INTRODUCTION TO THE SYK MODEL

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REFERENCES:

- SACHDEV-YE COND-MAT/9212030
- KITAEV - KITP TALK APRIL, 2015 & MAY 2015 (@ KITP TALK'S ARCHIVE)
- KITAEV-SUH - 1711.08467
- J.M. & STANFORD - 1604.07818
- ROSENHAUS - 1807.03334

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MOTIVATIONS :

1 - SOLVABLE IN LARGE N LIMIT

2 - - THERMALIZATION

  - CHAOS (SOLVABLE WITH CHAOS)

3 - SIMILAR TO NEAR EXTREMAL BLACK HOLES.

  → SAME PATTERN OF SYMMETRIES.

(3)

LARGE N MODELS.

	ANOMALOUS DIM OF $j>2$ OPERATORS	GRAVITY DUAL
– $O(N)$ VECTOR MODELS. e.g. $\int (\partial\vec{\phi})^2 + g^2((\vec{\phi})^2)^2$	$\gamma \sim \frac{1}{N}$	VASILIEV LIKE ?
– SYK, TENSOR MODELS.	$\gamma \sim 1$.	?

HARDER

– PLANAR GAUGE THEORIES.

e.g. LARGE N QCD

OR LARGE N, $\mathcal{N}=4$ SUSY YM $\longrightarrow$ $\Delta \sim g^2 N \gg 1$ $\leftarrow$ EINST GRAVITY.

$D\phi$ BRANE Q.M. $\rightsquigarrow$ Einst. gravity

④

SYK

HILBERT SPACE:

$\mathcal{H}$ = GENERATED BY N-MAJORANA FERMIONS ψ^i

$\{\psi^i, \psi^j\} = \delta^{ij} \qquad i,j = 1,\dots,N \,, \quad \psi_i^\dagger = \psi_i \qquad [(-1)^F, \psi^i] = -\psi^i$

N EVEN.

- $\mathrm{Dim}\,\mathcal{H} = 2^{N/2}$ → EXPONENTIAL IN N, BUT FINITE.

- ALL OPERATORS IN H

  ~~SPANNED BY~~ PRODUCTS OF $\psi_1^{s_1} \dots \psi_N^{s_N}$

- TO PUT ON COMPUTER: ψ_i → LIKE γ_i-MATRICES.

HAMILTONIAN

$$H = \sum_{i<j<k<l} j_{ijkl} \, \psi^i \psi^j \psi^k \psi^l.$$

↓

GAUSSIAN RANDOM VARIABLES, TIME INDEPENDENT.

$\langle j_{ijkl}^2 \rangle = J^2 / N^3$ → FOR CONVENIENCE

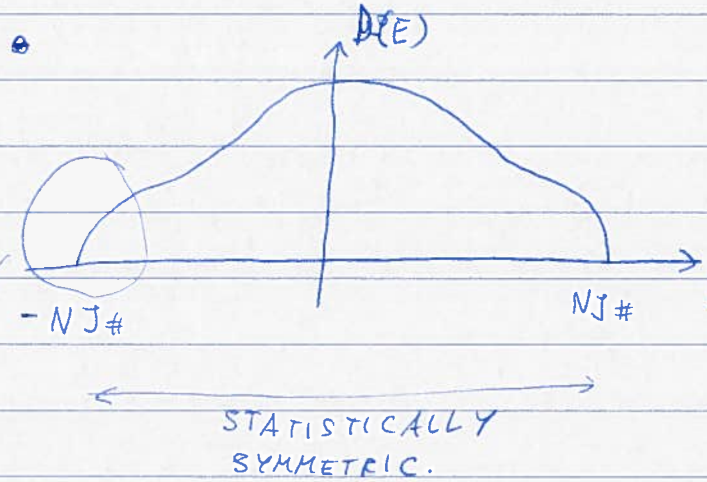
J → DIMENSIONS OF ENERGY.

(GENERALIZATION:

$$H = i^{(q/2)} \sum_{\{i_k\}} j_{i_1 \dots i_q} \psi^{i_1} \dots \psi^{i_q} \,, \quad \langle j_{i_1 \dots i_q}^2 \rangle \propto \frac{J^2}{N^{q-1}}$$
)

(5) SPECTRUM.

- DISCRETE → EXPONENTIALLY MANY LEVELS.



- WE STUDY LOW ENERGIES.
- BUT KEEP EXPONENTIALLY MANY LEVELS

$\begin{matrix} tJ \\ \beta J \end{matrix} \ll 1$ → WEAKLY COUPLED (LIKE $H=0$).

$\begin{matrix} tJ \\ \beta J \end{matrix} \gg 1$ → STRONGLY COUPLED

- $N \gg \begin{matrix} tJ \\ \beta J \end{matrix} \gg 1$ → LARGE N SOLUTION.

FOR tJ OR $\beta J \sim e^N$ → NEED NEW TECHNIQUES. NOT DISCUSSED HERE.

- TO LEADING ORDERS IN $1/N$ → CAN TREAT THE DISORDER AS AN EXTRA FIELD.

⇒ FLUCTUATIONS BETWEEN ≠ DISORDER REALIZATIONS ARE SMALL

(6)

- ONE CAN SHOW THAT A SIMPLE CLASS OF DIAGRAMS CAN BE RESUMED.

- INSTEAD WE WILL DERIVE THE SAME FROM A FUNCTIONAL INTEGRAL ARGUMENT.

$$Z = \int \mathcal{D}j \int \mathcal{D}\psi \, e^{\int dt \left[i\psi^j \partial_t \psi^j + j_{jklm} \psi^j \psi^k \psi^l \psi^m \right] - j^2 \frac{N^3}{J^2}}$$

j = TIME INDEPENDENT.

INTEGRATE OUT j

$$Z = \int \mathcal{D}\psi \, e^{\int dt \, \psi^j \dot{\psi}^j + \int dt dt' \left(\frac{\psi^j(t)\psi^j(t')}{N} \right)^4 \times J^2 N}$$

INSERT 1

$$1 = \int \mathcal{D}\tilde{\Sigma} \mathcal{D}\tilde{G} \, e^{i \iint \tilde{\Sigma}(t,t') \left(\tilde{G}(t,t') - \frac{1}{N} \sum_j \psi^j(t) \psi^j(t') \right) \times N} = \int \mathcal{D}G \, \delta\left(G - \sum_i \psi^i \psi^i\right)$$

- REPLACE

$$\left(\frac{\sum_i \psi^j \psi^j}{N} \right)^4 \to G^4(t,t')$$

$$Z = \int \mathcal{D}\tilde{\Sigma} \mathcal{D}\tilde{G} \int d\psi \, e^{\int dt \, \psi^j \dot{\psi}^j - \iint \psi^j(t) \psi^j(t') G(t,t') - \frac{N}{2} \int \Sigma G - \frac{J^2}{4} G^4}$$

QUADRATIC → INTEGRATE OUT

GET →

(7)

$$Z = \int \mathcal{D}\tilde{\Sigma}\,\mathcal{D}\tilde{G}\; e^{N\left\{\log \mathrm{Pf}(\partial_t - \tilde{\Sigma}) - \frac{1}{2}\int d\tau_1 d\tau_2 \left[\tilde{\Sigma}(t_1,t_2)\tilde{G}(t_1,t_2) - \frac{J^2}{4} G(t_1,t_2)^4\right]\right\}}$$

REMARKS

(1) N FACTOR $\to$ CLASSICAL, $N = \frac{1}{\hbar}$ AS A PARAMETER OF THIS THEORY

(2) N FERMIONS FUNTION OF 1 VARIABLE

$\rightsquigarrow \tilde{G}, \tilde{\Sigma}$ FUNCTIONS OF 2- VARIABLES.

(3) CLASSICAL EQUATIONS. $\to$ EQNS RESOMING FEYNMAN DIAG.

(WE WILL DISCUSS THEM MORE LATER).

(4) INSERTING IN ACTION $\quad Z = e^{\underbrace{N f(\beta J)}_{-\beta F}}$ $\leftarrow$ FREE ENERGY.

$\downarrow$

TRANSLATES INTO STATEMENTS OF SPECTRUM.

(5) $\tilde{G}, \tilde{\Sigma}$ $\rightsquigarrow$ LIKE THE "BULK GRAVITY"

- EXTRA DIMENSION (KINEMATIC SPACE = SPACE OF PAIRS OF POINTS)

- $G_N \sim 1/N \sim \hbar$

- NON-LOCAL

- WE WOULD LOVE TO GET BULK EINSTEIN'S EQUATIONS FROM $\mathcal{N}=4$ SYM BY A SIMILAR MANIPULATION!

(6) FOR $O(N)$ VECTOR MODELS (SAY ϕ^4)

SIMILAR STRUCTURE BUT $G(t,t')^4 \to \underset{\text{SAME } t = t'}{G(t,t)}$

$\Rightarrow \Sigma(t,t') = \delta(t-t')\,\Sigma(t,t) \to$ FUNCTIONS OF 1 VARIABLE

BECAUSE THE FIELD WE INTEGRATE OUT IS TIME DEPENDENT

$\phi^4 \to \lambda(t)\phi^2 + \lambda^2(t)/g^2$

7) GENERALIZES TO ANY DIMENSION OR BOSONS.

$$\int dt \to \int d^D x$$

$$\int dt\, \psi \dot{\psi} \to \int d^D x\, \psi \not{\partial} \psi \quad \text{or } (\partial\phi)^2.$$

$$\int dt_1\, dt_2\, G(t_1, t_2)^q \to \int d^D x_1\, d^D x_2\, G(\vec{x}_1, \vec{x}_2)^q$$

2 POINTS IN D DIMENSION

→ WE CAN ALSO DO A LATTICE OF SYK MODELS

→ INTERESTING FOR TRANSPORT QUESTIONS

& COND. MATT. APPLICATIONS.

– TENSOR MODELS WITHOUT DISORDER

9 CLASSICAL EQUATIONS

$$\frac{1}{\partial_t - \Sigma} = G \longrightarrow \frac{1}{i\omega - \Sigma(\omega)} = G(\omega)$$

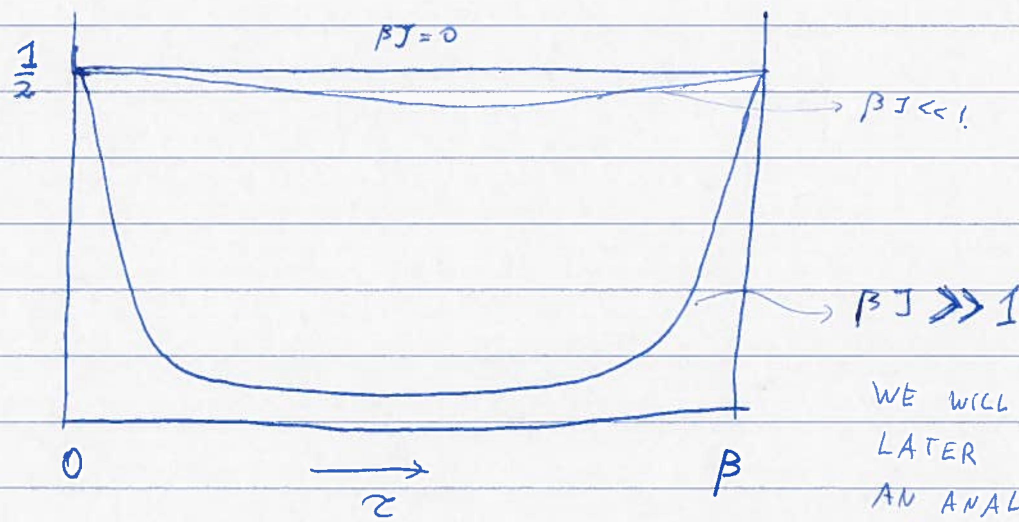
$$\Sigma(t_1,t_2) = J^2 G^3(t_1,t_2) \longrightarrow \Sigma(t_{12}) = J^2 G^3(t_{12})$$

TRANSL. SYMMETRY

- $G(t,t') = \frac{1}{N}\sum_i \langle \psi_i(t)\psi_i(t')\rangle \sim \langle \psi^i \psi^i \rangle$ (NO SUM).

- CAN BE SOLVED NUMERICALLY
  - SIMPLEST: IN EUCLIDEAN SPACE, FINITE TEMPERATURE

    $\tau = \tau + \beta$.



WE WILL LATER GIVE AN ANALYTIC FORMULA...

$G_{FREE} \sim \frac{1}{2}\,\mathrm{sign}(t-t')$ → FROM DEFINITION & TIME INDEPENDENT.

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LOW ENERGY, LOW TEMPERATURE ANALYSIS.

$1 \ll tJ \ll N.$, $1 \ll \beta J \ll N.$

STILL EXPONENTIALLY MANY STATES CONTRIBUTE.

$1 \ll Jt \ll J\beta$.

- SCALING ANSATZ.

$$G(\tau) = \frac{1}{\tau^{2\Delta}} \longrightarrow \Sigma(\tau) \propto G^3 \sim \frac{1}{\tau^{3.2.\Delta}}$$

FOURIER

$$G(\omega) \sim \omega^{2\Delta - 1} \qquad \Sigma(\omega) \sim \omega^{2.3.\Delta - 1}$$

EQN

$$(i\omega - \Sigma(\omega))G(\omega) = 1 \xrightarrow{\text{FIRST}} -\Sigma(\omega)G(\omega) = 1.$$

$$\omega^{2.4.\Delta - 2} \sim 1 \Rightarrow \boxed{\Delta = \frac{1}{4}}$$

CHECK $\omega < \Sigma(\omega) \sim \omega^{2.3.\frac{1}{4} - 1} \sim \omega^{\frac{1}{2}}$ FOR $\omega \to 0$. $\Rightarrow$ OK TO NEGLECT $i\omega$ vs $\Sigma(\omega)$

NEGLECTING $i\omega$ OR $\psi\dot{\psi}$ IN LAGR. $\rightarrow$ EQNS HAVE A SCALING SYMMETRY $\xrightarrow{?}$ CONFORMAL $\xrightarrow{?}$ REPARAMETRIZATION SYMMETRY

$T^{\mu}_{\mu} = 0 \rightsquigarrow T = 0 = H$ (SEEMS FUNNY, UNDERSTAND LATER)

(11)

$$-\Sigma(\omega)\, G(\omega) = 1 \quad \to \quad \Sigma * G = \delta$$
$$\Sigma = G^3.$$

INVARIANT UNDER.

$$G(t,t') \longrightarrow G_f(t,t') = \left(f'(t_1)\, f'(t_2)\right)^{\Delta} G\left(f(t_1), f(t_2)\right)$$
$$\Sigma \qquad\qquad\qquad 3\Delta \quad \Sigma'$$

- SYMMETRY OF THE LOW ENERGY EQUATIONS.

- $G = \dfrac{1}{|t_1 - t_2|^{2\Delta}}$

- IS <u>NOT</u> INVARIANT UNDER A GENERIC, f

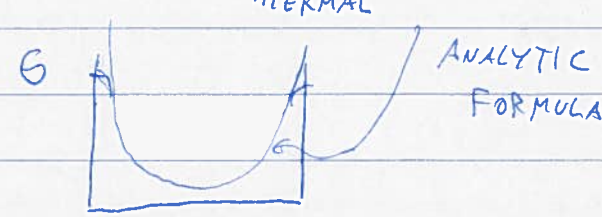
  (INV. UNDER SL(2).)

- $\mathrm{Diff}(S^1) \xrightarrow[\text{SPONTANEOUSLY}]{\text{BROKEN}} \mathrm{Diff}(S^1)/SL(2)$

  $f \to \dfrac{af+b}{cf+d}$

- ONE SIMPLE APLICATION.

  $f = \tan\left(\frac{\pi \tau}{\beta}\right)$ $\qquad \underbrace{\dfrac{1}{|t_{12}|^{2\Delta}}}_{\text{ZERO T.}} \longrightarrow G \sim \underbrace{\dfrac{1}{\left(\beta \sin \frac{\pi \tau}{\beta}\right)^{2\Delta}}}_{\text{THERMAL}}$



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INSERTING THIS INTO CLASSICAL ACTION

$$\log Z = \underbrace{\text{DIVERGENT}}_{N\beta J} + \underbrace{S_0(\Delta)}_{\text{CONSTANT}}$$

CONSTANT → "GROUND STATE ENTROPY"

$N\beta J$

J: CUTOFF SCALE FOR LOW EN. THEORY

GROUND STATE ENERGY.

$$= N\left[\#\beta J + s_0(\Delta)\right]$$

N: CLASSICAL PROP. TO N.

S_0 → DENSITY OF STATES.

$$\rho(E) \sim e^{\overbrace{S_0}^{N s_0}} f(E-E_0)$$

e^{S_0}: EXPONENTIAL CONTRIBUTION.

$f(E-E_0)$: eg $\sqrt{E-E_0}$

DOES NOT NECESARILY MEAN STATES WITH EXACTLY ZERO ENERGY (AS IN THE BPS CASE)

IN $N=2$ SUSY SYK → YES

$N=1$ " " → NO ~~SUSY~~

THE ZERO MODES OF LOW ENERGY ACTION POSE AN APPARENT PROBLEM.

$$\int \mathcal{D}G\,\mathcal{D}\Sigma \; e^{-N\underbrace{S_0}_{\text{CLASS.}} + \underbrace{(\delta\Sigma, \delta G)\,\boxed{S_2}\,(\delta\Sigma, \delta G)}_{\text{QUADRATIC ACTION.}}}$$

$\Sigma = \Sigma_{cl} + \delta\Sigma$.

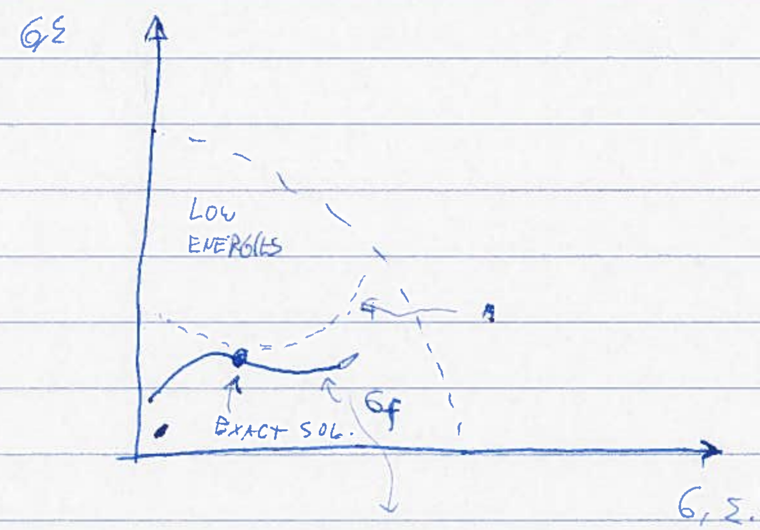
$G = G_{cl} + \delta G$.

$\Sigma, G \to G_f \quad \Sigma_f \quad \to$ ZERO MODES OF LOW ENERGY ACTION.

$\rightsquigarrow$ LEAD TO AN INFINITY.

(ORIGINAL THEORY WAS FINITE!)

$\rightsquigarrow$ NEED TO INCLUDE BREAKING OF REPARAM. SYMMETRY $\rightsquigarrow$ NOT EXACTLY ZERO MODES.



LOW ACTION VALLEY = VALLEY + SMALL POTENTIAL.

(14)

$$Z = \int \mathcal{D}G\,\mathcal{D}\Sigma \; e^{-S}$$

$$\approx \int \mathcal{D}f \underbrace{\int \mathcal{D}G_\perp \mathcal{D}\Sigma_\perp}_{\text{CONF INVARIANT.}} \; e^{-S[(G_0+\delta G_\perp)_f, \dots]}$$

$$\approx \int \mathcal{D}f \; e^{-\underbrace{S[(G_0)_f]}}$$

$$\uparrow$$

$$e^{N A_0 + \underbrace{S[f]}_{\text{ACTION FOR } f}}$$

- $S[f]$: - COMES FROM EFFECTS BREAKING REPARAM. SYMMETRY
  - SHORT t_{12}.
  - $\to$ LOCAL ACTION.
  - SHOULD BE INVARIANT UNDER $SL(2)$ $f \to \frac{af+b}{cf+d}$.

    "GAUGE SYMMETRY"

- SIMPLEST:

$$S = -\frac{N\alpha}{J}\int dt \; \{f(t), t\} \qquad \{f,t\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

N : CLASSICAL ACTION.

$\frac{1}{J}$ = DIMENSIONAL ANALYSIS.

α = NUMERICAL COEFICIENT THAT NEEDS TO BE COMPUTED.

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THIS EVEN CONTRIBUTES CLASSICALLY.

e.g. THERMAL PART FN.

$$\log Z = N\left[\beta J + \Lambda_0 + \# \frac{\alpha}{J\beta}\right]$$

$\int d\tau \{\tan \frac{\pi \tau}{\beta}, \tau\}$

→ LINEAR IN T NEAR EXTREMAL ENTROPY

- ALSO CONTRIB. TO CORRELATORS.

$$\langle 4pt \rangle = \int \mathcal{D}G\mathcal{D}\Sigma \; G(1,2)\, G(3,4)\; e^{S[G,\Sigma]}$$

LEADING ORDER IN βJ

$$\langle 4pt \rangle \sim \int \frac{\mathcal{D}f}{Vol(SL(2))} G_f(1,2)\, G_f(3,4)\; e^{+\frac{N\alpha}{J}\int \{f,t\}\, dt}$$

$$G_f(1,2) = \left[\frac{f'(1)f'(2)}{(f(1)-f(2))^2}\right]^{\Delta}$$

$$\langle 4pt \rangle = \langle 2pt \rangle\langle 2pt \rangle + \frac{1}{N}\left\{\left[\beta J \text{ OR } \beta t \times F \ldots\right] + \text{CONFOR}\right.$$

ENHANCED FOR LOW ENERGIES

FROM $G_\perp$

⇝ THERE ARE MANY LOW ENERGY CORRECTIONS DOMINATED BY THE SCHWARZIAN ACTION.

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- NEAR EXTREMAL ENTROPY ~ $S - S_0 \sim N \frac{T}{J}$ LINEAR.

- ENHANCED CORRECTIONS TO 4 pt FUNCTION.
  - INCLUDING CHAOS CONTRIBUTION.

$$\frac{\langle 4pt \rangle}{\langle 2p \rangle^2} \sim 1 - \frac{\beta J}{N} e^{\frac{2\pi t}{\beta}} \qquad \lambda_L = \frac{2\pi}{\beta}$$

- EXPONENTIAL TIME DEPENDENCE FROM SL(2) zero modes

- CHAOS FROM GEOMETRY OR SYMMETRIES ~ LIKE MOTION IN HYPERBOLIC SPACE.
(LYAPUNOV)

- QUANTUM TELEPORTATION IN THE TFD

- LOW ENERGY PHYSICS WHEN WE COUPLE 2 COPIES OF SYK ...