Entanglement In Quantum Field Theory

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There will be four topics today:

(1) The Reeh-Schlieder theorem (1961), which is the basic result showing that entanglement is unavoidable in quantum field theory.

(2) Relative entropy in quantum field theory.

(3) General proof of monotonicity of relative entropy (or strong subadditivity) in quantum mechanics.

(4) Density matrix for Rindler space.

We consider a quantum field theory in Minkowski spacetime M, with a Hilbert space $\mathcal{H}$ that contains a vacuum state Ω. There is an algebra of local operators, whose action can produce "all" states (or at least all states in a superselection sector) from the vacuum. For simplicity in the notation, we will assume that this operator algebra is generated by a hermitian scalar field $\phi(x)$. So states

$$\phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle$$

with arbitrary n and points $x_1, \ldots, x_n \in M$, are dense in $\mathcal{H}$.

The Reeh-Schlieder theorem says that actually, we get a dense set of states (in the vacuum sector of $\mathcal{H}$) if we restrict the points $x_1, \cdots, x_n$ to any possibly very small open set $\mathcal{U} \subset M$:

If this is false, there is a state χ in the vacuum sector such that

$$\langle\chi|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle = 0$$

whenever $x_1, \cdots, x_n \in \mathcal{U}$.

We will show that any such χ actually satisfies

$$\langle\chi|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle = 0$$

for all $x_i \in M$. Since states created by the ϕ's are dense (in the vacuum sector) this implies that $\chi = 0$.

Let us define

$$f(x_1, x_2, \cdots x_n) = \langle\chi|\phi(x_1)\phi(x_2)\cdots\phi(x_n)|\Omega\rangle.$$

We are given that this function vanishes if the x_i are in $\mathcal{U}$ and we want to prove that it vanishes for all x_i.

As a first step, pick a future-pointing timelike vector t and consider shifting x_n by a real multiple of t:

$$x_n \to x_n + u\mathrm{t}.$$

Let

$$g(u) = \langle \chi | \phi(x_1)\phi(x_2)\cdots\phi(x_{n-1})\phi(x_n + u\mathrm{t})|\Omega\rangle$$

with $x_i \in \mathcal{U}$. We have $g(u) = 0$ for sufficiently small real u because then $x_n + u\mathrm{t}$ is still in u. Also with H the Hamiltonian for translation in the t direction,

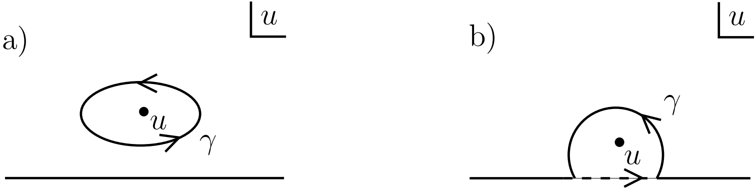
$$g(u) = \langle \chi | \phi(x_1)\phi(x_2)\cdots \exp(\mathrm{i}Hu)\phi(x_n)\exp(-\mathrm{i}Hu)|\Omega\rangle.$$

Since $H\Omega = 0$ this is

$$g(u) = \langle \chi | \phi(x_1)\phi(x_2)\cdots \exp(\mathrm{i}Hu)\phi(x_n)|\Omega\rangle$$

and since H is nonnegative, $g(u)$ is holomorphic in the upper half u-plane.

Such a function is zero. If we knew $g(u)$ to be holomorphic *on* the real axis and vanishing on a segment I of the real axis, we would say that $g(u)$ has a convergent Taylor series expansion around a point $p \in I$ and this expansion would have to be identically zero to make $g(u)$ vanish on the axis. To begin with we only know that $g(u)$ is holomorphic above the real axis, not on it, but we can get around this using the Cauchy integral formula:



So now we know that (keeping $x_1, \cdots, x_{n-1} \in \mathcal{U}$)

$$\langle \chi | \phi(x_1) \cdots \phi(x_{n-1}) \phi(x_n') | \Omega \rangle$$

vanishes if $x_n' = x_n + u\mathrm{t}$ where t is a timelike vector and u is any real number. Now we do this again, picking another timelike vector t' and replacing x_n' by $x_n'' = x_n' + u'\mathrm{t}'$ with u' real. Repeating the argument, we learn that

$$\langle \chi | \phi(x_1) \cdots \phi(x_{n-1}) \phi(x_n'') | \Omega \rangle = 0$$

for any such x_n''. But since any point in M can be reached from $\mathcal{U}$ by zigzagging backwards and forwards in various timelike directions, we learn that

$$\langle \chi \phi(x_1) \cdots \phi(x_{n-1}) \phi(x_n) | \Omega \rangle = 0$$

for $x_1, \cdots, x_{n-1} \in \mathcal{U}$ with no restriction on x_n.

The next step is to remove the restriction on x_{n-1}. We pick t as before and now consider a common shift of x_{n-1} and x_n in the t direction

$$(x_{n-1}, x_n) \to (x'_{n-1}, x'_n) = (x_{n-1} + u\mathrm{t}, x_n + u\mathrm{t})$$

Now we look at

$$h(u) = \langle\chi|\phi(x_1)\phi(x_2)\cdots\phi(x_{n-1} + u\mathrm{t})\phi(x_n + u\mathrm{t})|\Omega\rangle$$

It vanishes for small real u, and it can be written

$$h(u) = \langle\chi|\phi(x_1)\phi(x_2)\cdots\exp(\mathrm{i}uH)\phi(x_{n-1})\phi(x_n)|\Omega\rangle,$$

which implies that $h(u)$ is holomorphic in the upper half plane. Hence $h(u)$ is identically 0.

Repeating the process by shifting x_{n-1} and x_n in some other timelike direction, we learn that

$$\langle\chi|\phi(x_1)\cdots\phi(x_{n-1})\phi(x_n)|\Omega\rangle$$

vanishes for $x_1, \cdots, x_{n-2} \in \mathcal{U}$ with no restriction on x_{n-1}, x_n. The next step is to remove the restriction on x_{n-2}. We do this in exactly the same way, by considering what happens when we shift the last three coordinates by a common timelike vector. And so on.

So we end up proving the Reeh-Schlieder theorem: an “arbitrary” state (more exactly, a dense set of states in the vacuum sector of Hilbert space) can be created from the vacuum by acting with a product of local operators in a small open set $\mathcal{U} \subset M$.

Now I want to discuss the interpretation of this theorem. The first question that I want to dispose of is whether it contradicts causality. It certainly sounds unintuitive at first sight. Consider a state of the universe that on some initial time slice looks like the vacuum near $\mathcal{U}$, but contains the planet Jupiter at a distant region spacelike $\mathcal{V}$ separated from $\mathcal{U}$. Let J be a "Jupiter" operator whose expectation value in a state that contains the planet Jupiter in region $\mathcal{V}$ is close to 1, while its expectation value is close to 0 otherwise. The Reeh-Schlieder theorem says that there is an operator X in region $\mathcal{U}$ such that the state $X\Omega$ contains the planet Jupiter in region $\mathcal{V}$. So

$$\langle\Omega|J|\Omega\rangle \cong 0, \qquad \langle X\Omega|J|X\Omega\rangle \cong 1.$$

Is this a contradiction? We have $\langle X\Omega|J|X\Omega\rangle = \langle\Omega|X^\dagger JX|\Omega\rangle$. Since X is supported in $\mathcal{U}$ and J in the spacelike separated region V, $X^\dagger$ and J commute. So

$$1 \cong \langle X\Omega|J|X\Omega\rangle = \langle\Omega|JX^\dagger X|\Omega\rangle.$$

If X were *unitary* there would be a contradiction between the statements

$$0 \cong \langle\Omega|J|\Omega\rangle$$

and

$$1 \cong \langle\Omega|JX^\dagger X|\Omega\rangle,$$

because if X is unitary, then $X^\dagger X = 1$. But the Reeh-Schlieder theorem does not tell us that we can pick X to be unitary; it just tells us that there is *some* X in region $\mathcal{U}$ that will create the planet Jupiter in a distant region $\mathcal{V}$.

In comparing the above formulas, all we have found is that in the vacuum, the operators J and $X^\dagger X$ have a nonzero correlation function in the vacuum at spacelike separation. There is no contradiction there; spacelike correlations in quantum field theory are ubiquitous, even in free field theory.

The intuitive interpretation of the Reeh-Schlieder theorem involves entanglement between the degrees of freedom inside an open set $\mathcal{U}$ and those outside $\mathcal{U}$. To explain the intuitive picture, let us imagine that the Hilbert space $\mathcal{H}$ of our QFT has a factorization

$$\mathcal{H} = \mathcal{H}_{\mathcal{U}} \otimes \mathcal{H}_{\mathcal{U}'}$$

where $\mathcal{H}_{\mathcal{U}}$ describes the degrees of freedom in region $\mathcal{U}$ and $\mathcal{H}_{\mathcal{U}'}$ describes all of the degrees of freedom outside of $\mathcal{U}$. Then any state in $\mathcal{H}$, such as the vacuum state Ω, would have a decomposition

$$\Omega = \sum_i \sqrt{p_i}\psi^i_{\mathcal{U}} \otimes \psi^i_{\mathcal{U}'}$$

where we can assume the states $\psi^i_{\mathcal{U}}$ and also $\psi^i_{\mathcal{U}'}$ to be orthonormal and we assume the p_i are all positive (otherwise we drop some terms from the sum).

In general when we write

$$\Omega = \sum_i \sqrt{p_i} \psi^i_{\mathcal{U}} \otimes \psi^i_{\mathcal{U}'}$$

the $\psi^i_{\mathcal{U}}$ and $\psi^i_{\mathcal{U}'}$ do not form a basis of $\mathcal{H}_{\mathcal{U}}$ or of $\mathcal{H}_{\mathcal{U}'}$, because there are not enough of them. However, something like the Reeh-Schlieder theorem will be true for any state Ω such the $\psi^i_{\mathcal{U}}$ and the $\psi^i_{\mathcal{U}'}$ do form bases of their respective spaces. Using the fact that the $\psi^i_{\mathcal{U}'}$ are a basis of $\mathcal{H}_{\mathcal{U}'}$, we would be able to expand any state $\Psi \in \mathcal{H}$ as

$$\Psi = \sum_i \lambda^i_{\mathcal{U}} \otimes \psi^i_{\mathcal{U}'}, \qquad \lambda^i \in \mathcal{H}_{\mathcal{U}}.$$

Then because the $\psi^i_{\mathcal{U}}$ are a basis and the p_i are nonzero, we can define a linear operator X acting on $\mathcal{H}_{\mathcal{U}}$ by

$$X(\sqrt{p}_i \psi^i_{\mathcal{U}}) = \lambda^i$$

and we see that we have found an operator X acting only on degrees of freedom in $\mathcal{U}$ such that

$$X\Omega = \Psi.$$

A state

$$\Omega = \sum_i \sqrt{p_i}\psi^i_{\mathcal{U}} \otimes \psi^i_{\mathcal{U}'}$$

where the p_i are all positive and the $\psi^i_{\mathcal{U}}$, $\psi^i_{\mathcal{U}'}$ are bases might be called a "fully" entangled state. (I don't think this is standard terminology.) We call a state "maximally" entangled if the p_i are all equal (this is not possible for Hilbert spaces of infinite dimension, as in quantum field theory). The Reeh-Schlieder theorem means intuitively that the vacuum state Ω of a quantum field theory is fully entangled in this sense, between the inside and outside of an arbitrary open set $\mathcal{U}$.

However, the decomposition

$$\mathcal{H} = \mathcal{H}_{\mathcal{U}} \otimes \mathcal{H}_{\mathcal{U}'}$$

that we started with is certainly not literally valid in quantum field theory. If it were, then in $\mathcal{H}$ there would be an unentangled pure state $\psi \otimes \chi$, $\psi \in \mathcal{H}_{\mathcal{U}}$, $\chi \in \mathcal{H}_{\mathcal{U}'}$. This contradicts the fact that in quantum field theory there is a universal ultraviolet divergence in the entanglement entropy: the entanglement entropy of the vacuum between degrees of freedom in $\mathcal{U}$ and those outside of $\mathcal{U}$ is ultraviolet divergent, and the leading ultraviolet divergence is universal, that is it is the same for any state. The leading divergence is universal because any state looks like the vacuum at short distances.

Now let us discuss an important corollary of the Reeh-Schlieder theorem. Let $\mathcal{U}$ and $\mathcal{V}$ be spacelike separated open sets in Minkowski spacetime:

Let b be an operator supported in $\mathcal{V}$. Suppose that

$$\mathrm{b}\Omega = 0.$$

Then if a is supported in $\mathcal{U}$, we have

$$\mathrm{b}(\mathrm{a}\Omega) = \mathrm{a}\mathrm{b}\Omega = 0,$$

where I use the fact that $[\mathrm{a}, \mathrm{b}] = 0$ since $\mathcal{U}$ and $\mathcal{V}$ are spacelike separated. But the states $\mathrm{a}\Omega$ are dense in $\mathcal{H}$ (according to Reeh-Schlieder) so b identically vanishes.

Thus if $\mathsf{b} \neq 0$ is supported in a spacelike open set $\mathcal{V}$ that is small enough that it is spacelike separated from another open set $\mathcal{U}$, then

$$\mathsf{b}\Omega \neq 0.$$

The roles of $\mathcal{U}$ and $\mathcal{V}$ are symmetrical, so also for $\mathsf{a} \neq 0$ supported in $\mathcal{U}$,

$$\mathsf{a}\Omega \neq 0.$$

Let $\mathcal{A}_{\mathcal{U}}$ be the algebra of operators in region $\mathcal{U}$. We have proved two facts about the algebra $\mathcal{A}_{\mathcal{U}}$ acting on the vacuum sector $\mathcal{H}$:

States $\mathrm{a}\Omega$, $\mathrm{a} \in \mathcal{A}_{\mathcal{U}}$, are dense in $\mathcal{H}$. This is described by saying that Ω is a "cyclic" vector for the algebra $\mathcal{A}_{\mathcal{U}}$.

For any nonzero $\mathrm{a} \in \mathcal{A}_{\mathcal{U}}$, $\mathrm{a}\Omega \neq 0$. This is described by saying that Ω is a "separating" vector of $\mathcal{A}_{\mathcal{U}}$.

In short, the Reeh-Schlieder theorem and its corollary say that the vacuum is a cyclic separating vector for $\mathcal{A}_{\mathcal{U}}$.

Consider a quantum system with a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and let $\mathcal{A}$ be the algebra of operators on $\mathcal{H}_1$. A little thought shows that a general vector Ψ with its usual Schmidt decomposition

$$\Psi = \sum_i \sqrt{p_i} \psi_1^i \otimes \psi_2^i$$

is cyclic for $\mathcal{A}$ if the ψ_2^i are a basis of $\mathcal{H}_2$, and it is separating for $\mathcal{A}$ if the ψ_1^i are a basis for $\mathcal{H}_1$. So that is the meaning of the cyclic separating property if the Hilbert space is a tensor product.

This completes part 1 of the lecture.

Now, what about entanglement in quantum field theory? A mathematical machinery that can be useful for analyzing entanglement when the Hilbert space does not factorize is called Tomita-Takesaki theory. It applies whenever one has an algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ with a cyclic separating vector. My next goal will be an introduction to this. (Tomita-Takesaki theory has had many applications in recent years in quantum field theory, some of which we will hear about from other lecturers.)

The starting point in Tomita-Takesaki theory is that, given an algebra $\mathcal{A}$ with cyclic separating vector Ψ, we define an antilinear operator, the Tomita operator

$$S_\Psi : \mathcal{H} \to \mathcal{H}$$

by

$$S_\Psi \mathrm{a}\Psi = \mathrm{a}^\dagger \Psi.$$

The definition makes sense because of the separating property (if we could have $\mathrm{a}\Psi = 0$ with $\mathrm{a}^\dagger\Psi \neq 0$, we would get a contradiction) and it does define S_Ψ on a dense set of states in $\mathcal{H}$, because of the cyclic property (states $\mathrm{a}\Psi$ are dense in $\mathcal{H}$). A couple of obvious facts are that

$$S_\Psi^2 = 1$$

(which in particular says that S_Ψ is invertible) and

$$S_\Psi|\Psi\rangle = |\Psi\rangle.$$

The *modular operator* is a linear, self-adjoint operator defined by

$$\Delta_\Psi = S_\Psi^\dagger S_\Psi.$$

(The definition of the adjoint of an antilinear operator is $\langle\alpha|S|\beta\rangle = \langle\beta|S^\dagger|\alpha\rangle$.) Δ_Ψ is positive-definite because S_Ψ is invertible.

We will also need the relative modular operator. Let the state Ψ be cyclic separating, and let Φ be any other state. The relative Tomita operator $S_{\Psi|\Phi}$ is an antilinear operator defined by

$$S_{\Psi|\Phi} a|\Psi\rangle = a^\dagger|\Phi\rangle.$$

Again the well-definedness of the definition depends on the cyclic separating nature of Ψ, but no property of Φ is needed. In defining $S_{\Psi|\Phi}$, we assume that Ψ and Φ are unit vectors

$$\langle\Psi|\Psi\rangle = \langle\Phi|\Phi\rangle = 1.$$

The relative modular operator is defined by

$$\Delta_{\Psi|\Phi} = S^\dagger_{\Psi|\Phi} S_{\Psi|\Phi}.$$

It is still self-adjoint and positive semi-definite, but it is not necessarily invertible. If $\Phi = \Psi$ then the definitions reduce to the previous ones

$$S_{\Psi|\Psi} = S_\Psi, \quad \Delta_{\Psi|\Psi} = \Delta_\Psi.$$

Now we are ready to define relative entropy in quantum field theory. We fix an open set $\mathcal{U}$ (small enough so that the vacuum is cyclic separating), and consider the algebra $\mathcal{A}_\mathcal{U}$. Let Ψ be any cyclic separating vector for $\mathcal{A}_\mathcal{U}$, and Φ any other vector. The relative entropy between the states Ψ and Φ, for measurements in region $\mathcal{U}$ (as defined by Araki in the 1970's) is

$$\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi}|\Psi\rangle.$$

It is not immediately obvious that this has anything to do with relative entropy as defined in yesterday's lecture, but we will later see that this definition reduces to the more familiar one for the case of an ordinary quantum system. For now, let us just proceed and explore the consequences of this definition.

First let us discuss positivity properties of relative entropy, defined by

$$\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi}|\Psi\rangle.$$

First of all, if $\Phi = \Psi$ then we had

$$\Delta_\Psi\Psi = \Psi$$

so

$$\log\Delta_\Psi\Psi = 0$$

and hence the relative entropy between Ψ and itself is 0:

$$\mathcal{S}_{\Psi|\Psi}(\mathcal{U}) = 0.$$

But more than that, suppose that $\Phi = \mathsf{a}'\Psi$, where a' is unitary and $[\mathsf{a}', \mathcal{A}_\mathcal{U}] = 0$, so that measurements in region $\mathcal{U}$ cannot distinguish Φ from Ψ. One can show that in this case again $\Delta_{\Psi|\Phi} = \Delta_\Psi$ (exercise!) so again

$$\mathcal{S}_{\Psi|\mathsf{a}'\Psi}(\mathcal{U}) = 0.$$

Now consider a completely general state Φ. The inequality $-\log\lambda \geq 1-\lambda$ for a positive real number λ implies an operator inequality $-\log\Delta \geq 1-\Delta$, implying

$$\begin{aligned}\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) &= -\langle\Psi|\log\Delta_{\Psi|\Phi}|\Psi\rangle \geq \langle\Psi|(1-\Delta_{\Psi|\Phi})|\Psi\rangle \\ &= \langle\Psi|\Psi\rangle - \langle\Psi|S^\dagger_{\Psi|\Phi}S_{\Psi|\Phi}|\Psi\rangle = \langle\Psi|\Psi\rangle - \langle\Phi|\Phi\rangle = 0.\end{aligned}$$

So in general

$$\mathcal{S}_{\Psi|\Phi} \geq 0.$$

(For a converse to what we said before – the proof that $\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = 0$ only if $\Phi = \mathrm{a}'\Psi$ for some unitary a' that commutes with $\mathcal{A}_\mathcal{U}$ – I refer to section 3.3 of my notes.)

We gave another proof of positivity of relative entropy in the first lecture, but we do not yet know that they were proving the same thing; we will only learn that when we analyze Tomita-Takesaki theory for a factorized quantum system, later on.

Now we consider a smaller open set $\widetilde{\mathcal{U}} \subset \mathcal{U}$. Now we have two different algebras $\mathcal{A}_{\widetilde{\mathcal{U}}} \subset \mathcal{A}_{\mathcal{U}}$ and two different operators $S_{\Psi|\Phi;\widetilde{\mathcal{U}}}$ and $S_{\Psi|\Phi;\mathcal{U}}$ and associated modular operators $\Delta_{\Psi|\Phi;\widetilde{\mathcal{U}}}$ and $\Delta_{\Psi|\Phi;\mathcal{U}}$. The relative entropy beween Ψ and Φ for measurements in $\mathcal{U}$ is

$$\mathcal{S}_{\Psi|\Phi}(\mathcal{U}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\mathcal{U}}|\Psi\rangle.$$

The corresponding relative entropy for measurements in $\widetilde{\mathcal{U}}$ is

$$\mathcal{S}_{\Psi|\Phi}(\widetilde{\mathcal{U}}) = -\langle\Psi|\log\Delta_{\Psi|\Phi;\widetilde{\mathcal{U}}}|\Psi\rangle.$$

We want to prove that relative entropy is monotonic under increasing the region considered:

$$\mathcal{S}_{\Psi|\Phi}(\widetilde{\mathcal{U}}) \leq \mathcal{S}_{\Psi|\Phi}(\mathcal{U}).$$

This is an important statement for applications; for instance, it was used by A. Wall in proving the generalized second law of thermodynamics.

The states Ψ and Φ will be held fixed in this discussion, so to llighten the notation we will omit subscripts and denote the operators just as $S_{\mathcal{U}}$, $S_{\widetilde{\mathcal{U}}}$ and likewise $\Delta_{\mathcal{U}}$ and $\Delta_{\widetilde{\mathcal{U}}}$. The main point of the proof is to show that as an operator

$$\Delta_{\widetilde{\mathcal{U}}} \geq \Delta_{\mathcal{U}}.$$

As I will explain in a moment, this implies

$$\log \Delta_{\widetilde{\mathcal{U}}} \geq \log \Delta_{\mathcal{U}}. \qquad (*)$$

The inequality we want

$$-\langle \Psi | \log \Delta_{\widetilde{\mathcal{U}}} | \Psi \rangle \leq -\langle \Psi | \log \Delta_{\mathcal{U}} | \Psi \rangle$$

is just a matrix element of inequality $(*)$ in the state Ψ.

To show that if P and Q are positive self-adjoint operators and

$$P \geq Q \qquad (*)$$

then also

$$\log P \geq \log Q$$

let

$$R(t) = tP + (1-t)Q$$

so (by virtue of $(*)$), R is an increasing function of t, in the sense that $\dot{R}(t) \geq 0$. We have

$$\log R(t) = \int_0^\infty \mathrm{d}s \left(\frac{1}{s} - \frac{1}{s+R} \right).$$

So

$$\frac{\mathrm{d}}{\mathrm{d}t} \log R(t) = \int_0^\infty \mathrm{d}s \frac{1}{s+R} \dot{R} \frac{1}{s+R}.$$

The integrand is positive since it is BAB with A, B positive ($A = \dot{R}$, $B = 1/(s+R)$), so the integral is positive and thus $\frac{\mathrm{d}}{\mathrm{d}t} \log R \geq 0$. Hence $R(1) \geq R(0)$ or

$$\log P \geq \log Q.$$

(In case you think that we just proved was obvious, let me remark that for operators, the inequality $P \geq Q$ does not imply $P^2 \geq Q^2$. The function $P \to \log P$ is better, in that sense, than the function $P \to P^2$. Incidentally, $P^2 \geq Q^2$ for positive P, Q does imply $P \geq Q$.)

So monotonicity of relative entropy under increasing the region considered will follow from an inequality

$$\Delta_{\widetilde{\mathcal{U}}} \geq \Delta_{\mathcal{U}}.$$

If we try to understand this inequality, we may get confused at first. We have

$$\Delta_{\widetilde{\mathcal{U}}} = S^\dagger_{\widetilde{\mathcal{U}}} S_{\widetilde{\mathcal{U}}}, \qquad \Delta_{\mathcal{U}} = S^\dagger_{\mathcal{U}} S_{\mathcal{U}},$$

Here the two S's were defined, naively, by the same formula

$$S_{\widetilde{\mathcal{U}}} \mathsf{a}\Psi = \mathsf{a}^\dagger \Phi, \qquad S_{\mathcal{U}} \mathsf{a}\Psi = \mathsf{a}^\dagger \Psi$$

with the sole difference that a is in $\mathcal{A}_{\widetilde{\mathcal{U}}}$ in one case and in $\mathcal{A}_{\mathcal{U}}$ in the other. The algebra $\mathcal{A}_{\mathcal{U}}$ is bigger, so $S_{\mathcal{U}}$ is defined on more states. But states $\mathsf{a}\Psi$ with $\mathsf{a} \in \mathcal{A}_{\widetilde{\mathcal{U}}}$ are already dense in Hilbert space so actually $S_{\widetilde{\mathcal{U}}}$ and $S_{\mathcal{U}}$ coincide on a dense set of states.

If one is careless, one might assume that two operators that agree on a dense subspace of Hilbert space actually coincide. This is not true, however, for unbounded operators such as $S_{\widetilde{\mathcal{U}}}$ and $S_{\mathcal{U}}$. We have to remember that an unbounded operator is never defined on all states in Hilbert space, only (at most) on a dense subspace. The proper statement is that $S_{\mathcal{U}}$ is an extension of $S_{\widetilde{\mathcal{U}}}$, meaning that $S_{\mathcal{U}}$ is defined whenever $S_{\widetilde{\mathcal{U}}}$ is defined and, on states on which they are both defined, they coincide. In our problem, $S_{\mathcal{U}}$ is a proper extension, because there are states $a\Psi$, $a \in \mathcal{A}_{\mathcal{U}}$, that are not of the form $a\Psi$, $a \in \mathcal{A}_{\widetilde{\mathcal{U}}}$. Anyway, the fact that $S_{\mathcal{U}}$ is an extension of $S_{\widetilde{\mathcal{U}}}$ implies, as a general Hilbert space statement, that

$$S_{\widetilde{\mathcal{U}}}{}^{\dagger} S_{\widetilde{\mathcal{U}}} \geq S_{\mathcal{U}}{}^{\dagger} S_{\mathcal{U}},$$

which is what we need for monotonicity of relative entropy.

The intuitive idea of the inequality

$$S_{\widetilde{\mathcal{U}}}{}^{\dagger} S_{\widetilde{\mathcal{U}}} \geq S_{\mathcal{U}}{}^{\dagger} S_{\mathcal{U}}$$

is that the fact that $S_{\widetilde{\mathcal{U}}}$ is defined on fewer states than $S_{\mathcal{U}}$ is defined on corresponds to a constraint that has been placed on the states in the case of $S_{\widetilde{\mathcal{U}}}$, and this constraint raises the energy (i.e. the value of $\Delta = S^{\dagger} S$). I will give an analogy that aims to make this obvious. Instead of $S_{\mathcal{U}}$, we will consider the exterior derivative d mapping zero-forms (functions) on a manifold M to 1-forms. But we will assume that M has a boundary N, and we will consider two different versions of the operator d.

The first will be the derivative operator acting on differentiable functions that are constrained to vanish on the boundary of X:

$$\widehat{\mathrm{d}} : f(x_1, \ldots, x_n) \to \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x^n} \right), \quad f|_{\partial X} = 0.$$

We also consider the same operator d without the constraint that f vanishes on the boundary. Differentiable functions that vanish on the boundary are dense in Hilbert space, so $\widehat{\mathrm{d}}$ and d are each defined on a dense subspace of Hilbert space; moreover, obviously, d is an extension of $\widehat{\mathrm{d}}$ since it is defined whenever $\widehat{\mathrm{d}}$ is defined and they agree when they are both defined.

Associated to $\widehat{\mathrm{d}}$ is the Dirichlet Laplacian

$$\widehat{\Delta} = \widehat{\mathrm{d}}^\dagger \widehat{\mathrm{d}}$$

and associated in the same way to d is the Neumann Laplacian

$$\Delta = \mathrm{d}^\dagger \mathrm{d}.$$

Here $\widehat{\Delta}$ and Δ are nonnegative operators that coincide on a dense set of states, but $\widehat{\Delta}$ is more positive than Δ because of the constraint that the wavefunction should vanish on the boundary.

Indeed, the Neumann Laplacian Δ is associated to the energy function

$$\langle f|\Delta|f\rangle = \frac{1}{2}\int_M \mathrm{d}^n x\sqrt{g}|\mathrm{d}f|^2$$

but to get the Dirichlet Laplacian $\widehat{\Delta}$ we should add a boundary term to the energy to make the wavefunction vanish on the boundary. In fact, we can consider a family of operators Δ_t, $0 \leq t \leq \infty$ associated to the energy function

$$\langle f|\Delta_t|f\rangle = \frac{1}{2}\int_M \mathrm{d}^n x\sqrt{g}|\mathrm{d}f|^2 + t\int_N \mathrm{d}^{n-1}x\sqrt{g}|f|^2.$$

Clearly the operator Δ_t is an increasing function of t. For $t = 0$, Δ_t is the Neumann Laplacian, and for $t \to \infty$, Δ_t goes over to the Dirichlet Laplacian $\widehat{\Delta}$.

So $\widehat{\Delta} \geq \Delta$, which is analogous to our desired $\Delta_{\widetilde{\mathcal{U}}} \geq \Delta_{\mathcal{U}}$. A constraint on the state always raises the energy.

Just to make sure the analogy is clear, $\Delta_{\widetilde{\mathcal{U}}}$ is the operator associated to the energy function

$$\langle S_{\widetilde{\mathcal{U}}}\Lambda | S_{\widetilde{\mathcal{U}}}\Lambda\rangle$$

for a state Λ that should be in the domain of $S_{\widetilde{\mathcal{U}}}$. $\Delta_{\mathcal{U}}$ is similarly associated to

$$\langle S_{\mathcal{U}}\Lambda | S_{\mathcal{U}}\Lambda\rangle$$

for a state Λ that should be in the larger domain of $S_{\mathcal{U}}$. The second energy function is the same as the first except that it is defined on a larger space of states; we can get the second from the first by a constraint that removes some states. Such a constraint can only raiase the energy so $\Delta_{\widetilde{\mathcal{U}}} \geq \Delta_{\mathcal{U}}$. (See section 3.6 of my notes for a precise proof.)

Here is another analogy, now in finite dimensions. Let X be an $(n+m) \times (n+m)$ positive hermitian matrix, which we write in block form

$$X = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix}$$

For $\lambda > 0$, let

$$X_\lambda = \begin{pmatrix} A & B \\ B^\dagger & C + \lambda \end{pmatrix}.$$

Going from $\lambda = 0$ to $\lambda = \infty$ will be like going from Neumann to Dirichlet.

For $\lambda \to \infty$, the lower entries of a vector decouple and

$$\frac{1}{s + X_\lambda} \to \begin{pmatrix} 1/(s+A) & 0 \\ 0 & 0 \end{pmatrix}$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{1}{s+X_\lambda} = -\frac{1}{s+X_\lambda}\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\frac{1}{s+X_\lambda}$$

This is of the form $-CDC$ with C, D positive, so it is negative:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{1}{s+X_\lambda} \leq 0.$$

Hence

$$\left\langle \Psi \left| \frac{1}{s+X} \right| \Psi \right\rangle \geq \left\langle \Psi \left| \frac{1}{s+X_\lambda} \right| \Psi \right\rangle$$

for any Ψ and any $\lambda > 0$. Let us evaluate this for $\Psi = \begin{pmatrix}\psi\\ 0\end{pmatrix}$ and $\lambda \to \infty$ (using a result on the last slide):

$$\left\langle \Psi \left| \frac{1}{s+X} \right| \Psi \right\rangle \geq \left\langle \Psi \left| \frac{1}{s+X_\lambda} \right| \Psi \right\rangle \xrightarrow{\lambda\to\infty} \left\langle \psi \left| \frac{1}{s+A} \right| \psi \right\rangle .$$

Integrating over s from 0 to ∞, we learn

$$\langle\Psi|\log X|\Psi\rangle \leq \langle\psi|\log A|\psi\rangle. \qquad (*)$$

We will find that this inequality leads to monotonicity of relative entropy for a finite-dimensional quantum system.

To state it elegantly, define a unitary embedding $U : \mathbb{C}^n \to \mathbb{C}^{n+m}$ that takes ψ to $\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$. So $\Psi = U\psi$ and

$$\langle \Psi | \log X | \Psi \rangle = \langle U\psi | \log X | U\psi \rangle = \langle \psi | U^\dagger (\log X) U | \psi \rangle$$

Also

$$A = U^\dagger X U,$$

so our inequality $(*)$ on the previous slide becomes

$$\langle \psi | U^\dagger (\log X) U | \psi \rangle \leq \langle \psi | \log (U^\dagger X U) | \psi \rangle.$$

This completes part 2 of the lecture. I've explained what I regard as the most transparent explanation of monotonocity of relative entropy. But this argument as stated only applies to the special case of increasing the size of a region in spacetime.

If we had general monotonicity of relative entropy under partial trace, this would imply strong subadditivity of entropy. That in turn has had numerous applications in quantum field theory in recent years. But for this we need monotonicity of relative entropy in general, not just under increasing the size of a region.

What we will do now is to consider a general quantum system – finite-dimensional for simplicity – and imitate the ideas we've discussed up to this point. We'll define the Tomita-Takesaki operators, and fill a gap by explaining how the definition of relative entropy that we used today is related to yesterday's. Then we will imitate the proof of monotonicity that I just explained and arrive at a general proof of monotonicity under partial trace. (This proof is largely due to Petz and Nielsen.)

We start with a finite-dimensional Hilbert space $\mathcal{H}$ that is a tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces of the same dimension n. We let $\mathcal{A}$ be the algebra of $n \times n$ matrices acting on $\mathcal{H}_1$; an element $\mathrm{a} \in \mathcal{A}$ acts on $\mathcal{H}$ by $\mathrm{a} \otimes 1$. An arbitrary vector $\Psi \in \mathcal{H}$ has a decomposition

$$\Psi = \sum_{k=1}^{n} c_k |k\rangle \otimes |k\rangle'$$

where $|k\rangle$ and $|k\rangle'$, $k = 1, \ldots, n$ are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. (We will abbreviate $|j\rangle \otimes |k\rangle'$ as $|j, k\rangle$. We assume Ψ is a unit vector and likewise Φ later.) By now we know that Ψ is cyclic separating for the algebra $\mathcal{A}$ if and only if the c_k are all nonzero. Thus this is true for a generic vector.

The definition of the modular operator $S_\Psi : \mathcal{H} \to \mathcal{H}$ is

$$S_\Psi(\mathsf{a} \otimes 1)\Psi = (\mathsf{a}^\dagger \otimes 1)\Psi.$$

To make this into a formula for S_Ψ, pick i, j in the set $\{1, 2, \cdots, n\}$ and let a be the elementary matrix that acts by

$$\mathsf{a}|i\rangle = |j\rangle, \qquad \mathsf{a}|k\rangle = 0, \text{ if } k \neq i.$$

Its adjoint acts by

$$\mathsf{a}^\dagger|j\rangle = |i\rangle, \qquad \mathsf{a}^\dagger|k\rangle = 0, \text{ if } k \neq j.$$

So for $\Psi = \sum_i c_i|i, i\rangle$, we have

$$(\mathsf{a} \otimes 1)\Psi = c_i|j, i\rangle, \quad (\mathsf{a}^\dagger \otimes 1)\Psi = c_j|i, j\rangle.$$

So the definition of S_Ψ implies

$$S_\Psi(c_i|j, i\rangle) = c_j|i, j\rangle.$$

S_Ψ is antilinear, so

$$S_\Psi|j, i\rangle = \frac{c_j}{\bar{c}_i}|i, j\rangle.$$

The adjoint of S_Ψ is then

$$S_\Psi^\dagger |i,j\rangle = \frac{c_j}{\bar{c}_i} |j,i\rangle$$

and the modular operator $\Delta_\Psi = S_\Psi^\dagger S_\Psi$ is

$$\Delta_\Psi |j,i\rangle = \frac{|c_j|^2}{|c_i|^2} |j,i\rangle.$$

(In getting this formula, one has to remember that $S^\dagger$ is antilinear.)

We can describe the relative modular operator similarly. If Φ is a second state in $\mathcal{H}$, it has an expansion

$$\Phi = \sum_{\alpha=1}^{n} d_\alpha |\alpha\rangle \otimes |\alpha\rangle',$$

where $|\alpha\rangle$ and $|\alpha\rangle'$ (with $\alpha = 1, \cdots n$) are orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, in general different from the ones that appeared in the formula for Ψ. We will abbreviate $|\alpha, \beta\rangle$ for $|\alpha\rangle \otimes |\beta\rangle'$, $|\alpha, i\rangle$ for $|\alpha\rangle \otimes |i\rangle'$, etc. We will determine the relative modular operator $S_{\Psi|\Phi}$ straight from the definition

$$S_{\Psi|\Phi}(\mathsf{a} \otimes 1)\Psi = (\mathsf{a}^\dagger \otimes 1)\Phi.$$

For some $i, \alpha \in \{1, 2 \cdots, n\}$, define $\mathsf{a} \in \mathcal{A}$ by

$$\mathsf{a}|i\rangle = |\alpha\rangle, \qquad \mathsf{a}|j\rangle = 0 \;\text{ if } j \neq i.$$

Then

$$\mathsf{a}^\dagger|\alpha\rangle = |i\rangle, \qquad \mathsf{a}^\dagger|\beta\rangle = 0 \;\text{ if } \beta \neq \alpha.$$

So with $\Psi = \sum_i c_i |i, i\rangle$, $\Phi = \sum_\alpha d_\alpha |\alpha, \alpha\rangle$, we have

$$(\mathrm{a} \otimes 1)\Psi = c_i |\alpha, i\rangle, \qquad (\mathrm{a}^\dagger \otimes 1)\Phi = d_\alpha |i, \alpha\rangle.$$

So to get $S(\mathrm{a} \otimes 1)\Psi = (\mathrm{a}^\dagger \otimes 1)\Psi$, we need

$$S_{\Psi|\Phi} |\alpha, i\rangle = \frac{d_\alpha}{\bar{c}_i} |i, \alpha\rangle.$$

The adjoint is

$$S^\dagger_{\Psi|\Phi} |i, \alpha\rangle = \frac{d_\alpha}{\bar{c}_i} |\alpha, i\rangle$$

And therefore

$$\Delta_{\Psi|\Phi} |\alpha, i\rangle = \frac{|d_\alpha|^2}{|c_i|^2} |\alpha, i\rangle.$$

To make contact between the two definitions of relative entropy, we need to express $\Delta_{\Psi|\Phi}$ in terms of density matrices. The reduced density matrices of our state

$$\Psi = \sum_{k=1}^{n} c_k |k\rangle \otimes |k\rangle'$$

are

$$\rho_1 = \sum_i |c_i|^2 |i\rangle\langle i|, \qquad \rho_2 = \sum_i |c_i|^2 |i\rangle'\langle i|'.$$

Similarly the reduced density matrices of

$$\Phi = \sum_{\alpha=1}^{n} d_\alpha |\alpha\rangle \otimes |\alpha\rangle'$$

are

$$\sigma_1 = \sum_\alpha |d_\alpha|^2 |\alpha\rangle\langle\alpha|, \quad \sigma_2 = \sum_\alpha |d_\alpha|^2 |\alpha\rangle'\langle\alpha|'.$$

Comparing these formulas to what we found for the modular operators, we get

$$\Delta_\Psi = \rho_1 \otimes \rho_2^{-1}, \qquad \Delta_{\Psi|\Phi} = \sigma_1 \otimes \rho_2^{-1}.$$

Now we can compare Araki's definition of relative entropy

$$S(\Psi||\Phi) = -\langle\Psi|\log\Delta_{\Psi|\Phi}|\Psi\rangle$$

to the perhaps more familiar one of yesterday. The formula for $\Delta_{\Psi|\Phi}$ leads to $\log\Delta_{\Psi|\Phi} = \log\sigma_1 \otimes 1 - 1 \otimes \log\rho_2$. So $S(\Psi||\Phi)$ with Araki's definition is

$$-\langle\Psi|\log\sigma_1 \otimes 1|\Psi\rangle + \langle\Psi|1 \otimes \log\rho_2|\Psi\rangle$$

which is the same as

$$-\mathrm{Tr}_{\mathcal{H}_1}\rho_1\log\sigma_1 + \mathrm{Tr}_{\mathcal{H}_2}\rho_2\log\rho_2 = \mathrm{Tr}_{\mathcal{H}_1}\rho_1(\log\rho_1 - \log\sigma_1).$$

(In the last step, we use that $\mathrm{Tr}_{\mathcal{H}_2}\,\rho_2\log\rho_2 = \mathrm{Tr}_{\mathcal{H}_1}\,\rho_1\log\rho_1$ since ρ_1 and ρ_2 have the same eigenvalues.)

This indeed coincides with yesterday's definition.

Two remarks:

(1) Since the definitions are equivalent, today's proof of positivity of relative entropy makes sense verbatim in this situation and can indeed serve as a substitute for yesterday's.

(2) We've now derived the definition of relative entropy in two very different-looking ways: by considerations of classical probability theory at the beginning of yesterday's lecture and today by considerations of noncommutative algebras. I do think it is remarkable that they agree.

Now we want to understand the monotonicity of relative entropy in this setting. As discussed yesterday, this means that we consider a bipartite system with Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ (replacing what has been $\mathcal{H}_1$ so far) and with two density matrices ρ_{AB} and σ_{AB}. There are also reduced density matrices $\rho_A = \mathrm{Tr}_{\mathcal{H}_B} \rho_{AB}$, $\sigma_A = \mathrm{Tr}_{\mathcal{H}_B} \sigma_{AB}$ and we want to prove that

$$S(\rho_{AB}||\sigma_{AB}) \geq S(\rho_A||\sigma_A).$$

First we pass from $\mathcal{H}_{AB}$ to a "doubled" Hilbert space $\mathcal{H}_{AB} \otimes \mathcal{H}'_{AB}$ (playing the role played by $\mathcal{H}_1 \otimes \mathcal{H}_2$ until now) so that we can "purify" ρ_{AB} and σ_{AB} by deriving them as reduced density matrices asociated to pure states $\Psi_{AB}, \Phi_{AB} \in \mathcal{H}_{AB} \otimes \mathcal{H}'_{AB}$. Likewise Ψ_A, Φ_A are reduced density matrices associated to pure states $\Psi_A, \Phi_A \in \mathcal{H}_A \otimes \mathcal{H}'_A$. We can assume that Ψ_{AB}, Ψ_A are cyclic separating since as we have seen a generic vector has that property.

In quantum field theory, we had a small algebra $\mathcal{A}_{\widetilde{\mathcal{U}}}$ and a larger algebra $\mathcal{A}_{\mathcal{U}}$. In the present discussion, the analog of $\mathcal{A}_{\widetilde{\mathcal{U}}}$ is going to be the algebra $\mathcal{A}_A$ of matrices on $\mathcal{H}_A$ (acting on the first factor of $\mathcal{H}_A \otimes \mathcal{H}'_A$, in other words $\mathrm{a}_A \in \mathcal{A}_A$ acts on $\mathcal{H}_A \otimes \mathcal{H}'_A$ by $\Psi \to (\mathrm{a}_A \otimes 1)\Psi$) and the analog of $\mathcal{A}_{\mathcal{U}}$ is going to be the algebra $\mathcal{A}_{AB}$ of matrices on $\mathcal{H}_{AB}$ (acting similarly on the first factor of $\mathcal{H}_{AB} \otimes \mathcal{H}'_{AB}$).

In quantum field theory, $\mathcal{A}_{\widetilde{\mathcal{U}}}$ was naturally a subalgebra of $\mathcal{A}_{\mathcal{U}}$. The analog of this in the present context is that there is a natural embedding of $\mathcal{A}_A$ in $\mathcal{A}_{AB}$, namely

$$\varphi(\mathrm{a}) = \mathrm{a} \otimes 1.$$

Also, in quantum field theory, the small algebra and the large one naturally acted on the same Hilbert space, which was the Hilbert space of the quantum field theory. In the present context, the algebras $\mathcal{A}_A$ and $\mathcal{A}_{AB}$ act on different spaces $\mathcal{H}_A \otimes \mathcal{H}'_A$ and $\mathcal{H}_{AB} \otimes \mathcal{H}'_{AB}$. However, a natural map of the smaller space to the larger one presents itself, namely

$$U(\mathrm{a}\Psi_A) = \varphi(\mathrm{a})\Psi_{AB}.$$

Because Ψ_A is cyclic separating, this is a well-defined linear map from $\mathcal{H}_A \otimes \mathcal{H}'_A$ to $\mathcal{H}_{AB} \otimes \mathcal{H}'_{AB}$. A small calculation (see p. 42 of my notes for this and also for a remark on the next slide) shows that it is a unitary embedding.

A small calculation also shows that

$$U^\dagger \Delta_{AB} U = \Delta_A,$$

which is analogous to the relation between $\Delta_{\mathcal{U}}$ and $\Delta_{\widetilde{\mathcal{U}}}$ that we had in field theory. (It says that Δ_A and Δ_{AB} have the same matrix elements among the states on which Δ_A is defined.) So using our inequality from the end of part 2

$$\langle\psi|U^\dagger(\log X)U|\psi\rangle \leq \langle\psi|\log(U^\dagger XU)|\psi\rangle,$$

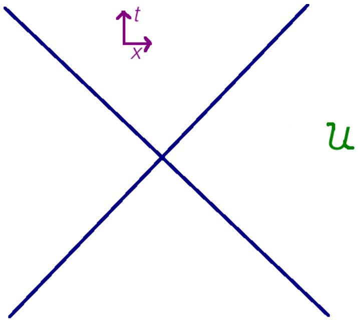
we get

$$\begin{aligned} S(\rho_A||\sigma_A) &= -\langle\Psi_A|\log\Delta_A|\Psi_A\rangle = -\langle\Psi_A|\log(U^\dagger\Delta_{AB}U)|\Psi\rangle \\ &\leq -\langle\Psi_A|U^\dagger(\log\Delta_{AB})U|\Psi_A\rangle \\ &= -\langle U\Psi_A|\log\Delta_{AB}|U\Psi_A\rangle \\ &= -\langle\Psi_{AB}|\log\Delta_{AB}|\Psi_{AB}\rangle = S(\rho_{AB}||\sigma_{AB}). \end{aligned}$$

This completes the proof.

Basically, this proof is the same as in the quantum field theory case except that we have to check a couple of details that are obvious in the quantum field theory case. Does this proof depend on tricky details, or is it obvious, given what we found in quantum field theory, that it would have to work? Opinions could differ on this, but philosophically, one might believe that quantum field theory isn't simpler than quantum mechanics and that what worked in quantum field theory should have an analog for a general quantum system.

Now we come to part 4 of the lecture. Going back to quantum field theory, in general for a state Ψ and a region $\mathcal{U}$, it is very hard to identify concretely the corresponding operator $S_{\Psi;\mathcal{U}}$. There is, however, one case in which this can be done and this example is very important for applications. This is the case that $\mathcal{U}$ is a "Rindler space" or wedge in Minkowski spacetime and Ψ is the vacuum state Ω. The Rindler wedge $\mathcal{U}$ is defined by the condition $|x| > t$ in the xt plane

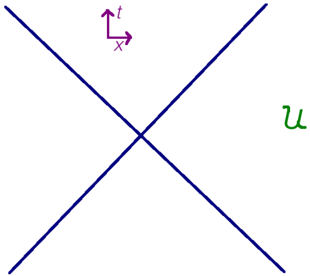


Transverse coordinates $\vec{y}$ will not play an important role.

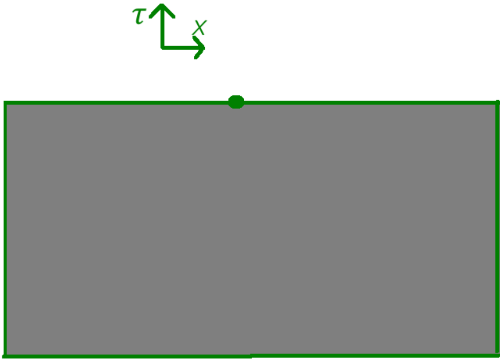
There is a rigorous approach due to Bisognano and Wichman (1971) which is based on holomorphy of correlation functions (for an introduction, see my notes, section 5.3). Instead today I will explain a very well-known path integral approach based on a presumed factorization of the Hilbert space

$$\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_r$$

where $\mathcal{H}_\ell$ and $\mathcal{H}_r$ are the degrees of freedom visible in the right and left Rindler wedges.



We continue, first of all to Euclidean time τ. The quantum vacuum state on an initial value surface $\tau = 0$ can be computed by a path integral on the lower half-space $\tau < 0$:



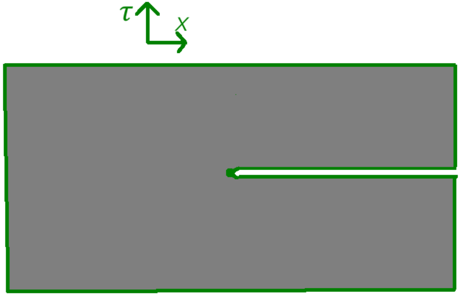
The green dot is supposed to be at $x = 0$ (any $\vec{y}$). It divides the initial value surface into left and right halves and we are going to assume a corresponding factorization of the Hilbert space

$$\mathcal{H} = \mathcal{H}_\ell \otimes \mathcal{H}_r$$

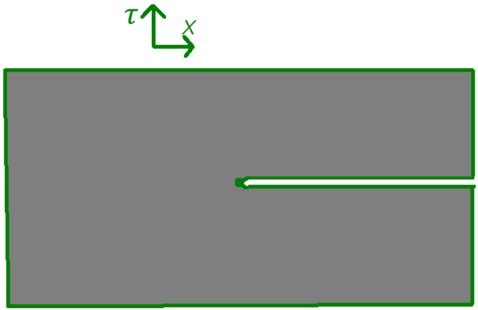
We are going to find a density matrix for the right half-space. For this, we think of the vacuum wavefunction as a functional $\Omega(\phi_\ell, \phi_r)$ that depends on field variables on the left and right half spaces. A density matrix $|\Omega\rangle\langle\Omega|$ for the pure state Ω would then be a function

$$|\Omega(\phi'_\ell, \phi'_r)\rangle\langle\Omega(\phi_\ell, \phi_r)|$$

of pairs of variables. A partial trace over $\mathcal{H}_\ell$ to get the density matrix ρ_r for the right half space is obtained, as usual, by setting $\phi'_\ell = \phi_\ell$ and integrating over ϕ_ℓ. This corresponds to a simple path integral procedure:



To explain this picture in more detail

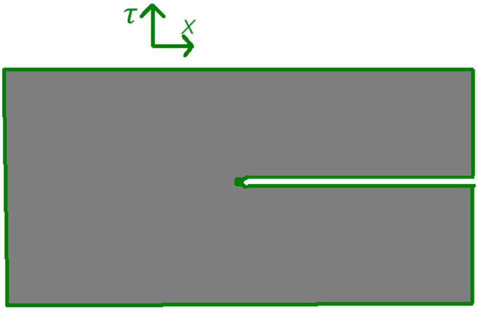


a path integral on the lower half plane has created the ket $|\Omega\rangle$ and a path integral on the upper half plane has created the bra $\langle\Omega|$. Then an integral over the field variables on the left half of the initial value surface has set $\phi'_\ell = \phi_\ell$. All this combines to a path integral on a Euclidean space with a cut on the right half of the initial value surface, as shown.

The result is a density matrix

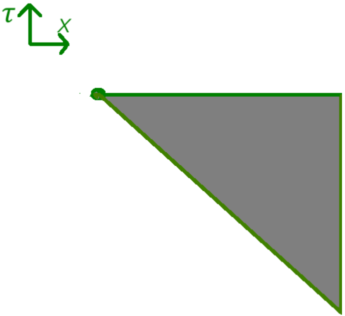
$$\rho(\phi'_r, \phi_r)$$

that depends on two sets of “right” variables, living just above and just below the cut.



We call the cut spacetime $W_{2\pi}$.

More generally we can consider a wedge of any opening angle η:



The wedge is obtained by rotating a half-space through an angle η. The rotation matrix acts by

$$R_\eta \begin{pmatrix} \tau \\ x \end{pmatrix} = \begin{pmatrix} \cos\eta & \sin\eta \\ -\sin\eta & \cos\eta \end{pmatrix} \begin{pmatrix} \tau \\ x \end{pmatrix}$$

In terms of real time $t = -\mathrm{i}\tau$, this formula reads

$$R_\eta \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \cosh(\mathrm{i}\eta) & -\sinh(\mathrm{i}\eta) \\ -\sinh(\mathrm{i}\eta) & \cosh(\mathrm{i}\eta) \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

The generator of a Lorentz boost is

$$K = \int_{t=0} \mathrm{d}x \mathrm{d}\vec{y}\, x T_{00}.$$

Formally we can write

$$K = K_r - K_\ell \qquad (*)$$

where K_r and K_ℓ are partial boost generators

$$K_r = \int_{t=0, x\geq 0} \mathrm{d}x \mathrm{d}\vec{y}\, x T_{00}, \qquad K_\ell = -\int_{t=0,\, x<0} \mathrm{d}x \mathrm{d}\vec{y}\, x T_{00}.$$

The purpose of the minus sign in $(*)$ is to ensure that both K_ℓ and K_r boost their respective wedges forwards in time.

The operator that implements a Lorentz boost by a real boost parameter θ is $\exp(-\mathrm{i}\theta K)$. Setting $\theta = -\mathrm{i}\eta$, we learn that, in real time language, the path integral on the wedge constructs the operator $\exp(-\eta K_r)$. (The path integral on the wedge propagates the degrees of freedom on the right half-space only, so here we use K_r, not K.) To get the density matrix of the right half-space, we set $\eta = 2\pi$ so

$$\rho_r = \exp(-2\pi K_r).$$

Likewise the density matrix of the left half-space is

$$\rho_\ell = \exp(-2\pi K_\ell).$$

We've learned that, when the Hilbert space factorizes $\mathcal{H} = \mathcal{H}_r \otimes \mathcal{H}_\ell$, the modular operator is $\Delta = \rho_r \otimes \rho_\ell^{-1}$. In this case that gives

$$\Delta_\Omega = \exp(-2\pi K_r)\exp(2\pi K_\ell) = \exp(-2\pi K).$$

Note that this only involves the well-defined operator K.

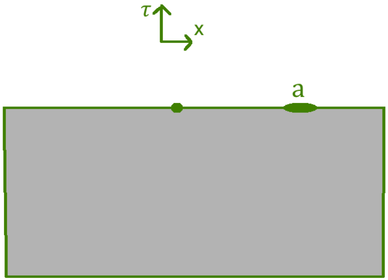
Now let us try to find the Tomita operator S_Ω. Its relation with Δ_Ω is $\Delta_\Omega = S_\Omega^\dagger S_\Omega$. Equivalently

$$S_\Omega = J_\Omega \Delta_\Omega^{1/2},$$

where J_Ω is antiunitary. To find J_Ω and S_Ω, we have to first understand $\Delta_\Omega^{1/2}$. We start by looking at a state

$$\mathrm{a}|\Omega\rangle$$

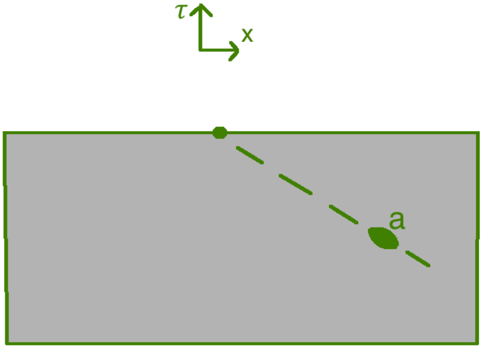
where a is any operator inserted on the right half of the initial value surface. Here is a path integral interpretation



Now we try to understand

$$\Delta_{\Omega}^{\alpha}\mathsf{a}|\Omega\rangle = \exp(-2\pi\alpha K_r + 2\pi\alpha K_\ell)\mathsf{a}|\Omega\rangle.$$

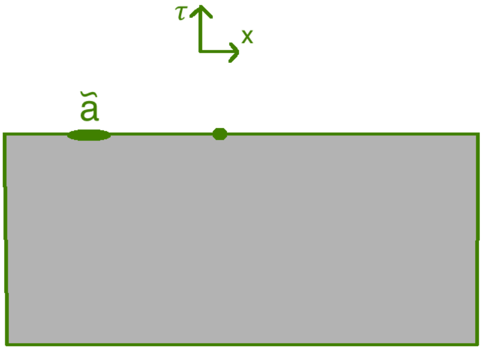
Here $\exp(-2\pi\alpha K_r)$ adds a wedge of opening angle $2\pi\alpha$ to the right of the picture and $\exp(2\pi K_\ell)$ removes a wedge of the same opening angle from the right of the picture. If we rotate the picture so that the boundary is still horizontal, it looks like this:



If we set $\alpha = 1/2$ to study

$$\Delta_\Omega^{1/2} \mathsf{a}|\Omega\rangle$$

we get this picture:

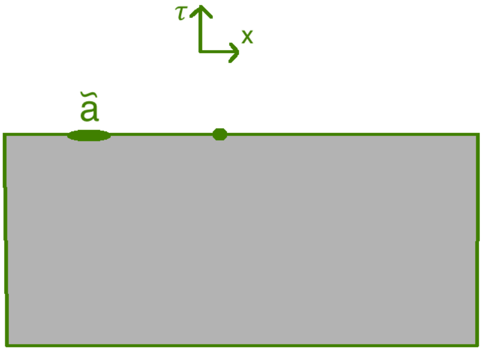


We cannot go any farther because there is no more wedge to remove on the left. So we cannot define

$$\Delta_\Omega^{\alpha} \mathsf{a}|\Omega\rangle$$

for $\alpha > 1/2$.

The picture



shows that

$$\Delta_\Omega^{1/2} a|\Omega\rangle = \widetilde{a}|\Omega\rangle$$

where $\widetilde{a}$ is a certain operator inserted on the left half space. We will discuss the relation between a and $\widetilde{a}$ in a moment.

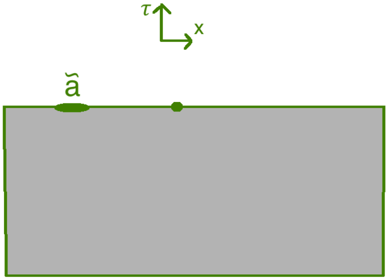
Now we can find the operator $S_\Omega = J_\Omega \Delta_\Omega^{1/2}$, which is characterized by

$$S_\Omega \mathsf{a}|\Omega\rangle = \mathsf{a}^\dagger|\Omega\rangle.$$

Suppose for simplicity that the operator algebra is generated by a hermitian scalar field ϕ. Then it is enough to consider the cases that a is $\phi(0, x, \vec{y})$ or $\dot{\phi}(0, \vec{x}, y) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(t, x, \vec{y})\Big|_{t=0}$. These operators are both hermitian so we want

$$S_\Omega \phi(0, x, \vec{y})|\Omega\rangle = \phi(0, x, \vec{y})|\Omega\rangle, \quad S_\Omega \dot{\phi}(0, x, \vec{y})|\Omega\rangle = \dot{\phi}(0, x, \vec{y})|\Omega\rangle.$$

From the picture



we have

$$\Delta_\Omega^{1/2}\phi(0,x,\vec{y})|\Omega\rangle = \phi(0,-x,\vec{y})|\Omega\rangle,\ \ \Delta_\Omega^{1/2}\dot{\phi}(0,x,\vec{y})|\Omega\rangle = -\dot{\phi}(0,x,\vec{y})|\Omega\rangle$$

where the minus sign in the second formula is there because $\Delta_\Omega^{1/2}$ rotated the picture through an angle π and reversed the sign of $\mathrm{d}/\mathrm{d}\tau$. So we need

$$J_\Omega\phi(0,x,\vec{y})J_\Omega^{-1} = \phi(0,-x,\vec{y}), \quad J_\Omega\dot{\phi}(0,x,\vec{y})J_\Omega^{-1} = -\dot{\phi}(0,x,\vec{y}).$$

In other words, J_Ω is an antiunitary operator that acts by $t,x,\vec{y} \to -t,-x,\vec{y}$.

The antiunitary operator that acts by $t, x, \vec{y} \to -t, -x, \vec{y}$ is what we might call CRT, a combination of charge conjugation C, a reflection R of one coordinate, and time-reversal T. So for the Rindler wedge

$$J_\Omega = \mathrm{CRT}.$$

The reason for the C is that J_Ω reverses the signs of conserved charges. To see this, consider a theory of two real scalar fields ϕ_1, ϕ_2 with conserved charge $Q = \int_{t=0} \mathrm{d}x\mathrm{d}\vec{y}(\phi_1\dot{\phi}_2 - \dot{\phi}_1\phi_2)$; this is clearly odd under J_Ω. Traditionally R and T are defined to commute with conserved charges so in traditional terminology $J_\Omega = \mathrm{CRT}$.

In even dimensions, CRT can be combined with π rotations of pairs of transverse coordinates $\vec{y}$ to get what is usually called CPT. In odd dimensions, there is no universal CPT symmetry; the universal discrete symmetry of Lorentz-invariant quantum field theory in any dimension is CRT.