Quantum Black Holes

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Hurdles for String Theory

- We don't have a super-LHC to probe the theory directly at Planck scale.
- We don't even know which phase of the theory may correspond to the real world. Analogy with water.

How can we be sure that string theory is the right path to quantum gravity in the absence of direct experiments?

A useful strategy is to focus on **universal** features that must hold in **all phases** of the theory.

A bit like 19th century physicists who could glean information about quantum theory of matter (identical particles, specific heat of molecules)

Quantum Black Holes

One of the most important clues about quantum gravity is the entropy of a black hole:

What is the **exact quantum generalization** of the celebrated Bekenstein-Hawking formula?

$$S = \frac{\mathbf{A}}{\mathbf{4}} + c_1 \log(A) + c_2 \frac{1}{A} \ldots + e^{-A} + \ldots$$

- **How to define it ? How to compute it?**
- **What ensemble?**
- Subleading corrections **depend sensitively** on the phase:

An **IR** window into the **UV.**

Exact Quantum Entropy

Any black hole in ***any*** phase (= compactification) of the theory should be interpretable as an ensemble of quantum states ***including*** *finite size quantum gravity corrections.*

- ***Universal*** and ***extremely stringent*** constraint
- ***Finite size corrections*** connect to a broader problem of ***Quantum Holography at finite N***.
- Is Quantum Gravity ***emergent*** or ***dual*** to CFT?
- ***Localization in supergravity*** provides novel tools to deal with nonperturbative quantum gravity effects.

Quantum Entropy and Holography

- Near horizon of supersymmetric BPS black holes in 4d has an $AdS_2 \times S^2$ factor

$$ds^2 = l_*^2 \left[(r^2 - 1) d\theta^2 + \frac{dr^2}{r^2 - 1} + d\psi^2 + \sin^2 \psi d\phi^2 \right]$$

$$F^I = -i\, e_*^I dr \wedge d\theta + P^I \sin\psi\, d\psi \wedge d\phi\,, \qquad X_*^I$$

- One can apply usual rules of holographic correspondence keeping in mind some of the important peculiarities of AdS_2

Attractor Mechanism

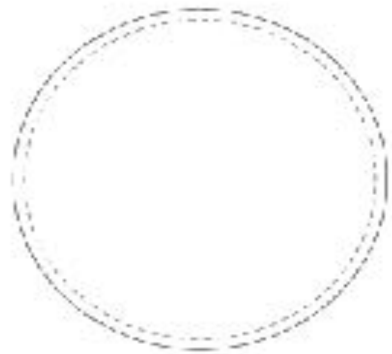
- In the classical theory, the near horizon values of the radius of *AdS*, the electric fields, and values of various of scalar fields are determined entirely in terms of the charges of the black hole by the attractor mechanism:

$$l_*^2(Q,P)\,, \qquad e_*^I(Q,P)\,, \qquad X_*^I(Q,P)$$

Ferrara Kallosh Strominger (96)

- In the quantum theory, we will hold these values fixed only at the boundary of AdS and allow for quantum fluctuations in the bulk of AdS.

$\mathbf{AdS_2/CFT_1}$ **Holography**



- Bulk AdS_2 is Poincaré disk. Put a cutoff at $r = r_0$
- Boundary CFT_1 is a finite dimensional Hilbert space of dimension $d(Q,P)$ and zero Hamiltonian $H=0.$

$$Z_{CFT}Q,P) := \mathbf{Tr} e^{(-2\pi r_0 H)} = d(Q,P)$$

Holographic Renormalization

- For the gauge field action we need a boundary term to make the variational problem well defined. It's equivalent to introducing a Wilson line.
- Divergences from infinite volume of AdS can be removed by boundary counterterms.
- The renormalized action (including Wilson line) is

$$S_{ren} = S_{bulk} + S_{bdry} - \frac{i}{2} Q_I \int A^I$$

Quantum Entropy

- Exponential of quantum entropy is given by *Sen (09)*

$$W(Q,P) := \langle \exp\left(-\frac{i}{2} Q_I \int A^I\right)\rangle_{ren}$$

- Path integral ***over all string fields*** *with an insertion of a Wilson line and with appropriate boundary conditions and renormalization.*
- Reduces to Bekenstein-Hawking-Wald for large *(Q, P)*
- Quantum holography requires

$$W(Q,P) = d(Q,P)$$

W(Q, P)	**d(Q, P)**
Black Hole (Q, P)	Brane (Q, P)
Quantum Entropy	Counting of States
AdS_2	CFT_1
Quantum Geometry	Hilbert Space

A quantum generalization of

Bekenstein-Hawking ***Boltzmann***

Can we compute both sides?

Are You Crazy?

- Attempting exact quantum calculations especially in bulk string field theory sounds foolhardy.

Strategy

- Integrate out massive string fields, use the Wilsonian local effective action for massless supergravity fields keeping all higher order terms. Reduces the problem to supergravity with an *arbitrary* local action. Still seems very difficult.
- *Be brave. Leave the secure domain of QFT and classical strings behind. Venture into the treacherous terrain of quantum gravity inhabited by fearsome dragons.*

Defining W(Q, P)	AdS/CFT
Counting d(Q, P)	D-branes, Duality
Degeneracy	Index, Modularity
Path integral	Localization
Sugra Localization	Off-Shell Sugra
Sugra Action	Nonrenormalization
Kloosterman phases	Chern-Simons terms
Wall-crossing	Mock Jacobi Forms

Choice of Ensemble

The gauge field behaves as

$$A^I_\theta \sim -\, ie^I r + \mu^I\,; \qquad\qquad F_{r\theta} = -\, ie^I$$

A natural choice is to hold the growing mode fixed. Thus hold the electric field (and hence the charge) fixed and let the chemical potential fluctuate.

Contrast this with the higher dimensional case.

Corresponds to *microcanonical* ensemble.

Sen (09, 10)

Degeneracy = Index

- The near horizon AdS_2 has *SU(1, 1)* symmetry.
- With four supersymmetries, the closure of algebra implies *SU(1, 1|2)* supergroup as symmetry.
- Hence, horizon must have *SU(2)* symmetry. Consistent with the fact that supersymmetric black holes in 4d are spherically symmetric.
- **Microcanonical** ensemble implies ***J = 0***.
- All horizon degrees of freedom are bosonic!

$$\mathbf{Tr}(\mathbf{1}) = N_B + N_F = \mathbf{Tr}(\textbf{-1})^F = N_B - N_F$$

Electric States in N=4 theory

Heterotic on T^6. Electric states with charge vector Q

Duality invariant $Q^2/2 = \Delta$

$$Z(\tau) = \frac{1}{q}\prod_{r=1}^{\infty}\frac{1}{(1-q^r)^{24}} \qquad (q := e^{2\pi i\tau})$$

$$= \sum_{n=-1}^{\infty} C(n)q^n = q^{-1} + 24 + \ldots$$

$$d(\Delta) = C(\Delta)$$

Dabholkar Harvey (89)

Modular Symmetry

- A holomorphic function $F(\tau)$ on the upper half complex plane is a modular form of weight k, if it transforms as

$$F\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k F(\tau)$$

for *a, b, c, d, k* integers and *ad-bc =1*

The matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *form the group SL(2, Z) under matrix multiplication.* ***Highly Symmetric.***

Our $Z(\tau)$ is a modular form of weight -12.

Hardy Ramanujan Rademacher

Fourier coefficients of modular forms admit a beautiful exact, convergent expansion in terms of complex analytic functions and phases:

$$C(N) = \sum_{c=1}^{\infty} \left(\frac{1}{c}\right)^{14} Kl(N, -1, c)\, \tilde{I}_{13}\left(\frac{4\pi\sqrt{N}}{c}\right)$$

where the modified Bessel function is defined by

$$\tilde{I}_\rho(z) := \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{dt}{t^{\rho+1}} \exp[t + \frac{z^2}{4t}]$$

Kloosterman Sum

$$Kl(n,m,c) := \sum_{\substack{d \in \mathbb{Z}/c\mathbb{Z} \\ da = 1\, \mathbf{mod}(c)}} e^{2\pi i\left(n\frac{d}{c} + m\frac{a}{c}\right)}$$

are the number theoretic phases in the formula earlier for *(n, m, c)* all integers. Satisfies nontrivial `Selberg identities' important for duality invariance.

$$Kl(np,m,c) = Kl(n,mp,c)$$

Dyonic States in N=8 theory

Type II on T^6. Dyons with charge vector (Q, P)

Duality invariant $\Delta = Q^2P^2 - (Q \cdot P)^2$

$$Z(\tau, z) = \prod_{r}^{\infty} \frac{(1 - q^r y)^2 (1 - q^r y^{-1})^2}{(1 - q^r)^4} \qquad (y := e^{2\pi i z})$$

$$= \sum_{n=-1}^{\infty} c(n, l) q^n y^l \quad ; \quad c(n, l) = C(4n - l^2)$$

$$\boxed{d(\Delta) = (-1)^{\Delta+1} C(\Delta)}$$

Maldacena Moore Strominger (99) Shih Strominger Yin; Pioline(05)

Jacobi forms

A Jacobi form of weight k and index m

$$\varphi_{k,m}(\tau, z)$$

`modular' with weight k

`elliptic' in z with index m

$$\varphi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)}\varphi(\tau, z)$$

$$(\forall \quad \lambda, \mu \in \mathbb{Z})$$

Our $Z(\tau, z)$ is a Jacobi form of weight 2 index 1.

Elliptic and modular properties again imply a bit more involved Hardy-Ramanujan-Rademacher expansion:

$$C(\Delta) = N \sum_{c=1}^{\infty} c^{-9/2} \tilde{I}_{7/2}\left(\frac{\pi\sqrt{\Delta}}{c}\right) K_c(\Delta)$$

Exact degeneracies are known also for dyonic states in N=4 in terms of Fourier coefficients of Siegel modular forms and exhibit an intricate structure of wall-crossings in the moduli space. They connect to the mathematics of Mock modular forms introduced by Ramanujan a century ago. More general N=2 degeneracies relate to Donaldson-Thomas invariants.

Generalized Kloosterman Sum $K_c(\Delta)$

$$\sum_{\substack{-c\le d<0;\\ (d,c)=1}} e^{2\pi i \frac{d}{c}(\Delta/4)}\; M^{-1}(\gamma_{c,d})_{\nu 1}\; e^{2\pi i \frac{a}{c}(-1/4)}$$

$$\nu = \Delta \mod 2$$

$$M^{-1}(\gamma)_{\nu\mu} = C\sum_{\epsilon=\pm}\sum_{n=0}^{c-1} \epsilon\, e^{\frac{i\pi}{2rc}\left[d(\nu+1)^2 - 2(\nu+1)(2rn+\epsilon(\mu+1)) + a(2rn+\epsilon(\mu+1))^2\right]}$$

*Intricate number theoretic phases, highly subleading in large area expansion, but essential for **integrality.** Quantum holography requires that the bulk must reproduce these nonperturbative phases.* ***And it does!***

Power of Quantum Holography (AdS_2 /CFT_1)

The path integral *W(Δ)* in the near horizon AdS_2 can be defined as a generalization of Wald entropy.

Includes nonlocal effects from massless loops.

Quantum Holography implies two things:

1. *W(Δ)* = *d(Δ)* (nontrivial prediction for a path integral)

   Path integral must be an integer!

2. *d(Δ)* = *W(Δ)* (nontrivial prediction for an index)

   Index must be positive! (index = *degeneracy.***)**

Positivity

Δ	4	7	8	11	12	15
$C(\Delta)$	-12	39	-56	152	-208	513

Note that $C(\Delta)$ are alternating in sign so that $d(\Delta) = (-1)^{\Delta+1} C(\Delta)$ is strictly positive.

Surprising for a number theorist or a field theorist because Fourier coefficients of modular forms or the index in a QFT don't have to be positive.

Sen (10) Dabholkar Gomes Murthy Sen (12)

*A **prediction from IR** quantum gravity for black holes which is **borne out by the UV**. Can we compute $W(\Delta)$?*

Harmonic Oscillator of Quantum Holography

The HRR expansion is ideally suited for a systematic exploration of quantum corrections to BH entropy.

$$\tilde{I}_{7/2}(z) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{ds}{s^{9/2}} \exp[s + \frac{z^2}{4s}] \quad ; (z = \frac{A}{4})$$

$$\sim \exp\left[z - 4\log z + \frac{c}{z} + \ldots\right]$$

The c=1 Bessel function sums *all perturbative* (in 1/z) corrections to entropy. The c>1 are non-perturbative

Can we compute all terms from the bulk to `prove' quantum holography for this harmonic oscillator?

Localization of Path Integrals

We are interested in a path integral of the form

$$Z = \int_{\mathscr{M}} d\mu \, e^{S_{ren}}$$

with a supersymmetric measure and action.

Localization techniques make it possible to evaluate such integrals. We have learnt a great deal about the noperturbative structure of QFT which was otherwise inaccessible without localization.

Duistermaat-Heckmann(82)..Witten (88) Nekrasov (02) Pestun (04) …

Can we localize **supergravity** path integrals?

Localization

Consider a supermanifold $\mathscr{M}$ with an integration measure $d\mu$. Let Q be an odd (fermionic) vector field on this manifold that satisfies:

1. $Q^2 = H$ for a compact bosonic vector field H .
2. $div_\mu(Q) = 0$ i. e. the measure is invariant under Q.

Note that Q is nilpotent on H-invariant configurations

Allows one to study `**Equivariant** Cohomology'

Field space is an (infinite-dimensional) supermanifold

Q is a supersymmetry, *H* is a Killing field.

Deformation Invariance

- Consider a deformation of the original integral

$$Z(\lambda) = \int_{\mathcal{M}} d\mu \; e^{S_{ren} - \lambda QV}$$

where V is an *H*-invariant fermionic function

$$H(V) = Q^2(V) = 0;$$

- One can then prove easily that

$$\frac{d}{d\lambda} Z(\lambda) = \int_{\mathcal{M}} d\mu \, Q(V \, e^{S_{ren} - \lambda QV}) = 0$$

Saddle-point Evaluation is Exact.

1. Find the critical manifold $\mathscr{M}_Q$ of the *QV* action.
2. Evaluate the action on this critical manifold to obtain the leading classical contribution.
3. Compute the one-loop contribution given by the determinants Z_{det} of the quadratic fluctuation operators of the *QV* action. Then,

$$Z(0) = Z(\infty) = \int_{\mathscr{M}_Q} d\mu_Q \, e^{S_{ren}(\mathscr{M}_Q)} Z_{det}(\mathscr{M}_Q)$$

The final answer is independent of the *QV* action.
Make a clever choice of the *QV* action.

Treating λ^{-1} as $\hbar$ we can now evaluate the path integral using `semiclassical' methods near saddle points of the QV action.

The path integral ***localizes*** onto the critical manifold $\mathscr{M}_Q$ of the QV action which is nothing but the space of Q-invariant configurations.

In many situations the critical manifold $\mathscr{M}_Q$ is finite dimensional. ***Enormous simplification*** reducing a path integral to an ordinary integral.

Challenges in Supergravity

- In Euclidean gravity the conformal factor has a wrong sign kinetic term.
- Since metric is dynamical what does it mean to have a background with a symmetry?
- At a more fundamental level, since all symmetries such as Q and H are ***gauge*** symmetries, how can we even get started with localization?
- Unlike in QFT, the action has higher derivative terms and is nonrenormalizable.

Strategy

- Use background field BRST quantization: field = background field + quantum field.
- Gauge parameters that don't vanish at infinity generate the Killing symmetries of the background. Use these symmetries to localize.
- Use off-shell superconformal supergravity.
- Nonrenormalization theorem: nonchiral D-terms don't contribute. Huge simplification.

de Wit Katmadas Zalk (12), Murthy Reys (13)

- A single prepotential ***F*** specifies the chiral terms.

Recap for Lecture II

Type II on T^6 gives *N=8* supergravity in 4d.

Massless bosons: metric + **28** vector fields + scalars

Consider BPS dyonic states with charge vector *(Q, P)*.

U-duality invariant $\Delta = Q^2P^2 - (Q \cdot P)^2$

- At weak coupling, point like states in 4d with degeneracy $d(\Delta)$ which can be computed using D-brane techniques.
- At strong coupling, these states collapse to form a a black hole in 4d with area $A(\Delta)$

Degeneracy

The degeneracy is given in terms of Fourier coefficients of an (indexed) partition function of *4 free bosons* and *4 free fermions* of an effective 2d CFT (worldvolume of a D1D5 string)

$$Z(\tau, z) = \prod_{r}^{\infty} \frac{(1-q^r y)^2 (1-q^r y^{-1})^2}{(1-q^r)^4} \qquad (y := e^{2\pi i z})$$

$$= \sum_{n=-1}^{\infty} c(n,l) q^n y^l \quad ; \qquad c(n,l) = C(4n - l^2)$$

$$\boxed{d(\Delta) = (-1)^{\Delta+1} C(\Delta)}$$

Quantum Entropy

- Near horizon geometry of the corresponding black hole is $AdS_2 \times S^2$
- We defined the quantum entropy $W(\Delta)$ as a path integral of N=8 supergravity fields.

$$W(\Delta) \sim \exp[\frac{A(\Delta)}{4}] = \exp[\pi\sqrt{\Delta}] \sim d(\Delta)\,, \qquad \Delta \gg 1$$

Bekenstein-Hawking = Cardy

Can we go beyond this leading semiclassical result and compute all quantum corrections?

$$W(\Delta) = d(\Delta)?$$

To attempt such a comparison it's useful to use

Hardy-Ramanujan-Rademacher expansion

Exact generalization of Cardy formula

$$d(\Delta) = \sum_{c=1}^{\infty} c^{-9/2} \tilde{I}_{7/2}(\pi\sqrt{\Delta})K_c(\Delta)$$

The c=1 Bessel function sums all perturbative corrections to entropy. The c>1 are non-perturbative

$$\tilde{I}_{7/2}(z) \sim \exp[z - 2\log z + \frac{c}{z} + \cdots]$$

Computing $W(\Delta)$

- The structure of the microscopic answer suggests that $W(\Delta)$ should have an expansion

$$W(\Delta) = \sum_{c=1}^{\infty} W_c(\Delta)$$

- We will find that $W_c(\Delta)$ arises from an $\mathbb{Z}_c$ orbifold saddle point of the path integral.
- The higher *c* are exponentially subleading. Unless one can evaluate each of them *exactly* it is not particularly meaningful to add them.

Localization enables us to do this.

Beauty of Off-Shell Supergravity

1. Supersymmetry transformations are written down once and for all (much like coordinate transformations) **independent of the action.**

– Algebra closes without using equations of motion.

– Essential for using SUSY inside a path integral.

2. It nicely separates the problem into two parts.

– Find the offshell localizing solutions once and for all independent of the physical action. √

– Evaluate the renormalized action on the localizing manifold for any given compactification.

Off Shell Multiplets of N=2 Sugra

- Vector Multiplet

$$\mathbf{X}^I = \left(X^I, A^I_\mu, Y^I_{ij} \,|\, \Omega^I_i \right) \,, \qquad I = 0, \ldots n_v$$

- Weyl Multiplet

$$\mathbf{w} = \left(e^a_\mu, w^{ab}_\mu, \phi^i_\mu, b_\mu, f^a_\mu, A_\mu, \mathscr{V}_\mu{}^i{}_j, T^{ij}_{ab}, D \,|\, \psi^i_\mu, \chi^i \right)$$

$i = 1,2$ is the SU(2) doublet index.

$(a\,,\mu) = 0,1,2,3$ are the tangent and spacetime indices

de Wit Holten Lauwers Van Proeyen (80, 81, 85)

Supersymmetry and Prepotential

- The supersymmetry transformations look like

$$\delta\Omega_i^I = 2\gamma^\mu \partial_\mu X^I \varepsilon_i + Y_{ij}^I \varepsilon^j + \sigma^{\mu\nu} \mathscr{F}_{\mu\nu}^{I-} \varepsilon_i$$

- The ***chiral*** couplings of vector multiplets to gravity including arbitrary higher-derivative terms are completely specifying by a ***single complex function*** of vector multiplet scalars and Weyl multiplet auxiliary field called the ***prepotential***.

$$F(\lambda X^I, \lambda^2 A) = \lambda^2 F(X^I, A) = \sum_{g=0}^{\infty} F^{(g)}(X^I) A^g$$

Choice of Q and V

- For our problem we choose a specific Killing spinor of the near horizon geometry $\varepsilon(x)$ such that

$$Q_\varepsilon^2 = (L - J) := H$$

$$\delta_\varepsilon^2(\phi) = \delta_H(\phi)$$

- Choose

$$V = \sum_i (Q_\varepsilon \Psi_i, \Psi_i)$$

- The deformation action has a bosonic part

$$QV \sim \sum_i (Q_\varepsilon \Psi_i, Q_\varepsilon \Psi_i)$$

Off-shell Localizing Solutions

- One finds ***off-shell*** localizing instantons in AdS_2 for supergravity coupled to n_v vector multiplets with scalars X^I and auxiliary fields $Y^I_{12} = Y^I_{21} := Y^I$

$$X^I = X^I_* + \frac{C^I}{r}, \qquad Y^I = \frac{2C^I}{r}, \qquad C^I \in R\,;\ (I = 0,1,\ldots n_v)$$

These solutions are ***universal*** in that they are *independent of the physical action* and follow entirely from the off-shell susy transformations.

Valid for any physical action.

Dabholkar Gomes Murthy (11), Gupta Murthy (12)

Supergravity Action

$$
\begin{aligned}
& 8\pi e^{-1}\mathcal{L} \\
= \quad & (-i(X^I \bar{F}_I - F_I \bar{X}^I)) \cdot (-\frac{1}{2}R) \\
+ \quad & \big[i\nabla_\mu F_I \nabla^\mu \bar{X}^I \\
+ \quad & \frac{1}{4} i F_{IJ} (F^{-I}_{ab} - \frac{1}{4}\bar{X}^I T^{ij}_{ab}\,\varepsilon_{ij})(F^{-abJ} - \frac{1}{4}\bar{X}^J T^{ij}_{ab}\,\varepsilon_{ij}) \\
- \quad & \frac{1}{8} i F_I (F^{+I}_{ab} - \frac{1}{4} X^I T_{abij}\,\varepsilon^{ij}) T^{ij}_{ab}\,\varepsilon_{ij} \\
- \quad & \frac{1}{8} i F_{IJ} Y^I_{ij} Y^{Jij} - \frac{i}{32} F\,(T_{abij}\,\varepsilon^{ij})^2 \\
+ \quad & \frac{1}{2} i F_{\widehat{A}} \widehat{C} - \frac{1}{8} i F_{\widehat{A}\widehat{A}} (\varepsilon^{ik}\varepsilon^{jl}\widehat{B}_{ij}\widehat{B}_{kl} - 2\widehat{F}^-_{ab}\widehat{F}^-_{ab}) \\
+ \quad & \frac{1}{2} i \widehat{F}^{-ab} F_{\widehat{A}I} (F^{-I}_{ab} - \frac{1}{4}\bar{X}^I T^{ij}_{ab}\,\varepsilon_{ij}) - \frac{1}{4} i \widehat{B}_{ij} F_{\widehat{A}I} Y^{Iij} + \text{h.c.} \big] \\
- \quad & i(X^I \bar{F}_I - F_I \bar{X}^I) \cdot (\nabla^a V_a - \frac{1}{2} V^a V_a - \frac{1}{4}|M_{ij}|^2 + D^a \Phi^i{}_\alpha D_a \Phi^\alpha{}_i) \,.
\end{aligned}
$$

Renormalized Action

The renormalized action for prepotential ***F*** simplifies:

$$\mathcal{S}_{ren}(\phi, q, p) = -\pi q_I \phi^I + \mathcal{F}(\phi, p)$$

$$\mathcal{F}(\phi, p) = -2\pi i \left[F\Big(\frac{\phi^I + ip^I}{2}\Big) - \bar{F}\Big(\frac{\phi^I - ip^I}{2}\Big)\right]$$

$$\phi^I := e_*^I + C^I$$

$\frac{1}{2}(\phi^I + ip^I)$ is the off-shell value of X^I at the origin of the Poincaré disk.

Dabholkar Gomes Murthy (11, 13)

Final Answer

$$W_1(Q,P) = \int [d\phi]\, e^{-\pi Q_I \phi^I + Im F(\phi + iP)}\, Z_{det}(\phi)\, Z_{inst}$$

$$Z_{det}(\phi) = \exp\left[-K(\phi + iP)\left(n_v - n_h + 23/12\right)\right]$$

$$e^{-K} := -\, i(X^I \bar{F}_I - \bar{X}^I F_I)$$

A finite dimensional integral determined entirely in terms of the prepotential (+ possibly point instantons).

Final integral

The prepotential for the truncated theory is

$$F(X) = -\frac{1}{2}\frac{X^1}{X^0}\sum_{a,b=2}^{7} C_{ab}X^a X^b \qquad (n_v = 7)$$

(dropping the extra gravitini multiplets of ***N=8***)

The *path* integral reduces to the Bessel integral

$$W_1(\Delta) = N\int \frac{ds}{s^{9/2}}\exp\left[s + \frac{\pi^2\Delta}{4s}\right]$$

$$W_1(\Delta) = \tilde{I}_{7/2}(\pi\sqrt{\Delta})$$

Degeneracy, Quantum Entropy, Wald Entropy

Δ	$C(\Delta)$	$W_1(\Delta)$	$\exp(\pi\sqrt{\Delta})$
3	8	7.972	230.765
4	-12	12.201	535.492
7	39	38.986	4071.93
8	-56	55.721	7228.35
11	152	152.041	22506.
12	-208	208.455	53252.
15	513	512.958	192401

Orbifold Contributions

- Including the M-theory circle, there is a family of geometries $\mathcal{M}_{c,d}$ that are asymptotically $AdS_2 \times S^1$:

$$ds^2 = (r^2 - \frac{1}{c^2})d\theta^2 + \frac{dr^2}{r^2 - \frac{1}{c^2}} + R^2 \left(dy - \frac{i}{R}(r - \frac{1}{c})d\theta + \frac{d}{c}d\theta \right)^2$$

- Freely acting $\mathbb{Z}_c$ orbifolds of BTZ black hole.

Related to the $SL(2,\mathbb{Z})$ family in AdS_3

Maldacena Strominger (98) Sen (09) Pioline Murthy (09)

Dijkgraaf Maldacena Moore Verlinde (00) Dabholkar Gomes Murthy (14)

Localization justifies keeping these subleading saddles.

Subleading Bessel Functions

- Contributions from these smooth orbifolds explain the Bessel functions **for all *c*** with correct argument because for each orbifold the localized solutions are the same but the renormalized action is reduced by a factor of c
- *What about the Kloosterman sums?*

How can a SUGRA path integral possibly reproduce this intricate number theoretic structure ?

Generalized Kloosterman Sum $K_c(\Delta)$

$$\sum_{\substack{-c\leq d<0;\\ (d,c)=1}} e^{2\pi i \frac{d}{c}(\Delta/4)}\; M^{-1}(\gamma_{c,d})_{\nu 1}\; e^{2\pi i \frac{a}{c}(-1/4)}$$

$$\nu = \Delta \mod 2$$

$$M^{-1}(\gamma)_{\nu\mu} = C \sum_{\epsilon=\pm}\sum_{n=0}^{c-1} \epsilon\, e^{\frac{i\pi}{2rc}\left[d(\nu+1)^2-2(\nu+1)(2rn+\epsilon(\mu+1))+a(2rn+\epsilon(\mu+1))^2\right]}$$

*Number theoretic phases essential for **integrality***

Chern-Simons-Witten Theory

- Our localization analysis so far ignored the topology.
- The Chern-Simons terms in the bulk and the boundary terms are sensitive to the global properties of $\mathcal{M}_{c,d}$
- Additional saddles specified by holonomies of flat connections. Various phases from CS terms assemble nontrivially into the Kloosterman sum.
- Closely related to knot invariants of Lens space $\mathcal{L}_{c,d}$ using the surgery formula.

Kloosterman and Chern-Simons

$$I(A) = \int_{\mathcal{M}_{c,d}} \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A^3 \right)$$

In our problem we have three relevant groups

$$U(1)^{n_v+1} \qquad SU(2)_L \qquad SU(2)_R$$

$$\sum_{\substack{-c \leq d < 0; \\ (d,c)=1}} e^{2\pi i \frac{d}{c}(\Delta/4)} \; M^{-1}(\gamma_{c,d})_{\nu 1} \; e^{2\pi i \frac{a}{c}(-1/4)}$$

Dabholkar Murthy Gomes (14)

Dehn Twisting

The geometries $\mathcal{M}_{c,d}$ are topologically a solid 2-torus and are related to $\mathcal{M}_{1,0}$ by Dehn-filling.

Relabeling of cycles of the boundary 2-torus:

$$\begin{pmatrix} C_n \\ C_c \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

C_1 is contractible and C_2 is noncontractible in $\mathcal{M}_{1,0}$

C_c is contractible and C_n is noncontractible in $\mathcal{M}_{c,d}$

Boundary Conditions and Holonomies

- The cycle C_2 is the M-circle and C_1 is the boundary of AdS_2 for the reference geometry
- This implies the boundary condition $\mathcal{M}_{1,0}$

$$\oint_{C_2} A^I = \text{fixed}\,, \qquad \oint_{C_1} A^I = \text{not fixed}$$

and a boundary term

$$I_b(A) = \int_{T^2} \mathrm{Tr} A_1 A_2 d^2 x$$

Contribution from $SU(2)_R$

$$\oint_{C_1} A = 2\pi i\gamma \frac{\sigma^3}{2} \qquad \oint_{C_2} A = 2\pi i\delta \frac{\sigma^3}{2}$$

$$\oint_{C_c} A = 2\pi i\alpha \frac{\sigma^3}{2} \qquad \oint_{C_n} A = 2\pi i\beta \frac{\sigma^3}{2}$$

Chern-Simons contribution is completely determined by the holonomies. For abelian the bulk contribution is zero for flat connections and only boundary contributes. For nonabelian

$$I_b[A_R] = 2\pi^2\gamma\delta \qquad I[A_R] = 2\pi^2\alpha\beta$$

Kirk Klassen (90)

$$\oint_{C_1} A_R = -\frac{2\pi i}{c}\frac{\sigma^3}{2}\,, \qquad \oint_{C_2} A_R = 0$$

Supersymmetric $\mathbb{Z}_c$ orbifold $\qquad J_R = 0$

$$\gamma = -1/c\,, \quad \delta = 0\,, \quad \alpha = -1\,, \quad \beta = -a/c$$

(using $\alpha = c\gamma + d\delta\,, \qquad \beta = a\gamma + b\delta$)

The total contribution to renormalized action is

$$S_{ren} = -\frac{2\pi i k_R}{4}\frac{a}{c} \qquad k_R = 1$$

Multiplier System from $SU(2)_L$

There is an explicit representation of the Multiplier matrices that is suitable for our purposes.

$$M^{-1}(\gamma)_{\nu\mu} = C \sum_{\epsilon=\pm} \sum_{n=0}^{c-1} \epsilon \, e^{\frac{i\pi}{2rc}\left[d(\nu+1)^2 - 2(\nu+1)(2rn+\epsilon(\mu+1)) + a(2rn+\epsilon(\mu+1))^2\right]}$$

Unlike $SU(2)_R$ the holonomies of $SU(2)_L$ are not constrained by supersymmetry and have to be summed over which gives precisely this matrix.

(Assuming usual shift of k going to k +2)

Knot Theory and Kloosterman

- This computation is closely related to knot invariants of Lens space $\mathcal{L}_{c,d}$ using the surgery formula of Witten. *Witten (89) Jeffrey (92)*
- This is not an accident. Lens space is obtained by taking two solid tori and gluing them by Dehn-twisting the boundary of one of them. But Dehn-twisted solid torus is our $\mathcal{M}_{c,d}$
- Intriguing relation between topology and number theory for an appropriate CS theory.

Remarkably AdS path integral reproduces all details.

A path integral (a complex analytic continuous object)

yields an integer (a number theoretic discrete object).

$W(\Delta)$ = integer!

An *IR* Window into the *UV*

- It counts with precision *nonperturbative* states with masses much higher than the string scale.
- If we did not know the spectrum of branes a priori we could in principle deduce it. e.g. in *N=6* models!

Summary

- ***Exact quantum entropy:*** For a class of black holes one can compute all perturbative and nonperturbative corrections to the Bekenstein-Hawking area formula.
- ***Localization in supergravity:*** A novel and powerful tool to analyze quantum effects in gravity even at a nonperturbative level.
- ***Quantum Holography:*** In these examples holography appears to hold at the quantum level including finite N corrections.

Quantum Gravity Dual or Emergent?

(1) AdS quantum gravity is exactly dual to CFT.

M theory has its **own** rules of computations

(with an as-yet-unknown nonperturbative formulation)

AdS/CFT is a special corner of this duality

(2) AdS quantum gravity is emergent from CFT.

Any `reasonable' CFT gives a nonperturbative definition of quantum gravity in AdS. How come?

Our computations seem to argue in favor of (1)

Not just UV-complete but UV-rigid

- *All* integers $d(\Delta)$ define a valid CFT_1
- Only a *sparse set* among them has a dual AdS_2
- Path integral $W(\Delta)$ can be computed *independently*.
- $W(\Delta) = d(\Delta)$ requires precise details of M-theory

(such as Chern-Simons terms and topological string)

Can we incorporate these quantum aspects of holography into `Bulk Reconstruction'?

Future Directions

- ***Localization in AdS_4*** *Dabholkar Drukker Gomes (14)*

The ABJM boundary partition function gives an Airy function analogous to the Bessel function. Localizing solutions give an integral very close to the Airy integral

- ***Black Holes in AdS_4*** *Benini Hristov Zaffaroni (15, 16)*

For black holes with ***AdS_2*** horizons in ***AdS_4*** the Bekenstein Hawking entropy agrees with the boundary

Can we do better? Develop localization methods to compute the measure in gauged supergravity.

- ***Localization in AdS_3***

Can we reproduce modular forms directly instead of their Fourier coefficients from localization in the bulk. Closer to *Ooguri-Strominger-Vafa and Denef-Moore*

- ***Topological Holography?***

Can the equivariant localization of the bulk path integral be matched with the boundary more directly to correspond to the elliptic genus?

- ***Duality Invariance*** *Gomes (17)*

Nontrivial Seleberg identities must hold for the Kloosterman sums for general charge configurations

Open Questions

- We have obtained a general answer for N=2 supergravity. We used N=2 truncation of N=8. It would be better if this computation can be performed without this truncation.
- In our N=8 example there are gravitini multiplets that we do not know how to treat. From onshell results one expects that their contribution will give the correct index for the Bessel function.
- Gauge fixing from Conformal Supergravity to Poincaré supergravity needs to done more carefully.

Localization in Gauge Theories

- Applying localization to gauge theory presents a number of new subtleties **even with rigid supersymmetry**. Supersymmetry algebra closes only up to gauge transformations:

$$\delta_\varepsilon^2 = \delta_H + \delta_g \,, \qquad Q^2 \neq H$$

- The gauge fixed action has no gauge symmetry. So we are left with 9 bosonic and 8 fermionic fields.

– Include ghosts, antighosts, Lagrange multipliers

– Define $$\hat{Q} = Q + Q_B$$

such that $$\hat{Q}^2 = Q^2 + \{Q, Q_B\} = H$$

Localization in Supergravity

The problem is exacerbated in supergravity:

- The `structure constants' of supergravity gauge algebra are field-dependent: soft algebra
- The metric is dynamical.

Earlier we dealt with them heuristically. One can set up a Background Field BRST formalism to deal with both these problems in a systematic way.

de Wit Murthy Reys (18)

Soft Algebras

Gauge transformation of fields

$$\delta\phi^i = R(\phi)^i_\alpha \xi^\alpha$$

Closure of Algebra

$$\delta(\xi_1)\,\delta(\xi_2) - \delta(\xi_2)\,\delta(\xi_1) = \delta(\xi_3)\,, \quad \xi_3{}^\alpha = f_{\beta\gamma}{}^\alpha\,\xi_1{}^\beta\,\xi_2{}^\gamma$$

Field-dependent structure functions satisfying

$$R^j{}_{[\alpha}\,\partial_j R^i{}_{\beta]} = \frac{1}{2} f_{\alpha\beta}{}^\gamma\, R^i{}_\gamma$$

Jacobi Identity

$$f_{[\alpha\beta}{}^\delta\, f_{\gamma]\delta}{}^\epsilon + R^j{}_{[\alpha}\,\partial_j f_{\beta\gamma]}{}^\epsilon = 0$$

BRST Quantization

Elevate gauge parameters to ghosts $\xi^\alpha \to \Lambda c^\alpha$

Field (ϕ) Ghost (***c***), Antighost (***b***), Lagrange Multiplier (***B***)

$$\delta_{\mathrm{B}}\phi^i = R(\phi)^i{}_\alpha \Lambda\, c^\alpha \qquad \delta_{\mathrm{B}} b_\alpha = \Lambda\, B_\alpha$$

$$\delta_{\mathrm{B}} c^\alpha = \frac{1}{2} f_{\beta\gamma}{}^\alpha\, c^\beta \Lambda\, c^\gamma \qquad \delta_{\mathrm{B}} B_\alpha = 0$$

Gauge fixing Lagrangian

$$\mathscr{L}^{\mathrm{GF}} = B_\alpha F(\phi)^\alpha - b_\alpha R(\phi)^j{}_\beta\, c^\beta\, \partial_j F(\phi)^\alpha$$

Nilpotent BRST Symmetry

$$\delta_B^2 = 0 \qquad \delta_B(\mathscr{L} + \mathscr{L}^{\mathscr{G}}\mathscr{F}) = 0$$

Background Field Split

- Split all fields (including the metric) into a background field and a quantum fluctuation:

$$\phi^i = \mathring{\phi}^i + \widetilde{\phi}^i$$

- Background gauge symmetry

$$\mathring{\delta}\mathring{\phi}^i = R(\mathring{\phi})^i{}_\alpha \,\mathring{\xi}^\alpha$$

$$\mathring{\delta}\widetilde{\phi}^i = R(\mathring{\phi} + \widetilde{\phi})^i{}_\alpha \,\mathring{\xi}^\alpha - R(\mathring{\phi})^i{}_\alpha \mathring{\xi}^\alpha$$

- Quantum gauge symmetry

$$\tilde{\delta}\mathring{\phi}^i = 0 \qquad \tilde{\delta}\widetilde{\phi}^i = R(\mathring{\phi} + \widetilde{\phi})^i{}_\alpha \,\xi^\alpha$$

Background Fields and BRST

- The combined gauge algebra closes.

$$[\,\mathring{\delta}\,,\mathring{\delta}\,] = \mathring{\delta} + \widetilde{\delta} \quad [\,\mathring{\delta}\,,\widetilde{\delta}\,] = \widetilde{\delta} \quad [\,\widetilde{\delta}\,,\widetilde{\delta}\,] = \widetilde{\delta}$$

- Elevate both the background and the quantum gauge parameters to ghosts:

$$\mathring{\xi}^{\alpha} \to \Lambda \mathring{c}^{\alpha}\,, \qquad \xi^{\alpha} \to \Lambda c^{\alpha}$$

- Nilpotent BRST charge $Q_B^2 = 0$
- Appropriate BRST-invariant gauge-fixing terms.

BRST on Ghosts

- $$\delta_{\mathrm{B}}\, \mathring{c}^{\,\gamma} = \tfrac{1}{2}\, f(\mathring{\phi})_{\alpha\beta}{}^{\gamma}\, \mathring{c}^{\,\alpha} \Lambda\, \mathring{c}^{\,\beta}$$
$$\delta_{\mathrm{B}}\, c^{\,\gamma} = \tfrac{1}{2}\, f(\phi)_{\alpha\beta}{}^{\gamma}\, (c + \mathring{c})^{\alpha} \Lambda\, (c + \mathring{c})^{\beta} - \tfrac{1}{2}\, f(\mathring{\phi})_{\alpha\beta}{}^{\gamma}\, \mathring{c}^{\,\alpha}\, \Lambda\, \mathring{c}^{\,\beta}$$

- $$\delta_{\mathrm{B}}\, b_{\alpha} = \Lambda\, B_{\alpha}\,, \qquad \delta_{\mathrm{B}}\, B_{\alpha} = 0$$

- Deform the BRST transformation by requiring that background fields and background ghosts are invariant:

$$Q_B \to \hat{Q}\,, \qquad \hat{Q}^2 = H$$

Deformation of BRST

- Choose background fields corresponding to the near horizon attractor geometry.
- Deform the BRST symmetry by demanding that ***both the background and background ghosts*** are invariant under the symmetry.

$$Q_B \to \hat{Q} \qquad \hat{Q}^2 = H$$

Background ghosts play the role of parameters of the background symmetries. Note that ghosts for fermionic supergravity symmetries are bosonic.

Equivariant Cohomology

- $$\hat{\delta}\, \mathring{\phi}^{\,i} = 0 \qquad\qquad \hat{\delta}\, \mathring{c}^{\,\alpha} = 0$$
  $$\hat{\delta}\, \widetilde{\phi}^{\,i} = R(\mathring{\phi} + \tilde{\phi})^{i}{}_{\alpha}\, \Lambda\, (c^{\alpha} + \mathring{c}^{\,\alpha})$$
  $$\hat{\delta}\, c^{\,\alpha} = \tfrac{1}{2}\, f(\phi)_{\beta\gamma}{}^{\alpha}\, (c + \mathring{c})^{\beta} \Lambda\, (c + \mathring{c})^{\gamma}$$
  $$-\tfrac{1}{2}\, f(\mathring{\phi})_{\beta\gamma}{}^{\alpha}\, \mathring{c}^{\,\beta}\, \Lambda\, \mathring{c}^{\,\gamma}$$
- $$\hat{\delta}^{2} = \delta_{\mathring{\xi}}\,, \qquad\qquad [\hat{\delta}\,, \delta_{\mathring{\xi}}] = 0$$
  $$\mathring{\xi}^{\alpha} \equiv \Lambda_{[2}\, f(\mathring{\phi})_{\beta\gamma}{}^{\alpha}\, \mathring{c}^{\,\beta}\, \Lambda_{1]}\, \mathring{c}^{\,\gamma}$$

Equivariant Cohomology instead of **BRST Cohomology**

One-loop Determinants *Pestun (07)*

- We still need to compute one-loop determinants for the quadratic fluctuations of the QV action.
- Group the fields such as (X_0, X_1) so that the linearized action of Q on these fields is simple:

$$Q\begin{pmatrix} X_0 \\ X_1 \end{pmatrix} = \begin{pmatrix} X_0' \\ X_1' \end{pmatrix}$$
$$Q\begin{pmatrix} X_0' \\ X_1' \end{pmatrix} = \begin{pmatrix} H_0 X_0 \\ H_1 X_1 \end{pmatrix} = H\begin{pmatrix} X_0 \\ X_1 \end{pmatrix}$$

(cohomological variables, blue bosonic, red fermionic)

$$V^{(2)}(X_0) = D(X_1')$$

Computation of Determinants

- $$QV^{(2)} = (X_B, K_B X_B) + (X_F, K_F X_F)$$

$$Z_{det} = \left[\frac{\det K_F}{\det K_B}\right]^{\frac{1}{2}} = \left[\frac{\det_{Coker(D)} H}{\det_{Ker(D)} H}\right]^{\frac{1}{2}}$$

- These ratios of determinants can be computed explicitly or by using Atiyah-Bott index formula.

$$Z_{det} = \exp\left[-K(\phi + iP)\left(n_v - n_h + 23/24\right)\right]$$

- Here K is the Kähler potential. $e^{-K} = l^2(\phi + iP)$ is the size of *AdS2* on the localizing manifold.

Jeon, Murthy, Reys (15, 18); David Gupta, Gava, Narain (17,18)

Atiyah-Bott Index Theorem

We are interested in $$\frac{\det_{Coker(D)} H}{\det_{Ker(D)} H} = \prod_n \lambda_n^{-a(n)}$$

Read off the multiplicities from an index $\textbf{ind}(D)(t)$:

$$\textbf{Tr}_{Ker(D)}(e^{-iHt}) - \textbf{Tr}_{Coker(D)}(e^{-iHt}) = \sum_n a(n) e^{-i\lambda_n t}$$

e^{-iRt} has a group action on spacetime (x) and field space (ϕ^i). Operator ***D*** is `transversally elliptic'

$$x \to \tilde{x} = e^{-iHt}(x)\,, \qquad \phi^i(x) \to \tilde{\phi}(\tilde{x}) = \gamma^{ij}\phi(x)$$

$$\boxed{\textbf{ind}(D)(t) = \sum_{x=\tilde{x}} \frac{(\textbf{str}\,(\gamma))}{\det(1 - \partial\tilde{x}/\partial x)}}$$

Determinant for a Hypermultiplet

In $AdS_2 \times S^2$ fixed points of our *H = L -J* are center of AdS_2 disk and North Pole and South Pole in S^2

$$\det(1 - \frac{\partial \tilde{x}}{\partial x}) = (1-q)^2(1-q^{-1})^2 \qquad q := e^{-it/l}$$

$$\textbf{str}(\gamma) = 4 - 2q - 2q^{-1} = -2q(1-q^{-1})^2$$

$$\textbf{ind}(D) = -4q(1-q)^{-2} = \sum_n 4nq^n$$

$$Z_{det} = \prod_{n \geq 1} (\frac{-in}{l})^{4n} \to (l^2)^{-\frac{1}{12}}$$