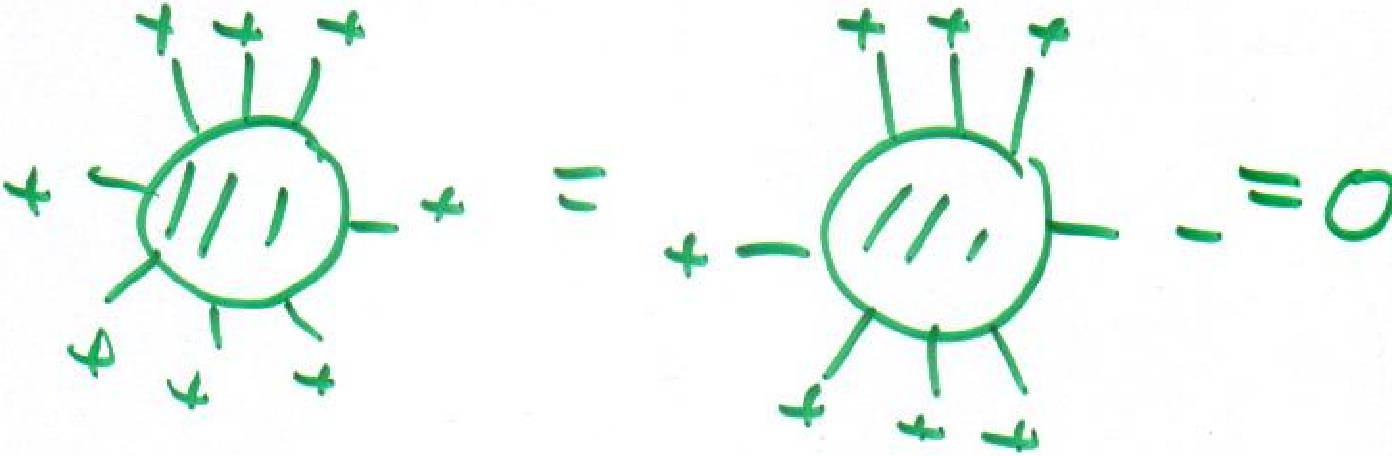
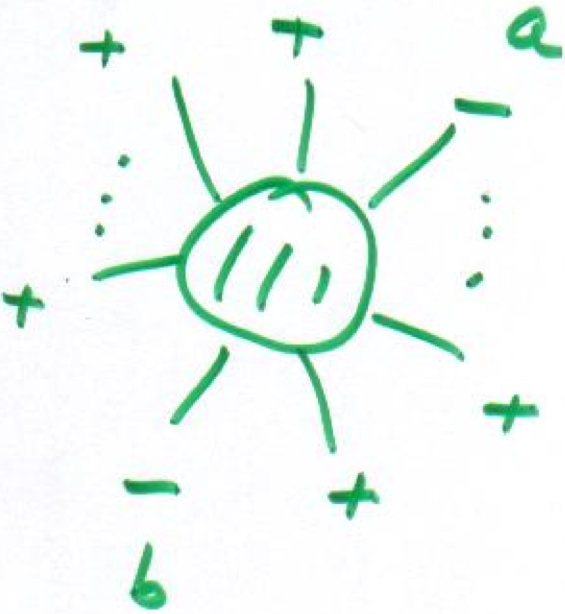
THE LAST THING I SAID LAST TIME WAS THAT IN YANG-MILLS THEORY AT TREE LEVEL



SO THE FIRST NONVANISHING AMPLITUDE IS THE MHV (MAXIMALLY HELICITY VIOLATING) AMPLITUDE



$$A = -i(2\pi)^4 \delta^4\left(\sum p_i\right) g^{n-2} \frac{\langle \lambda_a, \lambda_b \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_{i+1} \rangle}$$

WHERE

* HELICITIES CHECK
* A IS CALLED "HOLOMORPHIC"

COMPARE THIS WITH THE SAME AMPLITUDE WRITTEN IN THE STANDARD NOTATION

$$A(p_1, \epsilon_1; p_2, \epsilon_2; \ldots; p_n, \epsilon_n)$$

IT IS QUITE A MESS EVEN FOR $n=5$.

OUR GOAL, IN PART, IS TO ANSWER THE FOLLOWING QUESTIONS:

* WHY ARE MHV TREE AMPLITUDES SO SIMPLE? AND $\left.\begin{matrix}\text{ALL PLUS}\\ ++++-\end{matrix}\right\} = 0$

* WHAT GENERAL PROPERTY OF [redacted] YANG-MILLS TREE AMPLITUDES GENERALIZES THE SIMPLICITY OF MHV [redacted] AMPLITUDES?

* EXTEND TO LOOPS?

AS A PRELIMINARY, WE ASK THIS: CAN WE VERIFY CONFORMAL INVARIANCE OF MHV TREE AMPLITUDES?

HOW DOES CONFORMAL GROUP $SO(2,4)$ ACT? SOME GENERATORS ARE CLEAR

$$J_{ab} = \frac{1}{2}\left(\lambda_a \frac{\partial}{\partial \lambda^b} + (a \leftrightarrow b)\right)$$

$$\tilde{J}_{\dot{a}\dot{b}} = \frac{1}{2}\left(\tilde{\lambda}_{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{b}}} + (\dot{a} \leftrightarrow \dot{b})\right)$$

$$P_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

THEY ARE CLEARLY SYMMETRIES.

REMAINING GENERATORS ARE

$D =$ DILATATION

$K_{a\dot{a}} =$ SPECIAL CONFORMAL

HAVE

$$[D, P] = P, \qquad [D, K] = -K$$

SO K SCALES OPPOSITELY TO P.

AS $P_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$ WE GUESS

$$K_{a\dot{a}} = \frac{\partial}{\partial \lambda^a} \frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}}$$

$[P, K] = D + J + \tilde{J}$

AND THEN CLOSURE OF THE

ALGEBRA GIVES

$$D = \frac{1}{2}\left(\lambda^a \frac{\partial}{\partial \lambda^a} + \tilde{\lambda}^{\dot{a}} \frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}}\right) + 1$$

VERIFY THESE ANNIHILATE

$$A = \frac{-i g^{n-2} (2\pi)^4 \delta^4\left(\sum \lambda_i \tilde{\lambda}_i\right) \langle \lambda_a, \lambda_b \rangle^4}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_j \rangle}$$

FOR SIMPLICITY CONSIDER JUST D:

AS $\delta^4(\sum \lambda \tilde{\lambda})$ HAS $D=-4$,

$\langle \lambda_a, \lambda_b \rangle^4$ HAS $D=4$, THE

NUMERATOR IS INVARIANT.

AND

$$\frac{1}{\prod_{i=1}^{n} \langle \lambda_i, \lambda_j \rangle}$$

IS ANNIHILATED, FOR EACH i, BY

$$\frac{1}{2}\left(\lambda_i \cdot \frac{\partial}{\partial \lambda_i} + \tilde{\lambda}_i \cdot \frac{\partial}{\partial \tilde{\lambda}_i} \right) + 1.$$

SO WE HAVE ACHIEVED CONFORMAL INVARIANCE, BUT THE REPRESENTATION OF THE CONFORMAL GROUP IS EXOTIC!

$$J_{ab} = \frac{1}{2}\left(\lambda_a \frac{\partial}{\partial \lambda_b} + (a \leftrightarrow b)\right)$$ FIRST ORDER

$$P_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$ ZEROTH ORDER

$$K_{a\dot{a}} = \frac{\partial^2}{\partial \lambda^a \, \partial \tilde{\lambda}^{\dot{a}}}$$ SECOND ORDER

WE CAN PUT THE ACTION OF THE CONFORMAL GROUP IN A MORE STANDARD FORM IF WE JUST BORROW A TRICK FROM QUANTUM MECHANICS:

$$\tilde{\lambda}_{\dot{a}} \Leftrightarrow -\frac{\partial}{\partial \mu^{\dot{a}}}$$

$$\frac{\partial}{\partial \tilde{\lambda}^{\dot{a}}} \Leftrightarrow \mu_{\dot{a}}$$

SUDDENLY ALL GENERATORS BECOME FIRST ORDER:

$$J_{ab} = \frac{1}{2}\left(\lambda_a \frac{\partial}{\partial \lambda^b} + (ab)\right)$$

$$\tilde{J}_{\dot a \dot b} = \frac{1}{2}\left(\mu_{\dot a} \frac{\partial}{\partial \mu^{\dot b}} + (\dot a \dot b)\right)$$

$$D = \frac{1}{2}\left(\lambda^a \frac{\partial}{\partial \lambda^a} - \mu^{\dot a} \frac{\partial}{\partial \mu^{\dot a}}\right)$$

$$P_{a \dot a} = \lambda_a \frac{\partial}{\partial \mu^{\dot a}}$$

$$K_{a \dot a} = \mu_{\dot a} \frac{\partial}{\partial \lambda^a}$$

THIS REPRESENTATION IS EASY TO EXPLAIN. THE FOUR-DIMENSIONAL CONFORMAL GROUP IS $SO(4,2)$ WHICH IS THE SAME AS $SU(2,2)$ – EACH HAS 15 GENERATORS:

$$SO(4,2): \quad \frac{6 \cdot 5}{2} = 15$$

$$SU(2,2): \quad 4^2 - 1 = 15$$

SU(2,2) HAS AN OBVIOUS

4-DIM'L REPRESENTATION ACTING

ON

$$Z^I = (\lambda^a, \mu^{\dot{a}})$$

Z^I IS CALLED A TWISTOR AND

THE SPACE $\mathbb{C}^4$ OR $\mathbb{R}^4$ WITH

Z^I AS COORDINATES IS CALLED

TWISTOR SPACE.

THE ACTION OF $SU(2,2)$ ON THE Z^I IS GENERATED BY THE 15 TRACELESS MATRICES THAT CORRESPOND TO THE 15 FIRST ORDER OPERATORS J_{ab}, $\tilde{J}_{\dot{a}\dot{b}}$, D, $P_{a\dot{a}}$, $K_{a\dot{a}}$.

CONCRETELY, WE GO FROM A MOMENTUM SPACE SCATTERING AMPLITUDE TO A TWISTOR SPACE SCATTERING AMPLITUDE BY A FOURIER TRANSFORM:

$$\tilde{A}(\lambda,\mu) = \int \frac{d^2\tilde{\lambda}}{(2\pi)^2} \exp i[\tilde{\mu},\tilde{\lambda}] \, A(\lambda,\tilde{\lambda})$$

THE SAME FOURIER TRANSFORM TURNS A MOMENTUM SPACE WAVEFUNCTION $\psi(\lambda,\tilde{\lambda})$ TO A TWISTOR SPACE WAVEFUNCTION

$$\tilde{\psi}(\lambda,\mu) = \int \frac{d^2\tilde{\lambda}}{(2\pi)^2} \exp i[\tilde{\mu},\tilde{\lambda}] \, \psi(\lambda,\tilde{\lambda})$$

IF THE HELICITY IS h,

THEN

$$\left(\lambda \frac{\partial}{\partial \lambda} - \tilde{\lambda} \frac{\partial}{\partial \tilde{\lambda}}\right) \Psi = -2h\Psi,$$

AS WE SAW BEFORE,

SO

$$\left(\lambda \frac{\partial}{\partial \lambda} + \mu \frac{\partial}{\partial \mu}\right) \tilde{\Psi} = -(2+2h)\tilde{\Psi}$$

$$\begin{pmatrix} 1 & \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$$

$\Theta \lambda = \lambda$

$\Theta \mu = \mu$

$$\Theta = \lambda \frac{\partial}{\partial \lambda} + \mu \frac{\partial}{\partial \mu}$$

IF WE JUST LOOK AT THE OPERATOR ON THE LEFT HAND SIDE,

$$\lambda \frac{\partial}{\partial \lambda} + \mu \frac{\partial}{\partial \mu}$$

IT COINCIDES WITH

$$Z^I \frac{\partial}{\partial Z^I} \qquad Z^I = (\lambda, \mu)$$

AND GENERATES

$$Z^I \rightarrow t Z^I, \quad t \in \mathbb{C}^*$$

IF WE IDENTIFY

TWO SETS OF Z^I

THAT DIFFER BY

$Z^I \to t Z^I$, AND THROW

AWAY $\{Z^I = 0\}$, WE GET

THE "PROJECTIVE SPACE"

$\mathbb{CP}^3$ (OR $\mathbb{RP}^3$ IF THE

Z^I ARE REAL-VALUED)

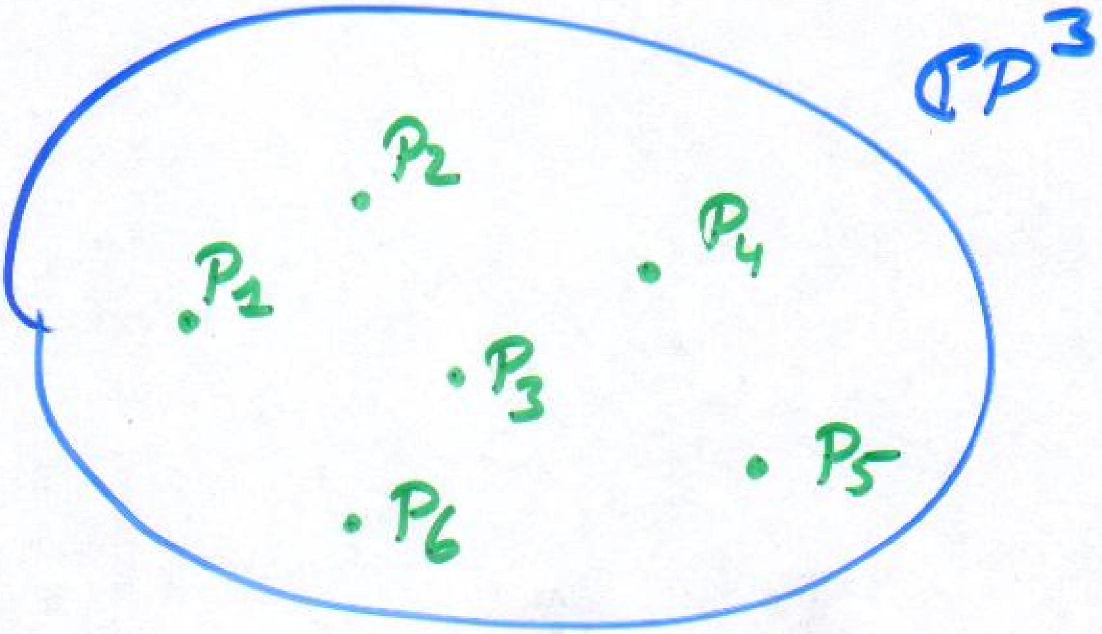
THE Z^I ARE CALLED

"HOMOGENEOUS COORDINATES"

$\mathbb{CP}^3$ (OR $\mathbb{RP}^3$) IS ALSO CALLED TWISTOR SPACE (OR PROJECTIVE TWISTOR SPACE IF WE WISH TO BE CAREFUL).

PHYSICALLY, THE Z^I AREN'T ALLOWED TO VANISH (IN FACT $Z = (\lambda, \mu)$ AND $\lambda \neq 0$) AND WAVEFUNCTIONS HAVE A KNOWN BEHAVIOR UNDER $Z^I \to t Z^I$ SO IT IS "CORRECT" TO USE PROJECTIVE TWISTOR SPACE.

SO IN AN n-GLUON YANG-MILLS SCATTERING PROCESS, THE EXTERNAL GLUONS ARE ASSOCIATED WITH POINTS IN PROJECTIVE TWISTOR SPACE:



IT TURNS OUT THAT THESE ARE VERY SPECIAL SETS OF POINTS

LET US SEE WHAT HAPPENS TO THE MHV TREE AMPLITUDES, WHICH WE RECALL

$$A(\lambda_i, \tilde{\lambda}_i) = (2\pi)^4 \delta^4(\sum \lambda_i \tilde{\lambda}_i) \, f(\lambda_i)$$

WHERE

$$f(\lambda_i) = -i g^{n-2} \frac{\langle \lambda_a, \lambda_b \rangle^4}{\prod_i \langle \lambda_i, \lambda_{i+1} \rangle}$$

THE ONLY IMPORTANT PROPERTY OF $f(\lambda_i)$ WILL BE ITS HOLOMORPHY

WE WRITE

$$(2\pi)^4 \delta^4\left(\sum \lambda_i^a \tilde{\lambda}_i^{\dot{a}}\right)$$

$$= \int d^4x^{a\dot{a}} \exp i\, x_{b\dot{b}} \sum_i \lambda_i^b \tilde{\lambda}_i^{\dot{b}}$$

SO THE AMPLITUDE IS

$$A(\lambda_i, \tilde{\lambda}_i) = \int d^4x^{a\dot{a}} \exp i\, x_{b\dot{b}} \sum_i \lambda_i^b \tilde{\lambda}_i^{\dot{b}} \; f(\lambda_i)$$

THE FOURIER TRANSFORM TO TWISTOR SPACE IS NOW VERY SIMPLE:

$$\tilde{A}(\lambda_i,\mu_i) = \prod_j \int \frac{d^2\tilde{\lambda}_j}{(2\pi)^2} \exp i\sum_j \mu_{j\dot{b}}\tilde{\lambda}_j^{\dot{b}}$$

$$\int d^4x^{a\dot{a}} \exp i x_{b\dot{b}} \sum_j \lambda_j^b \tilde{\lambda}_j^{\dot{b}}$$

$$f(\lambda_i)$$

$$= \int d^4x^{a\dot{a}} \prod_j \delta^2(\mu_{j\dot{b}} + x_{b\dot{b}}\lambda_j^{b})$$

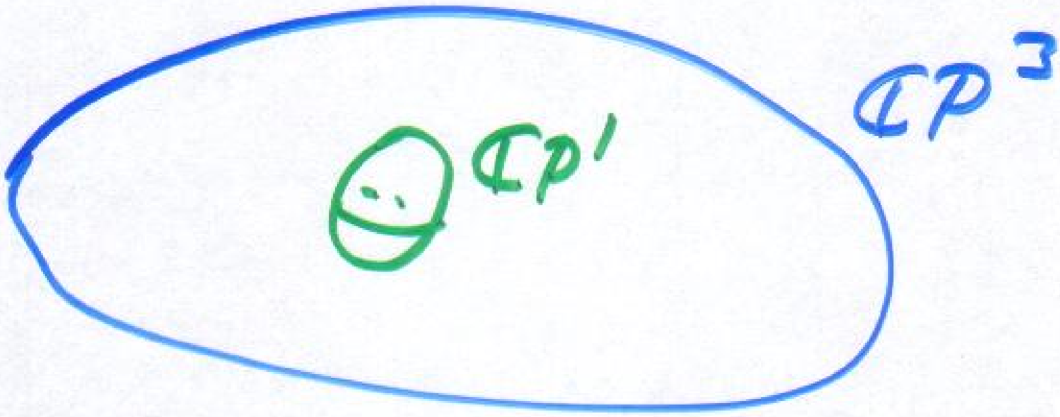
$$f(\lambda_i)$$

THIS HAS A SIMPLE PHYSICAL INTERPRETATION. PICK SOME $x^{b\dot{b}}$ IN MINKOWSKI SPACE, AND CONSIDER THE EQUATION

$$\mu_b + x_{bb'} \lambda^{b} = 0$$

THE SOLUTION SET, IF WE SET $x=0$, IS JUST $\mathbb{C}P^1$ WITH HOMOGENEOUS COORDINATES (λ^1, λ^2)

AND THAT IS REALLY TRUE FOR ANY x AS THE EQUATION LETS US SOLVE FOR μ IN TERMS OF λ.



THIS $\mathbb{CP}^1$ HAS "DEGREE ONE" AND IS AN EXAMPLE OF A "HOLOMORPHIC CURVE" IN $\mathbb{CP}^3$. THE SIMPLEST SUCH HOLOMORPHIC CURVES ARE DEFINED BY VANISHING OF A PAIR OF HOMOGENEOUS EQNS

IN THE Z^I:

$$f(z^1 \cdots z^4) = 0$$

$$g(z^1 \cdots z^4) = 0$$

IF f IS HOMOGENEOUS OF DEGREE d_1 AND g OF DEGREE d_2, THE CURVE HAS DEGREE $d_1 d_2$.

{IMPORTANT NOTE: ONLY THE SIMPLEST HOLOMORPHIC CURVES IN CP^3 ARE OF THIS SIMPLE TYPE}

FOR THE EQUATIONS

$$\mu_{\dot{b}} + x_{b\dot{b}}\lambda^{b} = 0 \qquad *$$

$d_1 = d_2 = 1$ (BOTH EQNS ARE LINEAR)

AND THE DEGREE IS

$$d = d_1 d_2 = 1$$

MOREOVER, EVERY DEGREE ONE

CURVE IN $\mathbb{C}P^3$ IS OF THE

FORM $*$ FOR SOME $x^{b\dot{b}}$.

THE AREA OF A HOLOMORPHIC CURVE OF DEGREE d, USING THE NATURAL METRIC ON $\mathbb{CP}^3$, IS $2\pi d$. SO THE CURVES WE FOUND WITH $d=1$ HAVE THE MINIMAL AREA AMONG (NONTRIVIAL) HOLOMORPHIC CURVES ... AND ARE ASSOCIATED WITH THE MINIMAL NONZERO YM TREE AMPLITUDES $\overbrace{++ + \dots +}^{+\text{'s}} - - \quad \cdots$ (MHV)

GOING BACK TO OUR AMPLITUDE

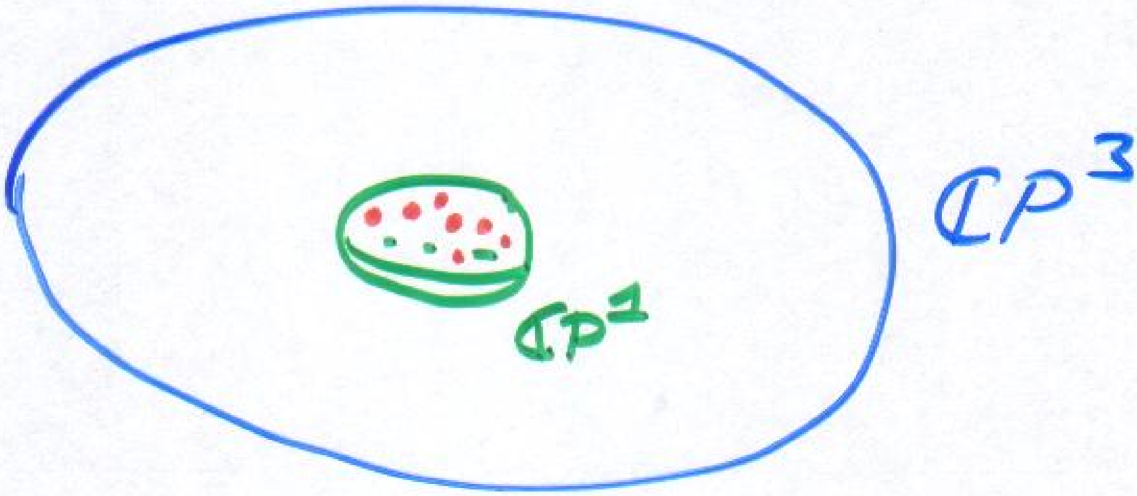
$$\tilde{A}(\lambda_i, \mu_i) = \int d^4x_{a\dot{a}} \prod_{j=1}^{n} \delta^2(\mu_{j\dot{b}} + x_{b\dot{b}}\lambda_j^{b}) \, f(\lambda_i)$$

THE δ-FUNCTIONS MEAN THAT THE AMPLITUDE VANISHES UNLESS

$$\mu_{j\dot{b}} + x_{b\dot{b}}\lambda_j^{b} = 0, \quad j = 1 \dots n$$

i.e. UNLESS SOME CURVE OF DEGREE 1 (DETERMINED BY SOME $x_{b\dot{b}}$) CONTAINS ALL n POINTS (λ_j, μ_j).

THE NONVANISHING CONFIGURATIONS LOOK LIKE THIS:



THE AMPLITUDE VANISHES UNLESS ALL n POINTS ARE CONTAINED IN A $\mathbb{CP}^1$.

NATURAL INTERPRETATION:

$\mathbb{CP}^1$ IS THE WORLDSHEET OF A STRING. IN SOME WAY OF DESCRIBING PERTURBATIVE GAUGE THEORY, THE AMPLITUDES ARISE FROM COUPLINGS OF THE GLUONS TO A STRING.

THE RESULT THAT MHV OR

$\underbrace{++ \cdots +}_{+} --$ AMPLITUDES

ARE SUPPORTED ON A

CURVE OF DEGREE 1

IS EQUIVALENT, AS WE'VE

SEEN, TO "HOLOMORPHY" OF

THESE AMPLITUDES.

WE CAN NOW PROPOSE AN ANSWER TO THE FOLLOWING PROBLEM: WHAT PROPERTY OF ARBITRARY YANG-MILLS TREE AMPLITUDES GENERALIZES THE FOLLOWING FACTS -

* VANISHING OF $\overset{+}{\widetilde{+++\cdots+}}$ AND $\overset{+}{\widetilde{+++\cdots+}}\,-$ AMPLITUDES

* HOLOMORPHY OF MHV, i.e. $\overset{+}{\widetilde{+++\cdots+}}\,--$ AMPLITUDES

THE CONJECTURE IS:

A TREE AMPLITUDE WITH

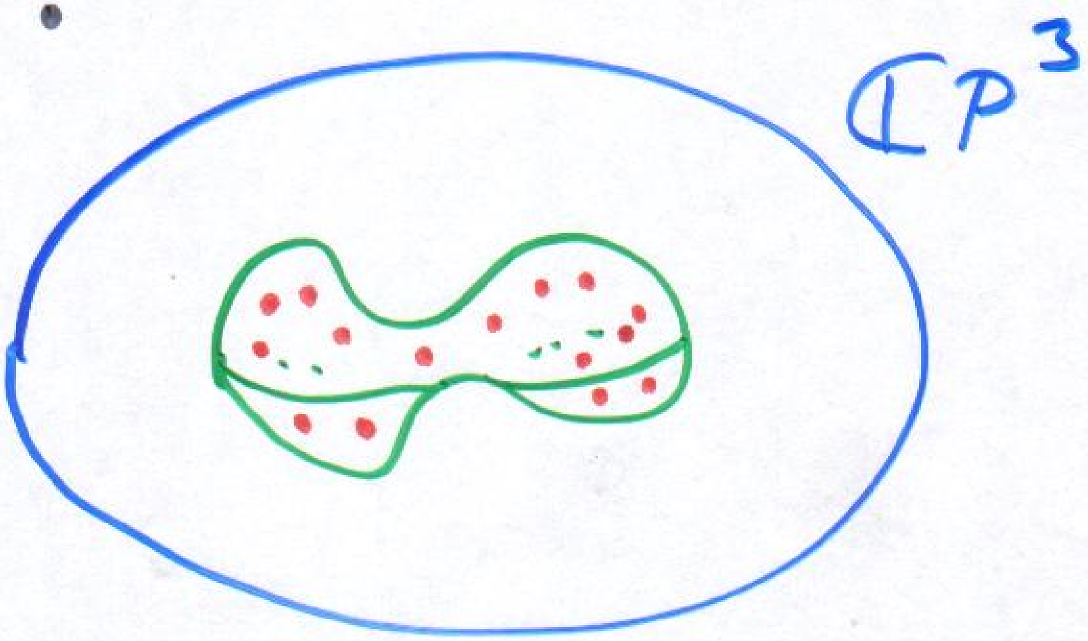
p GLUONS OF $+$ HELICITY

q GLUONS OF $-$ HELICITY

IS SUPPORTED ON A CURVE

OR STRING WORLDSHEET OF DEGREE

$q-1$:



IN THE LAST LECTURE, I'LL EXPLAIN WHAT KIND OF STRING THEORY CAN PRODUCE THIS ANSWER.

IN THE MEANTIME, LET US JUST EXPLAIN THE POWERS OF $g = g_{YM}$, THE YANG-MILLS GAUGE COUPLING.

IN YANG-MILLS THEORY, A TREE LEVEL SCATTERING AMPLITUDE WITH p GLUONS OF + HELICITY AND q OF − HELICITY IS PROPORTIONAL TO

$$g^{p+q-2}$$

IN STRING THEORY, WE GET

$$\alpha^{p}\beta^{q}\exp(-(q-1)c)$$

WHERE α AND β ARE CONSTANTS IN THE "VERTEX OPERATORS" OF + OR − HELICITY GLUONS – AND c IS THE ACTION FOR A STRING WORLDSHEET OF DEGREE 1.

THE TWO AGREE IF

$$\alpha = g, \ \beta = g^{-1}$$

$$\exp(-c) = g^2$$

OUR PROPOSAL IS THUS

THAT

PERTURBATIVE YANG-MILLS THEORY CAN BE INTERPRETED AS THE INSTANTON EXPANSION OF A STRING THEORY.