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# Exact Quantum Entropy of Black Holes

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Abstract:

Quantum entropy of a black hole is a quantum generalization of the celebrated Bekenstein-Hawking area formula. For supersymmetric black holes in string theory, quantum entropy can be placed in a broader context of quantum holography and defined in terms of a supergravity path integral in the near horizon spacetime. Quantum gravity corrections to the Bekenstein-Hawking formula in the bulk correspond to finite $N$ corrections in the boundary.

In these lectures I describe examples where both the bulk and boundary partition functions are computable exactly including all perturbative and nonperturbative corrections. Supersymmetric localization proves to be a valuable tool in these nonperturbative explorations of quantum gravity. Surprisingly, the supergravity path integral in the bulk evaluates to an integer in agreement with the boundary. This 'integrality from the bulk' provides highly nontrivial evidence for quantum holography and suggests intriguing connections with number theory and topology.

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## Contents



For a review of quantum black holes in string theory see [1, 2]; for more details of counting of black hole degeneracies see [3]; for a review of modular forms and the relevant number theory in connection with black hole counting see [4]; for a review of localization methods in quantum field theory see [5, 6, 7]. Relevant references for the main topics are described in the text below.

# 1. Lecture I

In the first lecture I will motivate the study of quantum entropy of black holes and then define it using $AdS_2/CFT_1$ holography. I will describe the counting of microscopic degeneracies of a class of black holes and aspects of analytic number theory that facilitate the comparison between the boundary and the bulk. Finally, I will review localization techniques in supersymmetric field theory and explain how they can be applied to the supergravity path integral to evaluate quantum entropy.

## 1.1 Quantum Entropy and $AdS_2/CFT_1$ Holography

For supersymmetric black holes, the near horizon spacetime contains an $AdS_2$ factor. Using the framework of $AdS_2/CFT_1$ holography one can define the exponential of the quantum entropy of a black hole with charge vector $\Gamma$ by a formal path integral $W(\Gamma)$ of massless supergravity fields [8, 9] in the bulk $AdS_2$. The action for the path integral is determined by the effective Wilsonian action obtained by integrating out the massive string fields. This definition is a generalization of the Bekenstein-Hawking-Wald entropy of a black hole in that it includes the nonlocal contributions from quantum loops of massless particles in addition to the local contributions from integrating out massive string states.

The path integral $W(\Gamma)$ in the bulk defined above is dual to the quantity $d(\Gamma)$ in the boundary which corresponds to the number of microstates of the black hole. This follows from the fact that the near horizon limit corresponds to the low energy limit in the boundary. The microstates of the supersymmetric black hole are separated from the excited states by a mass gap. Thus, the Hilbert space of the boundary theory is finite-dimensional and consists of the microstates of the black hole. Moreover, for a $CFT_1$, conformal invariance implies that the Hamiltonian is zero and hence the partition function $d(\Gamma)$ is simply the dimension of the Hilbert subspace corresponding to the number of black hole microstates.

In some situations the degeneracy $d(\Gamma)$ depends only on a single duality invariant combination of charges[1] which we denote by the integer $\Delta$.

## 1.2 Counting Functions for BPS Black Holes

The quantum states that correspond to supersymmetric black holes have a representation as bound states of branes at weak coupling. Using techniques from brane dynamics it has been possible to compute the indexed degeneracies of these bound states, and in a number of examples they are given by the Fourier coefficients of modular forms [4].

As a simple example, consider the half-BPS states of $\mathcal{N} = 4$ supersymmetric theory in four dimensions. The indexed partition function for these states is given by the partition function of 24 left-moving bosons of the heterotic string world-sheet [10, 11]:

$$Z(\tau) = \frac{1}{q} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{24}} \qquad (q := e^{2\pi i \tau})\,. \tag{1.1}$$

This function is a modular form of weight $-12$:

$$Z(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^{-12} Z(\tau) \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,, \tag{1.2}$$

and admits a Fourier expansion

$$Z(\tau) = \sum_{N=-1}^{\infty} c(N) q^N = q^{-1} + 24 + \dots\ . \tag{1.3}$$

[1] In general, it is necessary to specify additional arithmetic duality invariants which are more subtle to classify. One can choose charges for which these duality invariants are trivial.

The degeneracy of half-BPS states is given by $d(\Delta) = c(\Delta)$ with $\Delta$ is a duality invariant combination of the charges. In number theory, the partition function above is well-known in the context of the problem of partitions of integers. One can identify

$$c(N) \;=\; p_{24}(N+1) \qquad (N \geq 0)\,. \tag{1.4}$$

where $p_{24}(I)$ is the number of colored partitions of a positive integer $I$ using integers of 24 different colors.

Similarly the counting function for one-eighth BPS black holes in $\mathcal{N}=8$ is given by a weak Jacobi form [12] with a slightly more complicated Fourier expansion.

### 1.3 Hardy-Ramanujan-Rademacher Expansion

There is a beautiful result in 'analytic number theory' due to Hardy, Ramanujan, and Rademacher which allows one to express Fourier coefficients of a modular form as a convergent expansion in terms of complex analytic functions. A derivation can be found, for example, in [13, 14].

The main ideas can be explained concretely in the example above. The modular properties of $Z(\tau)$ and the fact that it has negative modular weight imply that $c(N)$ admits the Hardy-Ramanujan-Rademacher expansion for $N \geq 0$:

$$c(N) = \sum_{c=1}^{\infty} \left(\frac{2\pi}{c}\right)^{14} Kl(N,-1,c)\, I_{13}\left(\frac{4\pi\sqrt{N}}{c}\right)\,, \tag{1.5}$$

where

$$I_\rho(z) := \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{dt}{t^{\rho+1}}\exp[t+\frac{z^2}{4t}] \tag{1.6}$$

is the modified Bessel function of index $\rho$, and

$$Kl(n,m,c) := \sum_{\substack{d\in\mathbb{Z}/c\mathbb{Z}\\ da=1\,\mathrm{mod}(c)}} e^{2\pi i\left(n\frac{d}{c}+m\frac{a}{c}\right)}. \tag{1.7}$$

is the Kloosterman sum defined for integers $n, m, c$.

For large $z$, the leading Bessel function with $c=1$ has the asymptotics:

$$I_\nu(z) \sim \frac{e^z}{z^{\nu+\frac{1}{2}}}\,(1+\frac{a_1}{z}+\frac{a_2}{z^2}\ldots) \tag{1.8}$$

where $a_i$ are constants determined by $\nu$. The leading exponential corresponds to the well-known Cardy formula in conformal field theory and can be identified with the exponential of the Bekenstein-Hawking-Wald entropy. The subleading Bessel functions with $c>1$ are exponentially suppressed in comparison. For the Fourier coefficients of the weak Jacobi forms of interest later, there is a similar but more complicated generalized Hardy-Ramanujan-Rademacher expansion [14, 15].

Such an expansion is particularly well-suited for comparing the integer $d(\Delta)$ with the path integral $W(\Delta)$ which *a priori* is a complex analytic object. Our goal in the remaining lecture will be to see how the supergravity path integral for $W(\Delta)$ can reproduce these intricate details of the Hardy-Ramanujan-Rademacher expansion.

### 1.4 Localization in Supergravity

Loalization is a powerful tool for evaluating a complicated supersymmetric path integral by '*localizing*' it to a submanifold in field space [16, 17, 18, 19]. The heuristic idea behind localization follows from the observation that the Berezin integral over a fermionic variable $\theta$ vanishes:

$$\int d\theta = 0\,. \tag{1.9}$$

If an integrand of an integral is invariant under a supercharge $Q$, then the integral along the orbit of the supercharge in field space parametrized by a fermionic coordinate $\theta$ should vanish by the identity above. If $Q$ does not act freely, then this argument works everywhere except near the fixed points of Q. The path integral then receives contributions only from the localizing manifold which is the submanifold left fixed by the supercharge $Q$. In many cases the localizing manifold is finite-dimensional. The path integral then reduces to an ordinary integral. This method has been used successfully over the years to perform a number of nontrivial computations in quantum field theory, for example in [20, 21].

One of the goals of these lectures is to explain how localization methods could be extended to supergravity path integrals. There are a number of new conceptual and technical issues that arise when the metric is dynamical. There has been considerable progress in evaluating the path integral for $W(\Delta)$ for a class of supersymmetric black holes using localization techniques [22, 23, 24] guided by the Hardy-Ramanujan-Rademacher expansion of the corresponding $d(\Delta)$. Developing these methods further would be a way to learn about nonperturbative aspects of quantum gravity that would otherwise be inaccessible.

### 1.5 Localizing submanifold and the Bessel Function

For a large class of models in $\mathcal{N} = 2$ supergravity, it has been possible to find the localizing solutions explicitly [22, 25]. The dimension of the localizing submanifold is finite and equals the number of vector multiplets. This result is universal in that it follows purely from the off-shell supersymmetry transformations and is independent of the specific form of the physical action and the compactification [22, 25]. The path integral thus reduces to a finite dimensional integral with the integrand determined by one-loop determinants around the saddle and the physical action evaluated on the localizing manifold.

The physical action in general is rather complicated and includes all higher derivative terms coming from integrating out massive string fields. Using the supersymmetry of near horizon geometry one can argue that the nonchiral terms which involve integration over the entire superspace evaluate to zero [26, 27]. Thus, only the chiral terms coming from integration over half of the superspace contribute. Such terms in the action are summarized in terms of a single complex function of the vector multiplet scalars called the prepotential [28, 29]. The localized integral can then be expressed in terms of the prepotential and has a particularly simple form reminiscent of the OSV conjecture [30, 31, 31, 32, 33].

In the examples that we discuss in these lectures, the prepotential is known exactly including all higher-derivative corrections. Moreover, the finite dimensional integral also simplifies and yields precisely the integral representation of the Bessel function!

# 2. Lecture II

In the second lecture I will describe the nonperturbative contributions to the quantum entropy. It is meaningful to include these highly subleading corrections because the evaluation of the path integral around the leading localizing saddle point is one-loop exact. I will then give an assessment of the current status and discuss future directions in the explorations of quantum holography.

## 2.1 Nonperturbative Corrections and Orbifolds

Besides the localizing saddle discussed above, there are additional saddle points which are obtained by the ones above by smooth $\mathbb{Z}_c$ orbifolds of $AdS_2 \times S^1$ (where $S^1$ is the M-theory circle) labeled by a positive integer $c$ [14, 34, 35, 36, 37]. The analysis is identical to the above except that the physical action evaluated on the orbiolded saddles is $1/c$ times the action of the unorbifolded saddle. Consequently, the orbifolded saddle points give the subleading Bessel functions in the expansion (1.5) with the argument reduced by a the factor $1/c$.

## 2.2 Chern-Simons and Kloosterman

The Kloosterman sums multiplying the Bessel functions for $c > 1$ in (1.5) are highly subleading. However, they are conceptually quite important because their precise form is essential for the integrality of $d(\Delta)$. How can a path integral reproduce these subtle number theoretic details?

It turns out that the supergravity action includes Chern-Simons terms for various gauge fields as well as the higher derivative gravitational Chern-Simons terms. In addition, the definition of the quantum entropy path integral includes boundary Wilson lines. The localizing solutions described above follow from solving local differential equations and are insensitive to the topology. There are additional saddle points coming from flat connections of the gauge fields which are sensitive to the topology of $AdS_2 \times S^1$ orbifolds. The Chern-Simons action and the boundary terms evaluated on these flat connections lead to charge-dependent topological phases which combine nicely to yield the Kloosterman sums. For more recent work see [38, 39, 40]. Quantum holography thus implies an interesting connection between the number theory of Fourier coefficients of modular forms in the boundary and the topological information encapsulated by the Chern-Simons-Witten theory in the bulk [41, 42].

## 2.3 BRST Quantization in Gravity

An important conceptual difference between localization in theories with local supersymmetry compared to theories with rigid supersymmetry is that the localizing supersymmetry is a gauge symmetry. Moreover, if the metric is dynamical it is not clear what one means by Killing symmetries required to set up the localization computation.

This issue has been addressed recently [43]. One can set up background field BRST quantization for theories with supergravity gauge symmetry on spaces with asymptotic boundaries like $AdS_2$. The nilpotent BRST charge $Q_{BRST}$ acts both on the background and the quantum fields as well as background and quantum ghosts. The background fields and in particular the metric can be restricted to be invariant under residual Killing symmetries inherited from the boundary. Then the background ghost must also be restricted accordingly and play the role of parameters of the background geometry. Requiring the background ghosts to be invariant allows one to deform $Q_{BRST}$ into the supercharge $Q$ which can be then used for localization.

### 2.4 Computation of Determinants

To complete the localization computation, it is necessary to evaluate the one-loop determinants of various fields around the localizing saddle point. Since one is interested only in ratios of fermionic and bosonic determinants, one can use Atiyah-Bott index theory to evaluate them [44, 45, 46] following the work of Pestun [21] in gauge theories. Computation of these determinants is essential for reproducing the correct index of the Bessel functions.

### 2.5 Assessment and Future Directions

Various perturbative and nonpertubative terms in the computation of $W(\Delta)$ add up to reproduce the Hardy-Ramanujan-Rademacher expansion of $d(\Delta)$. The nonperturbative corrections are crucial for obtaining integrality in agreement with the quantum degeneracies. They reveal an intriguing connection between topology, number theory, and quantum gravity.

It seems likely that in the near future the localization in supergravity could be developed further to address a number of nonperturbative questions in quantum gravity in much the same way localization in quantum field theory has been used successfully to learn about the nonperturbative structure of gauge theories. I will discuss some of the technical and conceptual open questions. Apart from various technical advances that are now beginning to be developed systematically one can foresee a number of interesting applications.

1. *Localization in the bulk of $AdS_4/CFT_3$ holography*:

   The partition function of the boundary ABJM $CFT_3$ has been computed and yields an Airy function at large $N$. In this context, the Airy function plays a role analogous to the Bessel function encountered in $AdS_2$. It has been possible to find the localizing solutions of the corresponding gauged supergravity in the bulk [47]. The resulting finite dimensional integral has the right form to yield the integral representation of Airy function. The computation of one-loop determinants is more complicated. It would be interesting to systematically compute these determinants and possibly compare even the finite $N$ corrections.

2. *Black holes in $AdS_4$ and $AdS_5$*:

   These black holes also have an $AdS_2$ factor but are qualitatively different from black holes in flat space in some ways and have only half as much supersymmetry. Nevertheless, it seems possible in principle to apply localization methods to these black holes as well. It would be interesting to see if localization can yield concrete results in agreement with the boundary computations [48, 49].

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