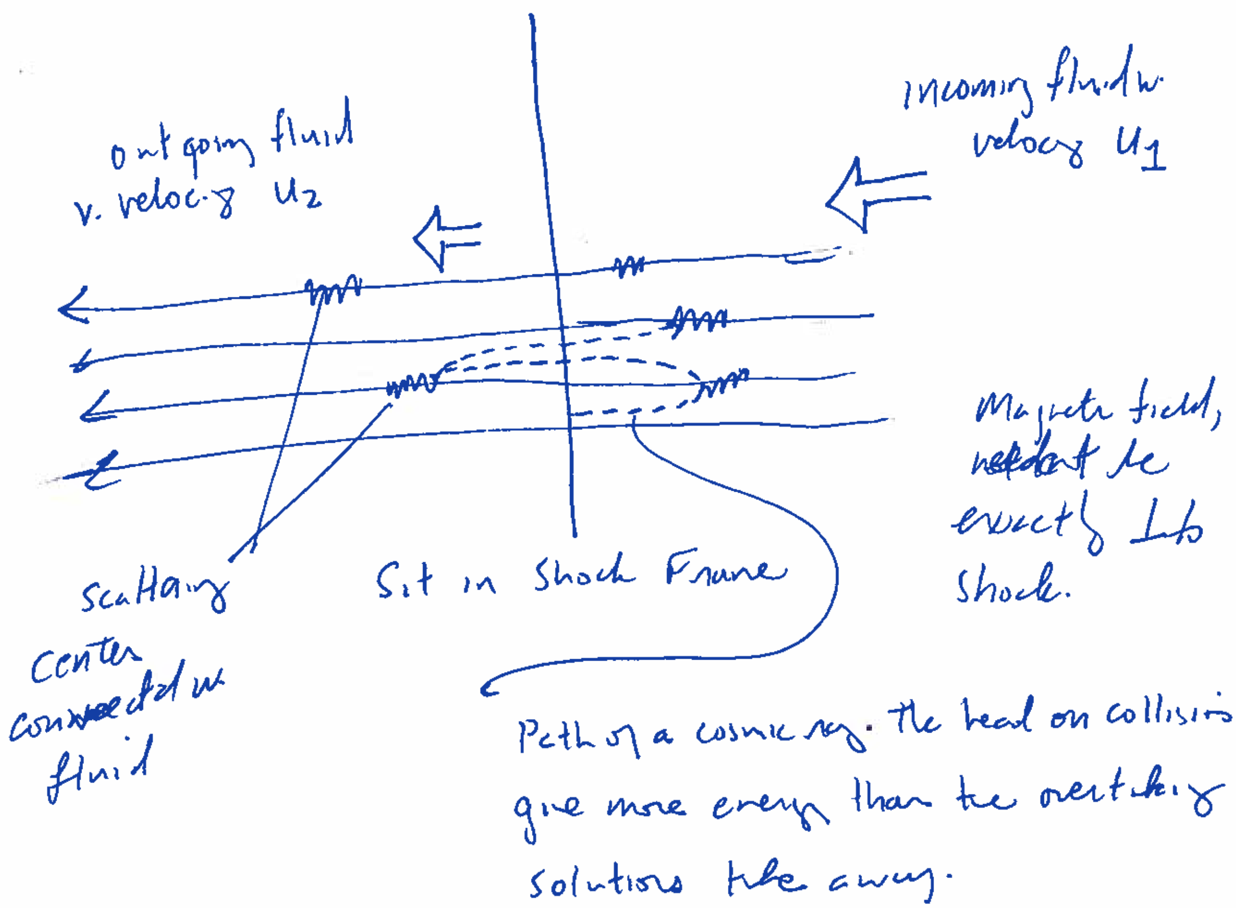
$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial z} \frac{v^2}{3\nu} \frac{\partial f}{\partial z} = 0 \qquad (26)$$

i.e. the coefficient of spatial diffusion along the magnetic field is related to the pitch angle scattering frequency as

$$\boxed{D_{\parallel} = \frac{v^2}{3\nu}} \qquad (27)$$

Application to diffusive shock acceleration

This follows the early treatment of Blandford & Ostriker (1978). Separate work by Tony Bell, PhD thesis (1978). Much work since 1960's suggesting shocks could be efficient particle accelerators.



First order Fermi process

What is the spectrum?

Assume the cosmic rays are convected with the fluid due to frequent scattering. The transport eqn. (27) that we derived generalizes to

$$\frac{\partial f}{\partial t} + u\cdot\nabla f - \frac{p}{3}(\nabla\cdot u)\frac{\partial f}{\partial p} = \nabla\cdot D_{\parallel}\hat{b}\hat{b}\,\nabla f \qquad (28)$$

Why is this right? To see why, ignore diffusion momentarily, and integrate over p-space, assuming angular isotropy (this is in fact an equation for the isotropic part of f).

$$\int p^2 dp\left[\frac{\partial f}{\partial t} + u\cdot\nabla f - \frac{p}{3}(\nabla\cdot u)\frac{\partial f}{\partial p}\right] = 0$$

Use $\int p^2 dp\, f = n_{cr}$

$$\frac{\partial n_{cr}}{\partial t} + u\cdot\nabla n_{cr} = (\nabla\cdot u)\int \frac{p^3}{3}\frac{\partial f}{\partial p}dp = -(\nabla\cdot u)\int p^2 f\, dp$$

$$= -n_{cr}\nabla\cdot u$$

which is the standard continuity equation.

Next, multiply by $\frac{pv}{3} \sim \frac{cp}{3}$ and integrate, using

$$P = \frac{1}{3}\int d^3p\; p v f$$

The result is

$$\frac{\partial P_{cr}}{\partial t} + u\cdot\nabla P_{cr} = \frac{\nabla\cdot u}{3}\int p^2 \frac{cp^2}{3}\frac{\partial f}{\partial p}\,dp = -\frac{4}{3}\nabla\cdot u\int p^2 \frac{cp}{3} f\,dp$$

$$= -\frac{4}{3} P\,\nabla\cdot u$$

which is the pressure evolution equation for an ultra relativistic gas.

Back to the shock problem. Assume a steady state, let $\nabla \to z$, and idealize the shock as a discontinuity. Eqn. (28) becomes

$$u\frac{\partial f}{\partial z} - (u_1 - u_2)\delta(z)\frac{p}{3}\frac{\partial f}{\partial p} = \frac{\partial}{\partial z} D_{\parallel}\frac{\partial f}{\partial z} \qquad (29)$$

Solve on either side of the shock & connect the results by applying jump conditions. Since $\nabla\cdot u = 0$ upstream & downstream,

$$u_1 f_1 - D_1\frac{\partial f_1}{\partial z} = \text{constant}$$

$$u_2 f_2 - D_2\frac{\partial f_2}{\partial z} = \text{constant.}$$

The general solution of $\quad uf - D\frac{\partial f}{\partial z}$

is

$$A + B e^{\int^z dz' \frac{u}{D}}$$

In our problem, $u<0$ so in region 2 ($z<0$), the solution exponential increases unless $B \equiv 0$.

Thus, in region 2 we set $f = f_2$

While in region 1 we set

$$f = f_1 + (f_2 - f_1)\, e^{\int_0^z dz' \frac{u_1}{D_1(z')}} \qquad (30)$$

which corresponds to exponential decay away from the shock over a lengthscale D_1/u_1, while f is continuous across the shock.

It remains to calculate the momentum spectrum. We do this by a Gaussian pillbox argument, integrating eqn (29) across the shock.

The result is

$$(u_1 - u_2)\frac{p}{3}\frac{\partial f}{\partial p} + D\frac{\partial f}{\partial z}$$

is continuous. Or,

$$\frac{(u_1 - u_2)}{3}\, p\frac{\partial f_2}{\partial p} + u_1 f_2 = 0$$

Here I've assumed the upstream injection spectrum f_1 is negligible at the energies considered here.

u_1 & u_2 are related by the shock compression ratio $r = \frac{\rho_2}{\rho_1} = \frac{u_1}{u_2}$

$$p\frac{\partial f_2}{\partial p} = \frac{-3u_1 f_2}{u_1(1-\frac{1}{r})} = -\frac{3r}{r-1} f_2$$

$$f_2 \sim p^{-3r/r-1}$$

For a strong shock in a $\gamma = 5/3$ gas, $r \rightarrow \frac{\gamma+1}{\gamma-1} = 4$

$$f_2 \propto p^{-4}$$

Which is related to the energy distribution function by

$$N(E)\,dE = f(p)\,p^2\,dp$$

$$N(E) = f(p)\,p^2\,\frac{dp}{dE} \sim E^{-2} \quad \text{for } E = cp.$$