Holographic Entanglement Entropy, PiTP 2018
Matthew Headrick

I. EEs in general QFTs — basic facts + examples

- Natural “subsystem” in relativistic QFT:
  causal domain $D(A)$ of spatial region $A$
  on Cauchy slice

Every causal curve in $D(A)$ intersects $A$
Every op $O$ in $D(A)$ can be written by Heisenberg
EOM in terms of ops in $A$

$\Rightarrow \langle O \rangle$ determined by $\rho_A$

$D(A') = D(A) \Rightarrow \rho_{A'} = \rho_A \Rightarrow S(A') = S(A)$
\[ \rho_A = \frac{1}{2} e^{-2\pi K} \]

\( K = \) generator of boosts in \( x-t \) plane within \( D(A) \)

\[ = \int_0^\infty dx \, x \, T_{tt} \]

Thermal state at temp \( \frac{1}{2\pi} \) w.r.t Hamiltonian \( K \)

Local physical temp \( T(x) = \frac{1}{2\pi x} \) (Unruh)

Estimate of \( S(A) \) for field of mass \( m \)

Entropy density

\[ s(T) = \begin{cases} 0, & T < m \\ \# T, & T > m \end{cases} \]

\[ S(A) \approx \int_0^\infty dx \, s(T(x)) = \int_0^\infty dx \, s(\frac{1}{2\pi x}) \approx \int_0^\infty dx \, \frac{1}{x} \]

\[ \xi = \frac{1}{m} = \text{correlation length} \]

Only fields within distance \( \xi \) of entanglement surface
are significantly entangled
diverges $\Rightarrow$ cut off at $x = \epsilon$
fIELDS close to a highly surface are very entangled!

$$S(A) \approx \int_0^\epsilon \frac{dx}{x} = \ln \frac{3}{\epsilon}$$

If the theory has UV fixed pt that is a CFT, then for $T \gg \mu$, $s(T) = \frac{2\pi c}{6} T$, so

$$S(A) = \frac{c}{6} \ln \frac{3}{\epsilon} + \ldots$$

In higher $D$, $x^i, \cdots$ in terms of area $\sigma$, $A = \{x^i > 0\}$,

$$S(A) \propto \frac{\sigma}{\epsilon^{D-2}}$$

$\epsilon$ coeff. is not meaningful
Near any smooth ext. surf., $A$ looks like half-space

$\Rightarrow$ $S(A)$ UV divergent like $\frac{\sigma}{\epsilon^{D-2}}$

• 2d CFT: 1 interval

$\epsilon \to \infty \Rightarrow$ IR divergence
cut off by considering interval $[x]$
2 endpoints \[ S(A) = \frac{c}{3} \ln \frac{1}{\epsilon} + \ldots \]

\[\text{dim. analysis} \Rightarrow S(A) = \frac{c}{3} \ln \frac{1}{\epsilon} + \text{non-universal const.} \]

I can be checked by variational calculus, which give zero in to allow generalizations.

Now suppose CFT is at temp $T$:

\[ S(A) = \frac{c}{3} \ln \frac{\sinh \pi TL}{\pi T \epsilon} \]

\[ S(A) \]

\[ \log \downarrow \]

\[ \epsilon \ln \text{circ} \text{ slope } = \frac{2\pi}{6} \epsilon TL = s(T) \]

CFT on $S'$ in vac:

\[ S(A) = \frac{c}{3} \ln \left( \frac{2R}{\epsilon} \sin \frac{L}{2R} \right) \]

\[ S(A) \]

\[ \log \downarrow \]

\[ 2\pi R \] \[ \Rightarrow L \]

Note: \[ S(A^c) = S(A) \]

(pure state)

\[ \frac{dS(A)}{dL} < 0 \text{ for } L > \pi R \]

\[ \Rightarrow \text{cond. ent. } < 0 \]
• Capped theory: interval

Expect

\[ S(A) = \begin{cases} \frac{\varepsilon}{3} \ln \frac{L}{\varepsilon} & L \ll \varepsilon \\ \frac{\varepsilon}{3} \ln \frac{3}{\varepsilon} & L \gg \varepsilon \end{cases} \]

\[ \varepsilon \to 0 \]
\[ \varepsilon \to \infty \]

< only these parts are entangled

\[ S(A) \]
\[ \begin{array}{c}
\frac{\varepsilon}{3} \ln \frac{3}{\varepsilon} \\
\varepsilon
\end{array} \]
\[ \frac{\varepsilon}{3} \begin{array}{c}
\ln \frac{3}{\varepsilon} \\
\varepsilon
\end{array} \]
\[ \to L \]

(checked semi-analytically for free bosons + fermions)

(Note that all curves so far have been concrete \( \subset SSA \))

• CFT: 2 in intervals

\[ I(A \cup B) = S(A) + S(B) - S(A \cup B) \]

Only calculated exactly for free fermions:

\[ I(A \cup B) = \frac{1}{\varepsilon} \ln \frac{(L_A + \Delta)(L_B + \Delta)}{\Delta(L_A + L_B + \Delta)} \]
Qualitative features hold in any CFT (+ more generally)
- Finite (volume of $\Delta$) $\leq$ diameter at local at
east points
- $\geq 0$ (makes sense: $\geq 0$ by subadditivity, if 0
then $\exists \alpha^2 \neq 0 \beta^2 \Rightarrow$ no correlations)
- Conformally invariant, i.e. fun of cross-ratio of
east points ($\leq$ finite)
- Non-decreasing fun of $L_B \quad (\leq SSA, I(A; B) > I(A; B))$
  $\Rightarrow$ non-increasing fun of $\Delta$ (makes sense, since $I(A; B)$
  measures correlation)
- $\to \infty$ as $\Delta \to 0$
- $\to 0$ as $\Delta \to \infty$ (are determined by highest
  normality, etc)
II. Ryu-Takayanagi formula

- Motivate (a historically) for an example we understand small:
  \[ \rho_A = \frac{1}{Z} e^{-\beta H} = \frac{1}{Z} \sum_i e^{-\beta E_i} |i\rangle_A \langle i|_A \]

TFD: \[ |\text{TFD}\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\beta E_i/2} |i\rangle_A |i\rangle_{A^c} \]

A is another copy of same system.
Unique purification of \( \rho_A \) invariant under:
\[ A \leftrightarrow A^c \text{ exchange } \times T\text{-reversal} \]

By construction, \( S(A) = S_{\text{Ent}} \)

In holographic QFT, \( \rho_A \) is described for \( T \gg T_{\text{HPP}} \)
by black hole \( S(A) = S_{\text{BH}} = \frac{1}{4G_N} \text{area (horizon)} + O(G_N^0) \)

|TFD⟩ described by non-analytic exclusion:
So in this case,

\[ S(A) = \frac{1}{4} C_N \text{area } (\text{hor}) + O(C_N^0) \]

What distinguishes horizon?

Fixed point of Killing sym.

But if we want to generalize to examples without Killing symmetries, we need weaker condition:

Horizon = minimal area surface between \( A + A^c \)

(on five-slice)

What picks out five-slice? Symmetry: invariant under

five-reflection

Want to, + will, get rid of this symmetry assumption as well.

So we guess:

\[ S(A) = \frac{1}{4} C_N \text{area } (\text{in}(A)) + O(C_N^0) \]

\( \text{in}(A) = \text{min. area surf. on t-ref. invariant five-slice} \)

between \( A + A^c \)

What does “between” mean? It means \( \neq \) bulk
region $r(A)$ that interpolates between $A$ and $\partial(A)$, i.e. such that $\partial r(A) = A \cup -m(A)$

Mathematically, $m(A)$ is homologous to $A$ and $r(A)$ is called "homology region". (It's causal domain is "entanglement wedge")

In black hole example, $\partial A = \emptyset$

In general, $\text{homology} \Rightarrow \partial r(A) = \partial A$

given $\frac{1}{z^2}$ factor in metric, min. surf. ends

on body

Div. area

$$\sigma \int \frac{dz}{z^{D-1}}$$

$$\int dz = \frac{\sigma}{c^{D-2}} \quad (D > 2)$$

or $\ln \epsilon \quad (D = 2)$

Just as we saw in field theory!
First example of how this simple formula geometrizes physics of entanglement in field theory.
Each part of minimal surface represents a different aspect of entanglement.
Here, part of \( \omega(A) \) near body represents UV divergent entanglement across \( DA \).

- **Examples (D=2)***

  \[ \int_{\text{AdS radius}} dS^2 = \frac{L^2}{\ell^2} \left( dx^2 + dz^2 \right) \]

  \[ \frac{L}{\ell} = \frac{2}{3} \gamma \]

  \( \omega(A) = \text{semi-circle} \)

  \[ \text{length} = 2L \ln \frac{L}{\ell} \]

  \[ \Rightarrow S(A) = \frac{c}{2} \ln \frac{L}{\ell} \]

**Finite temp.:***

\[ S(A) = \frac{c}{3} \ln \frac{\sinh \frac{L}{\pi T \ell}}{\pi T \ell} \]
For $L \ll \frac{1}{r}$, $r(A)$ stays close to hole does not
"white" horizon $\Rightarrow$ vac. result

For $L > \frac{1}{r}$, $r(A)$ skims but does not intersect
horizon $\Rightarrow$ linear $s(t)$ growth of $S(A)$

Note that minimal surfaces are nested + do not
cross horizon

This nesting is general: $A = A' \Rightarrow r(A) < r(A')$
This gives us license to ignore what's on other side
of horizon. This is good — $S(A)$ should not depend
on choice of purification of $\Lambda$

There are reasons to believe that $r(A)$ is the dual
of $\Lambda$

Circle:

Again, reproduces
new result:

$$S(A) = \frac{c}{3} \ln \frac{2R}{\epsilon} \sin \frac{L}{2R}$$

(Actually, all of the examples so far are cheats, since
they are all black holes in different coordinate systems, so we're just using BH. But all the qualifiers tend to hold in style D and with less sym

How do we get \( S(A_c) = S(A) \) here? Because \( m(A_c) = -m(A) \). \( A \sim -A_c \), so if \( m \sim A \) then \( -m \sim A_c \)

But if shebe is mixed, we should not get \( S(A_c) = S(A) \) always.

Because of horizon, \( A \) is not homologous to \( A_c \).
So \( \omega(A) \), \( \omega(A^c) \) are in different topological classes \( \Rightarrow \) in general \( S(A) \neq S(A^c) \)

Capped theory

Toy model: cut off \( z \) at \( z = \frac{3}{2} \) 

well \( z = \frac{3}{2} \)

\( \eta \) does not carry entropy

\( \Rightarrow \) does not count as "body" in homology cond.

Half-line

\[
S(A) = \frac{1}{\epsilon} \int_\varepsilon^3 d\tilde{z} \frac{1}{z} = \frac{c}{6} \ln \frac{3}{\varepsilon}
\]

as expected from field theory.

\( \text{min. surf. is cut off by IR cutoff of bulk geom.} \)
\[
L < 3 : \quad S(A) = \frac{c}{5} \ln \frac{c}{e}
\]

\[
L > 3 : \quad S(A) = \frac{c}{5} \ln \frac{3}{e}
\]

Phase transition:

This is a large-\(c\) phase transition — like Hawken-Page, finite-volume \(\text{age } N\) deficient

Must be smoothed out by non-perturbative quantum corrections — not understood

2 intervals in CFT vac

2 locally minimal surfaces for \(AB\): 

\[
m(AB) = m(A) m(B)
\]
\( Q \) is smaller when \( A, B \) are far apart
\( Q \) when close together
\( \Rightarrow I(A; B) = 0 \)
\( \Rightarrow I(A; B) > 0 \)

Again, a large-\( c \) phase transition
Obeys all properties of \( I(A; B) \) predicted on
general grounds before, except \( I(A; B) > 0 \)

\[ I(A; B) = 0 \Rightarrow \rho_{AB} = \rho_A \otimes \rho_B \Rightarrow \langle O_A O_B \rangle_c = 0 \]
which is not true

Have to include quantum corrections

\[ I(A; B) = O(c^0) \]

\( c^0 \) can be calculated, or \( \neq 0 \)

Notice that existence of conf by \( Q \) enforces

\[ I(A; B) > 0 \text{ subadditivity} \]
- General properties

We already saw that the minimal surface automatically realizes two important properties of $EE$:

1. pure $\Rightarrow S(A) = S(A^c)$

2. $S(AB) \leq S(A) + S(B)$

Similarly, it automatically realizes the much more difficult strong subadditivity inequality:

$$S(AB) + S(BC) \geq S(B) + S(ABC)$$
\[ S(AB) + S(BC) = \frac{1}{4 \ln N} \left[ \text{area}(m(AB)) + \text{area}(\neg(BC)) \right] \]
\[ = \frac{1}{4 \ln N} \left[ \text{area} \left( \lnot(B) \right) + \text{area} \left( \lnot(ABC) \right) \right] \]
\[ = \frac{1}{4 \ln N} \left[ \text{area} \left( \lnot(B) \right) + \text{area} \left( \lnot(ABC) \right) \right] \]
\[ = S(B) + S(ABC) \]

Note: \( \lnot(B) \sim B \) via \( \lnot(B) = \lnot(AB) \cap \lnot(BC) \)
\( \lnot(ABC) \sim ABC \lor \lnot(ABC) = \lnot(AB) \cup \lnot(BC) \)

I. Mutual information (MMI):
\[ S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC) \]
\[ \text{i.e.} \quad I(A; BC) \geq I(A; B) + I(A; C) \]

e.g. \[ \begin{array}{ccc}
A & \quad & B \\
\quad & \quad & C
\end{array} \]
Ex: Can happen that $I(A; B) = I(A; C) = 0$ and $I(A; BC) > 0$

Does not generally hold for quantum or even classical states, e.g. $\rho_{ABC} = \frac{1}{2} (|100⟩⟨100| + |111⟩⟨111|)$
III. Covariant holographic EE

So far we've restricted to T-reversal invariant spacetimes and regions on fixed slice. (Note that this includes all static spacetimes, which is most common case studied.)

In general a special slice $\Sigma$ on which we can look for a minimal surface. Can try simply

$$m(A) = \min \text{ surf. } \sim A$$

without restriction to any slice.
Doesn't work — can make area arbitrarily small by wigglng in 4 directions

Two possible natural ways to resolve this:

1) $m(A) = \text{extremal surf}$

   : = \text{extremum of area functional}

   if \exists several, take 1 least area

2) other ways to express extremal:

a) expansion of

   will geodesics = 0

   in all 4 directions

b) extrinsic curvature tensor

   $K_{\mu \lambda}$ with

   $K_{\mu \lambda} g^{\mu \lambda} = 0$

   w.r.t. to $u$

   (generalization of $K = 0$ for hypersurfaces)

Hilbert-Rongamuli-Takahayagi (HRT) formula
bifurcate horizon is extremal so this case works

Notice that it depends only on $D(A)$

2) Minimize over over space, maximize over time

$$\max_{\sigma \in \mathcal{A}} \min_{\nu \in \mathcal{M}_\nu} \mu(\nu)$$

bulk Cauchy slice (again, depends only on $D(A)$)

If $\sigma$ wiggles in time, $\min_{\nu \in \mathcal{M}_\nu} \mu(\nu)$ will be small, so $\sigma$ has to be smooth

$$\min_{\nu \in \mathcal{M}_\nu} \mu(\nu) = \frac{4\pi r^2}{c^4}$$
"Maximin"

Both seem reasonable. Which is right?

Luckily we don't have to decide — they're equivalent.

Specifically, Wall showed:

1) A maximin surface exists (under certain conditions)

2) The maximin surf. is the HRT surf.

Bonus: HRT surf. exists!

We won't discuss proof of (1).

Proof of (2) is nice; we'll give outline:

Two parts:

2a) Maximin surf. is external

   Inference: extremum w. variations in space +
   for directions ⇒ w. variations in all directions

2b) Any other external surf. m. has

   \[ \text{area}(m) \geq \text{area}(m_0) \]
Lemma: \( D(y) \) meets belly on \( D(A) \)

\[ m_2 \sim D(A) \cap \sigma_0 \quad \text{via} \quad r_a := D(y) \cap \sigma_0 \]

By Raychaudhuri + Einstein + NEC, \( \Theta \leq 0 \)

but \( \Theta = 0 \) on \( m_1 \) \[ \implies \Theta \leq 0 \implies \text{area}(m_2) \leq \text{area}(m_1) \]

since \( m_0 \) is minimal on \( \sigma \), \( \text{area}(m_2) \geq \text{area}(m_0) \)

By a similar argument, one shows that \( \text{HRT} \Rightarrow \text{RT} \)

RT surface is automatically external

Can \( \exists \) external surf off + reflection invariant slice?

\[ m_2 \sim A \]
\[ \text{area}(m_2) \geq \text{area}(m_1) \]
\[ \text{area}(m_2) \geq \text{area}(m_1) \]

SSA: static proof doesn't apply because
\( m(AB), m(BC) \) aren't on same slice

Lemma: \( \exists \) common maximal slice \( \sigma \) for

\[ B + ABC \ (\text{resty of maximal slices} \Rightarrow \text{resty of entanglement wedge(s)}) \]

On \( \sigma \), find \( \text{min. area} \) \( m(AB), m(BC) \)

By RT SSA argument,

\[ \text{area}(m(AB)) + \text{area}(m(BC)) \geq S(\text{CD}) + S(ABC) \]

By maximin,

\[ S(\text{AD}) \geq \text{area}(m(AB)) \]

\[ S(\text{BC}) \geq \text{area}(m(BC)) \]

\[ \text{causality} \]

On \( \text{hdy} \)

\[ J^+(\text{DA}) \]

\[ J^-(\text{DA}) \]

\[ A^c \]

\[ D(\text{A}) \]

\[ D(\text{A}^c) \]

\[ \text{entanglement horizon} \]

\[ m(\text{DA}) \cap \text{hdy} = J^+(\text{DA}) \]
= only bulk signals sent from $\tilde{r}(A')$ can reach $m(A)$, possibly changing $S(A)$

so HRT respects bulk causality

(essentially a version of Cao-Wald for external surfaces)

In spacetime $\Sigma$, black hole formed from collapse,

$S(A') = S(A)$: $\Sigma \neq \emptyset$ by bulk Cauchy slice

$\Sigma = \text{bulk Cauchy slice}$  $\Rightarrow A^c \approx A$

In particular, $S(\Sigma) = 0$

More generally, since $S(A)$ depends only on $D(A)$, if

$A = \text{whole connected component}$ then $S(A) > t$-indep

So HRT gives entropy not unitarily-evolving

"fine-grained" density matrix