Lecture 2

(Lecture 1 was blackboard).

a-maximization and RG flows

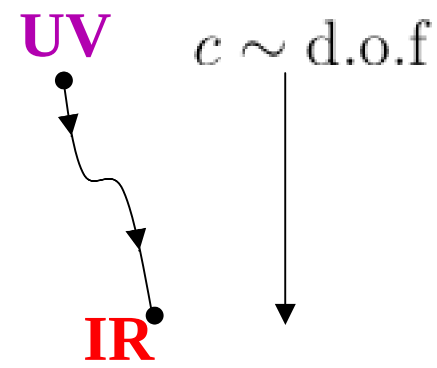
or

Towards a c-theorem in Four Dimensions

Based on works with Brian Wecht,

and, to appear, with Barnes, Wecht, and Wright.

RG Flows and the c-theorem



• Quantity, c, counts # massless d.o.f. of CFT.
Endpoints of all RG flows should satisfy:

$$c_{UV} > c_{IR}$$

• Stronger: can extend to monotonically decreasing c-function,
$\dot{c}(g(t)) < 0$ along entire RG flow to IR. $(t = -\log \mu)$

• Strongest (Recall 2d, Zamolodchikov): RG flows are **gradients**:

$$\dot{g}^I(t) \equiv -\beta^I(g) = -G^{IJ}\frac{\partial c(g)}{\partial g^J} \Longrightarrow \dot{c}(g) = -G^{IJ}\frac{\partial c}{\partial g^I}\frac{\partial c}{\partial g^J} < 0$$

positive definite!

What about in four dimensions?

Cardy conjecture for quantity that counts # massless d.o.f. of 4d **CFT**s: coefficient ***a:*** $\langle T^\mu_\mu \rangle = c(\text{Weyl})^2 + a(\text{Euler}).$

plausible count of massless d.o.f.!

no!

yes!?

$$(\text{Weyl})^2 = -\frac{1}{16\pi^2}\left(R^2_{\mu\nu\rho\sigma} - 2R^2_{\mu\nu} + \frac{1}{3}R^3\right)$$

Vanishes if conformally flat

$$(\text{Euler}) = \frac{1}{16\pi^2}\left(R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2\right)$$

Topological

Same coeff. c

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = c\Pi_{\mu\nu\rho\sigma}\frac{1}{x^4} + c'\Pi_{\mu\nu}\Pi_{\rho\sigma}\frac{1}{x^4}$$

c not monotonic. Seen in perturbation thy and via susy exact results.

Free fields:

$$a \sim n_s + \frac{11}{2} n_f + 62 n_V$$

always positive here

$$c \sim n_s + 6 n_f + 12 n_V$$

Perturbative corrections computed in various theories e.g. Banks Zaks. c sometimes increases, sometimes decreases. a always decreases, in every example.

Application: QCD

$$a_{UV} = 62(N_c^2 - 1) + 11 N_c N_f$$

pions

$$a_{IR} = N_f^2 - 1$$

Asymptotic freedom suffices to ensure $a_{UV} > a_{IR}$

4d "a-theorem" **conjecture:** **Endpoints** of all 4d RG flows satisfy $a_{UV} > a_{IR}$ and also $a_{CFT} > 0$ **True in every known example!!!** Powerful result **if** really generally true. Proof?

Cardy's heuristic proof:

$$a \sim \int_{S^4} \langle T^\mu_\mu \rangle \qquad \dot{a} \sim -\int_{S^4} \langle T^\mu_\mu T^\nu_\nu \rangle < 0.$$

He pointed out it doesn't really work. Subtractions, etc...
Forte and Lattore suggest it works better for Euclidean AdS...

Some promising attempts but, unlike 2d, NO general and generally accepted, complete proof as of yet.

Given **striking** successes of $a_{UV} > a_{IR}$ and $a_{CFT} > 0$

"weak version" perhaps stronger claims true?

Medium?: Perhaps can define a monotonically decreasing "a-function" along entire RG flow to IR, critical at precisely CFTs, where it coincides with a?

Strongest?: RG flow as gradient flow of a-function, with postive definite metric on space of coupling constants?

Investigated by Osborn: $\widetilde{a}(g) = a_{conf}(g) + W_I \beta^I$ ← **coincides with a at endpoints.**

$$\frac{\partial \widetilde{a}}{\partial g^I} = (G_{IJ} + \partial_{[I} W_{J]})\beta^J$$

Shown in perturbation theory. Medium claim if G positive definite. Strongest if W exact. Neither manifestly true. Both found true in all examples.

$$\widetilde{a}(g) = a_{conf}(g) + W_I \beta^I \qquad \frac{\partial \widetilde{a}}{\partial g^I} = (G_{IJ} + \partial_{[I} W_{J]})\beta^J$$

Perturbative metric and 1-form W on coupling constant space:

$$ds^2 = 4n_V \frac{dg^2}{g^2} + \frac{dY^2}{16\pi^2} + \frac{d\lambda^2}{16\pi^2}$$

$$W = 2n_V \frac{dg}{g} + \frac{Y dY}{16\pi^2} + \frac{\lambda d\lambda}{16\pi^2}$$

Obtained by Osborn and collaborators (to higher orders too) via making coupling constants spatially dependent and re-doing renormalization analysis. Get additional beta fns, e.g. $\beta_{\mu\nu} = G_{IJ}(g)\partial_\mu g^I \partial_\nu g^J$ **how metric above is defined. Find it's positive definite, good. Explicit checks, but no proof.**

Aside on AdS/CFT:

$$AdS_5 \times H_5 : \qquad a = c \sim \frac{N^2}{vol(H_5)}$$

Henningson, Skenderis
Gubser

very restricted subset of N=1 SCFTs!

Holographic RG flows: $$ds^2 = e^{2A(r)} dx_\mu dx^\mu - dr^2$$

$$a = c = \frac{universal}{A'(r)^3}$$

Prove monotonic iff weak-energy condition satisfied in bulk. Freedman, Gubser, Pilch, Warner.

$$\dot{c} = -G_{IJ}\beta^I \beta^J \qquad G_{IJ} = cG_{IJ}^{bulk}$$

$$-\frac{dc}{dA} \qquad \frac{d\phi^I}{dA}$$

Anselmi, Giradello, Porrati, Zaffaroni.

Suggest strongest claim, at least in this restricted context.

Exact results, using supersymmetry.

First consider RG fixed points, = SCFTs. $U(1)_R \subset SU(2,2|1)$

This conserved superconformal R-symmetry is very useful!

$$\Delta = \frac{3}{2}R \longleftarrow$$

exact dimension of all chiral primary fields.
(inequality for non-chiral primary)

$$a = 3TrR^3 - TrR, \quad \text{and} \quad c = 3TrR^3 - \frac{5}{3}TrR.$$

Anselmi, Erlich, Freedman, Grisaru, Johansen

Can exactly compute central charges in terms of 't Hooft anomalies! Readily computable! Use power of 't Hooft anomaly matching!

How these results are obtained: "multiplet of anomalies"

$$\overline{D}^{\dot\beta} T_{\alpha\dot\beta} = D_\alpha L_T \longrightarrow T^\mu_\mu - i\frac{3}{2}\partial_\mu R^\mu = L_T|_{\theta^2} \qquad (*)$$

$$L_T = \frac{1}{4}\beta(g^{-2})W^2_{gauge} + \tau_{IJ}W^I_{flav}W^J_{flav} + \frac{c}{24\pi^2}\mathcal{W}^2 - \frac{a}{24\pi^2}\Xi$$

(coeffs of current 2-pt funs.)

$$\Xi \equiv \mathcal{W}^2 + (\overline{D}^2 + R)(G_{\alpha\dot\alpha}G^{\alpha\dot\alpha} + 2\overline{R}R)$$

topological, chirally projected super Euler/Pontrjagin density.

$$\mathcal{W}^2|_{\theta^2} = Weyl^2 + F_R^2 + i(F_R\widetilde{F}_R + R\widetilde{R})$$

$$\Xi|_{\theta^2} = Euler + i(F_R\widetilde{F}_R + R\widetilde{R})$$

ignoring coefficients here.

Im.(*) $\longrightarrow$ RRR and R 't Hooft anomalies as sums of a and c, with fixed coeffs!

Get coeffs either via above, or just use known results for free V and Q to fix coeffs.

$$a = \alpha TrR^3 + \beta TrR, \quad \text{and} \quad c = \gamma TrR^3 + \delta TrR.$$

Free N=1 vector superfield: $R(\lambda_\alpha) = 1$

$$a_V = 2 = \alpha + \beta, \quad c_V = \frac{4}{3} = \gamma + \delta.$$

(we rescale a and c by 32/3 compared to others, to conserve toner ink)

Free N=1 chiral superfield: $R(\psi_Q) = \frac{2}{3} - 1 = -\frac{1}{3}$

$$a_Q = \frac{2}{9} = -\frac{\alpha}{27} - \frac{\beta}{3}, \qquad c_Q = \frac{4}{9} = -\frac{\gamma}{27} - \frac{\delta}{3}$$

Fixes universal coefficients, once and for all, for all theories:

$$a = 3TrR^3 - TrR, \quad \text{and} \quad c = 3TrR^3 - \frac{5}{3}TrR.$$

Also exact results along entire RG flow to IR:

$$R(Q_i) = \frac{2}{3}\Delta(Q_i) = \frac{2}{3} + \frac{1}{3}\gamma_i(g(t))$$

running R charges

Exact beta functions proportional to R-charge violation:

$$\beta_{NSVZ}(g^{-2}) \sim T_2(G) + \sum_i T_2(r_i)(R_i - 1)$$

$U(1)_R$ anomaly

$$\beta(h) = \frac{3}{2}h(R(W) - 2)$$

h=superpotential coupling, beta fn related to R-charge violation of W.

RG flow to SCFT fixed point: $U(1)_R \rightarrow U(1)_{R_*}$

conserved in IR SCFT

Knowing $U(1)_{R_*}$ exactly determines $\Delta_*(Q_i)|_{SCFT}$!

SQCD example:

UV

$$R^\mu = R^\mu_* + X^\mu$$

Not conserved. Irrelevant in IR.

Conserved, doesn't flow.

IR SCFT

R flows to conserved R-symmetry in IR: $R^\mu \to R^\mu_*$

Unique anomaly free R-symmetry in IR: $R(Q) = R(\widetilde{Q}) = 1 - \dfrac{N_c}{N_f}$

$$a_{UV} = 2(N_c^2 - 1) + 2N_c N_f \left(\frac{2}{9}\right)$$

Gives exact anomalous dims and central charges for IR SCFT! (Will soon discuss away from endpoints.)

$$a_{IR} = 2(N_c^2 - 1) + 2N_c N_f \left(-\frac{3N_c^3}{N_f^3} + \frac{N_c}{N_f}\right)$$

$$a_{UV} > a_{IR}$$

For a single chiral superfield of R-charge = R:

$$a(R) = 3(R-1)^3 - (R-1) \qquad c(R) = 3(R-1)^3 - \frac{5}{3}(R-1)$$

Plotting, it's clear why a=good and c=bad, e.g. in SQCD example. Free field value, R=2/3, is local maximum of a. Gauge interactions lower R. Reducing flavors extends flow of R to smaller values, so a is reduced further, good for a-theorem. What about a>0? OK too!

Unitary bound: all gauge invariant spinless operators must have $\Delta \geq 1$ saturates bound iff it's a free field (decoupled operator).

For gauge invt. chiral primaries, unitary bound is thus: $R \geq \frac{2}{3}$

SQCD: $R(M) = 2R(Q) = 2\left(1 - \frac{N_c}{N_f}\right)$

Hits unitarity bound for $N_f \leq \frac{3}{2}N_c$

Hitting unitarity bound signals accidental symmetry.
Here get entire Seiberg dual, free magnetic.

Lot's of other SCFTs. "Generic" for enough matter (to avoid W dyn) but still asymp. free. Big Landscape of SCFTs, and RG flows.

Mention a couple of other SCFT examples:

SQCD with fundamentals + adjoint X, no W.

SQCD with some of the flavors coupled to singlets: $W = SQ'\widetilde{Q}'$

Etc.....

Examples are practically endless! Can explore these and RG flows among them, using susy exact results,

if we know the superconformal R-symmetry ....

Finding the superconformal $U(1)_{R_*}$ of general SCFTs:

Problem: R can mix with flavor symmetries, $R = R_0 + \sum_I s_I F_I$

which is the superconformal one $U(1)_{R_*} \subset SU(2,2|1)$?

Solution (KI, B. Wecht '03) **: to uniquely determine** $U(1)_{R_*}$

locally maximize: $a_{trial}(R) = 3\mathrm{Tr}\ \mathrm{U}(1)_{\mathrm{R}}^3 - \mathrm{Tr}\ \mathrm{U}(1)_{\mathrm{R}}$

over all conserved, possible R-symmetries.

't Hooft anomalies,
exactly computable!

(We proved this, using susy + CFT unitarity)

Value of $a_{trial}(R)$ at local maximum is the conformal anomaly

a_{SCFT} appearing in Cardy's conjecture!

"a-maximization"

Why a-maximization: the $U(1)_R \subset SU(2,2|1)$ is the unique

$R = R_0 + \sum_I s_I F_I$ that maximizes $a = 3TrR^3 - TrR$ Equiv to:

(*) $9TrR_*^2 F_I = TrR_* F_I$ ← **extremal condition**

(**) $TrR_* F_L F_J < 0$ ← **local maximum**

Apply for all flavor currents. Yield a unique solution for R.

(*) comes from anomaly of flavor super-current in superbkgd curvature + field strength coupled to R-current. Argue

$$\overline{D}^2 J_I = \frac{k_I}{384\pi^2} \mathcal{W}^2$$

(*) hinges on having single term, no Ξ

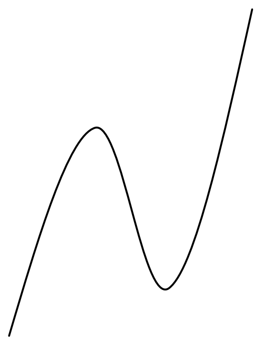
(**) from trace anomaly with fields coupled to flavor currents:

$T^\mu_\mu - i\frac{3}{2}\partial_\mu R^\mu = \tau_{IJ} W_I W_J|_{\theta^2}$ yields $TrR_* F_I F_J = -\frac{1}{3}\tau_{IJ} < 0$

Quick example: Consider a free chiral superfield Φ

$$a = 3(r-1)^3 - (r-1)$$

This has extrema at $r=2/3$ and $r=4/3$.



max! **min!**

correct value for free field.

And it's basically just as easy for interacting theories!

General observation: Since we're maximizing a cubic function, R-charges, chiral primary operator dimensions, and central charges must **always** be quadratic irrationals:

$$\frac{n+\sqrt{m}}{p}$$

Quantized, so cannot depend on any continuous moduli!

Let's write our a-extremum condition out more explictly, for a general susy gauge theory with gauge group G, matter in reps r_i and W=0.

(*) $$0 = 9TrR^2F_I - TrF_I = \sum_i |r_i|(F_I)_i \left(9(R_{Q_i}-1)^2-1\right)$$

Must hold for any anomaly free flavor charges $\sum_i T_2(r_i)(F_I)_i = 0$

hence $$R_{Q_i} = 1 \pm \frac{1}{3}\sqrt{1+\lambda\frac{T_2(r_i)}{|r_i|}}$$

parameter fixed by cond. that R-symm be anomaly free.

Check for Banks Zaks type perturbatively accessible RG fixed points

$$\gamma_{Q_i} = 1 - \sqrt{1+\lambda\frac{T_2(r_i)}{|r_i|}} = -\lambda\frac{T_2(r_i)}{|r_i|} + \dots$$

Precisely reproduces explicitly computed anomalous dimensions!

a-maximization almost proves the *a*-theorem!

Since relevant deformations generally break the flavor symmetries,

$$\mathcal{F}_{IR} \subset \mathcal{F}_{UV}$$

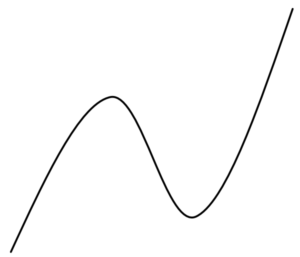
Maximizing over a subset then implies that $a_{IR} < a_{UV}$

ALMOST……NOT QUITE SO FAST!

Loopholes:

1) Accidental symmetries.

2) Only a local max.



Trying to close these loopholes. a-thm checks also in examples where accidental symmetries are crucial (Kutasov, Parnachev, Sahakyan).

A big zoo of examples: SCFTs obtainable from SQCD with fundamentals + adjoints

(KI, B. Wecht)

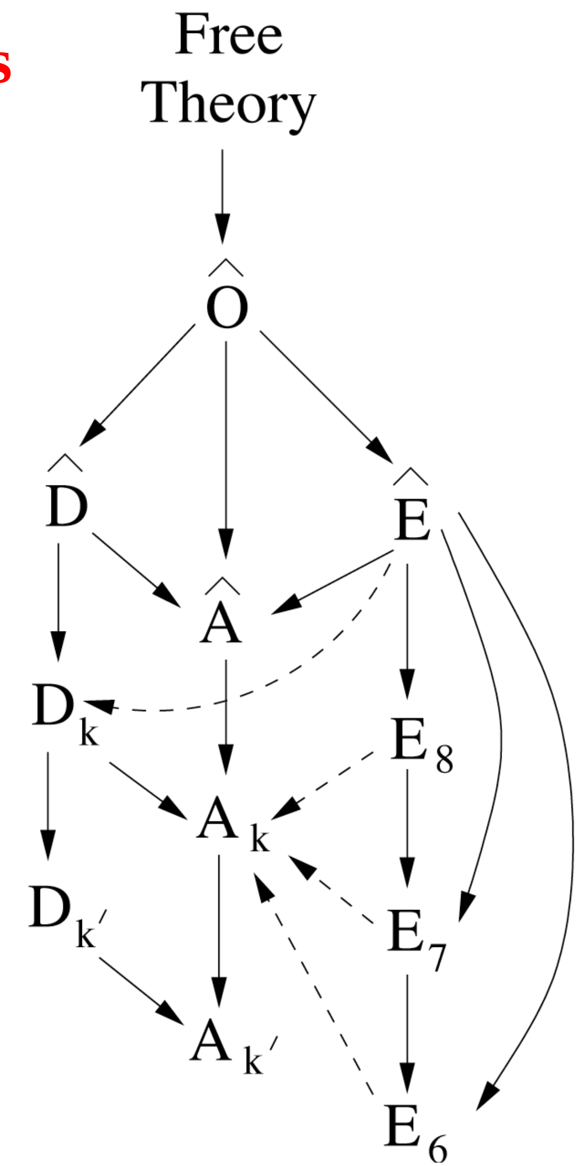
Determine operator dimensions, and classify which superpotentials are relevant, and when. Find it coincides with **Arnold's ADE** classification!

$$W_{\widehat{O}} = 0 \qquad W_{\widehat{A}} = TrY^2$$

$$W_{\widehat{D}} = TrXY^2 \qquad W_{\widehat{E}} = TrY^3$$

$$W_{E_8} = Tr(X^3 + Y^5)$$ etc.

Lots of new SCFTs!



All flows indeed compatible with the *a*-conjecture, $\boxed{a_{UV} > a_{IR}}$

Extend $a_{UV} > a_{IR}$ to a monotonically decreasing a-function along entire RG flow?

Kutasov: Recall our argument that $a_{IR} < a_{UV}$ because $\mathcal{F}_{IR} \subset \mathcal{F}_{UV}$.

Implement $\mathcal{F}_{IR} \subset \mathcal{F}_{UV}$ IR constraint on R-charges via **Lagrange multipliers**, interpreted as **running couplings** along RG flow to IR.

a-maximization, holding Lagrange multipliers fixed gives candidate

a-function along RG flow: $a(\lambda) = a(R, \lambda)|_{R(\lambda)}$.

extremizing solution.

E.g.

$$a = 2|G| + \sum_i |r_i|[3(R_i - 1)^3 - (R_i - 1)] - \lambda(T(G) + \sum_i T(r_i)(R_i - 1))$$

This Lagrange multiplier enforces anomaly freedom in the IR.

Extremizing w.r.t. the R-charges, yields $R_i(\lambda) = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda T(r_i)}{|r_i|}}$

Interpret as R-charges along flow from UV to IR: $\lambda = 0 \rightarrow \lambda_*$

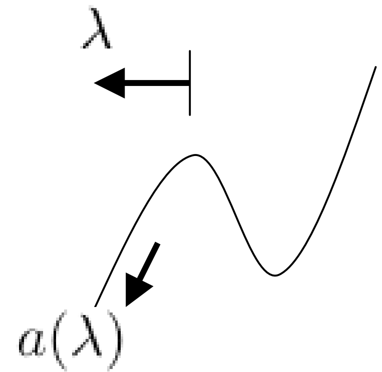
Since $R_i = \frac{2}{3} + \frac{1}{3}\gamma_i$ this gives $\gamma_i(\lambda) = 1 - \sqrt{1 + \frac{\lambda T(r_i)}{|r_i|}}$

Expand: $\gamma_i(\lambda) = \sum_{p=1}^{\infty} \frac{(2p-3)!!}{p!} \lambda^p \frac{T(r_i)^p}{|r_i|^p}$ Taking $\lambda = \frac{g^2|G|}{2\pi^2}$

again, this **precisely** reproduces the 1-loop anomalous dimensions. Also the scheme indep. parts of higher loop ones up to three loops!!! (computed, by Jack, Jones, North). Believe above are **exact** anomalous dimension, in some scheme.

Computing $a(\lambda) = a(R, \lambda)|_{R(\lambda)}$:

$$a(\lambda) = 2|G| - \lambda T(G) + \frac{2}{9}\sum_i |r_i| \left(1 + \frac{\lambda T(r_i)}{|r_i|}\right)^{3/2}.$$



Interpolating a-function. **Monotonically decreasing** along RG flow:

$$\frac{da}{d\lambda} = -[T(G) + \sum_i T(r_i)(R_i - 1)] \sim \beta_{NSVZ}(g) < 0.$$

Example: SQCD with fundamentals + an adjoint X:

$$R_Q(\lambda) = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda}{2N_c}} \qquad R_X(\lambda) = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda N_c}{N_c^2 - 1}}$$

$$a(\lambda) = 2(N_c^2 - 1) - \lambda N_c + \frac{2}{9}(2N_c N_f)\left(1 + \frac{\lambda}{2N_c}\right)^{3/2} + $$
$$+\frac{2}{9}(N_c^2 - 1)\left(1 + \frac{\lambda N_c}{N_c^2 - 1}\right)^{3/2}$$

Flow from UV to IR, $\lambda : 0 \longrightarrow \lambda_*$

Where R=anomaly free
i.e. NSVZ beta funct =0
i.e. where a' =0.

Must modify when any gauge invt. operators hit R=2/3 unitarity bound to include accidental symmetry effects. Can take this into account if we know how symmetry acts on operators, using 't Hooft matching. Minimal possibility: just operator hitting bound becomes free.

More generally, Lagrange-multipliers for every interaction:

$$a(R_i, \lambda_I) = 3Tr\ R^3 - Tr\ R + \sum_J \lambda_J \widehat{\beta}_J(R)$$

$$\widehat{\beta}_G(R) = -(T(G) + \sum_i T(r_I)(R_i - 1)) \sim \beta_{NSVZ}(g^{-2})$$

$$\widehat{\beta}_W = R(W) - 2 = \frac{2}{3h}\beta(h)$$

IR constraints on R-charges = proportional to exact beta functions.

Extremizing over the R-charges, holding Lagrange multipliers fixed, gives **running R-charges** $R_i(\lambda)$ and thus anomalous dimensions. (Always, non-trivially, recover the leading perturbative expressions.) Plugging $R_i(\lambda)$ into above expression gives **a-function** such that

$$\frac{\partial a(\lambda)}{\partial \lambda_I} = \widehat{\beta}_I$$

Suggests gradient flow!

Compute 4d analog of Zamolodchikov metric for our **a-function**:

$$\frac{\partial a}{\partial g} = G_{gg}\beta_{NSVZ}(g) \qquad \text{and} \qquad \frac{\partial a}{\partial h} = G_{hh}\beta_W(h)$$

(Work to leading order, so ignore cross terms.)

Use leading order dictionary between the λ_I and the couplings, foung by comparing $R_i(\lambda)$ with perturbative anomalous dims.

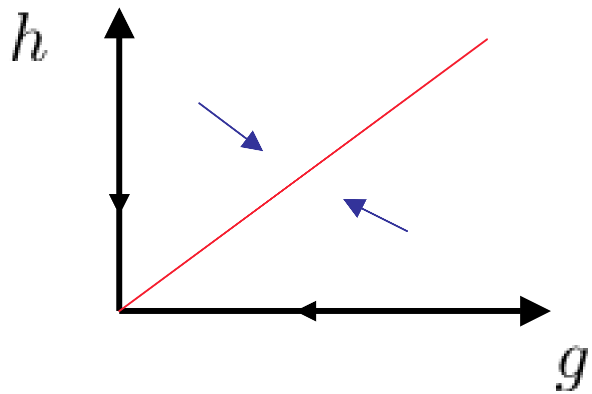
We find precise numerical agreement, with leading, perturbative metrics found by Freedman and Osborn!

(Computed from $\beta_{\mu\nu} = G_{IJ}\partial_\mu g^I \partial_\nu g^J$ making couplings spatially dependent.)

gauge $G_{gg} = \frac{4|G|}{g^2} + O(g^0), \qquad G_{hh} = \frac{1}{24\pi^2} + O(h^2)$ **Yukawa**

Looks promising for this a-function, and stronger a-theorem claim!

Quick example: N=1 with 3 adjoints and $W = h\mathrm{Tr}\left(\Phi_1[\Phi_2,\Phi_3]\right)$



N=4 fixed line, attractive in IR

$$a = 2|G| + 3|G|(3(R-1)^3 - (R-1)) - \lambda_G(T(G) + 3T(G)(R-1)) + \lambda_W(3R-2)$$

extremize

$$R = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda_G T(G)}{|G|} - \frac{\lambda_W}{|G|}}$$

Running R charges.

Flows attracted to N=4 fixed line: $\lambda_G T(G) = \lambda_W$

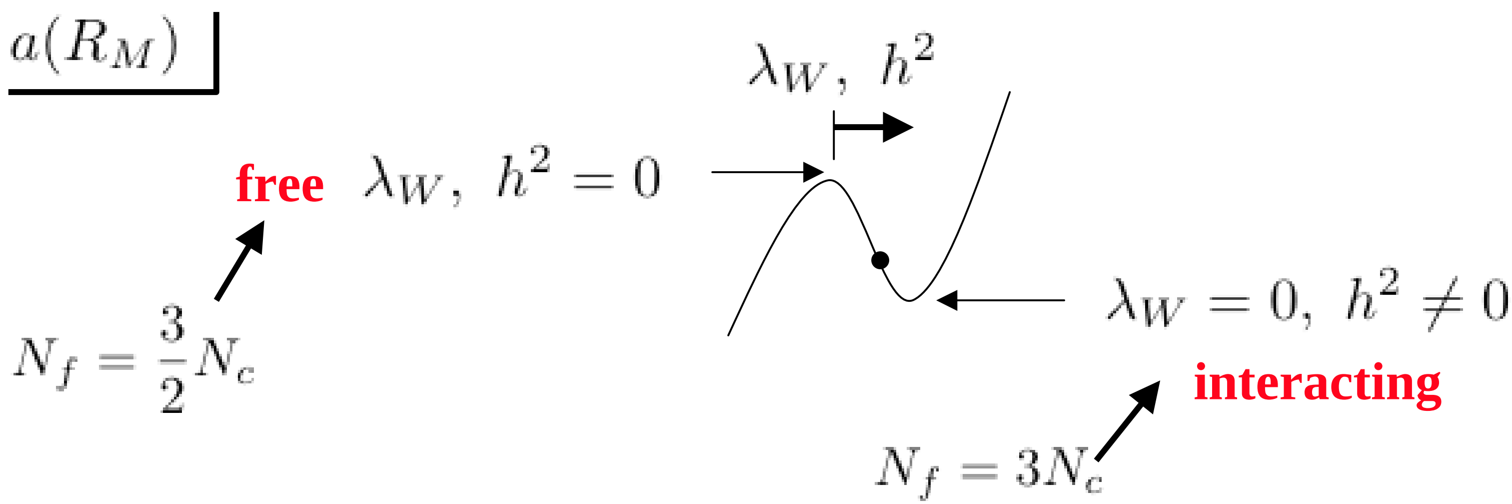
$a(\lambda_G,\lambda_W)$ **decreases along flows to fixed line. At weak coupling, can obtain**

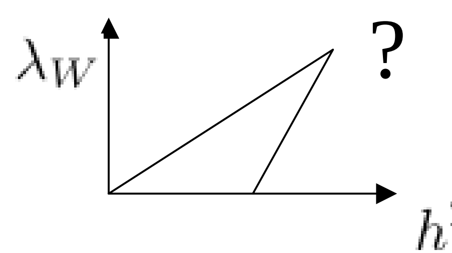
$\lambda_G(g,h)$ **and** $\lambda_W(g,h)$ **dictionary explicitly. Map flows beyond weak coup.?**

Would like to better understand $\lambda_G(g,h)$ and $\lambda_W(g,h)$ relations.

E.g. for magnetic dual of SQCD find $R_M = 1 \pm \frac{1}{3}\sqrt{1-\frac{\lambda_W}{N_f^2}}$

$\lambda_W = 0$ for both free theory, R=2/3, and interacting case R=4/3.





$$G_{hh} = \frac{d\lambda_W(h^2)}{d(h^2)}$$ ← positive definite?

(over simplifying a bit, G a matrix)

Conclude/Summary:

Use **a-maximization** to fix the superconformal R-symmetries.
Get new, exact results for previously mysterious SCFTs.

Find lots of new SCFTs and explore RG flows among them.

The a-theorem is almost proved for SCFTs. Loopholes closing.

Can obtain new, general results about 4d SCFTs, e.g. R and central charges always of general form: $\dfrac{n+\sqrt{m}}{p}$

Evidence for stronger 4d analogs of 2d c-theorem: monotonically decreasing a-function, and gradient RG flow. Interesting agreement with earlier perturbative computations of metric. But no proof yet that metric is always postive definite.