Strings in $AdS_5 \times S^5$: Making Some Sense of a Messy Spectrum

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A major challenge in the exploration of the AdS/CFT correspondence is the solution of string theory in a special curved spacetime background ($AdS_5 \times S^5$). We have developed a perturbative approach to the computation of the spectrum of the string in this background that allows us to go beyond previously-explored limits and see, in the string spectrum, the degeneracy-lifting effects of spacetime curvature. This lifting of degeneracy has its counterpart in the detailed structure of the anomalous dimension of certain operators in the dual gauge theory. The two approaches yield identical results to second order in the natural expansion parameter and then diverge.

[a]Based on work with John Schwarz, Tristan McLouhghlin and Ian Swanson of CalTech

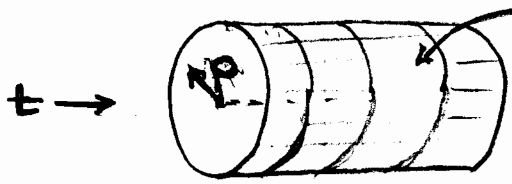
AdS/CFT CORRESPONDENCE: OVERVIEW

String theory in special 10d space with scale $\hat{R}$:

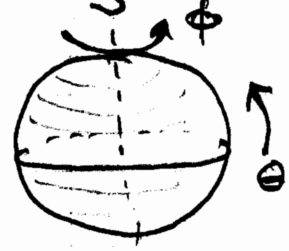
$$ds^2 = \hat{R}^2\left(-ch^2\rho\, dt^2 + d\rho^2 + sh^2\rho\, d\Omega_3^2\right) + \hat{R}^2\left(\cos^2\theta\, d\phi^2 + d\theta^2 + \sin^2\theta\, d\hat{\Omega}_3^2\right)$$

$AdS_5 \quad \times \quad S^5$

⊕ R-R 5 form!



(3+1)-d "bdry" at $\rho = \infty$



$SO(4,2) \supset SO(1,1) \times SO(4) \leftarrow$ isometry $\rightarrow SO(6) \supset SO(2) \times SO(4)$

$\hookrightarrow SU(4)$

> Strings have <u>discrete</u> spectrum of energies on this space (units of $\hat{R}^{-1}$) - hard to compute

(JM) Proposed <u>equivalence</u> to (3+1)-d $SU(N_c)$ SYM theory: ($\mathcal{N}=4$)

$$L = \frac{1}{4g^2}\, \mathrm{tr}\, F_{\mu\nu}^2 + (4 \text{ gluinos}) + \frac{1}{2g^2}\, \mathrm{tr}\left((D\phi^I)^2 - [\phi^I, \phi^J]^2\right)$$

χ^a

$I = 1, \dots, 6 \Rightarrow SO(6)$ R-symmetry

$\mathcal{N} = 4$ SUSY $\Rightarrow \beta(g) = 0 \Rightarrow SO(4,2)$ conformal inv.

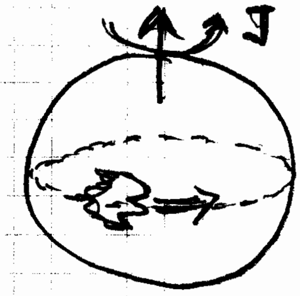
> Gauge invariant operators → discrete spectrum of dimensions $\Delta(g)$ - equal to $E_{\text{string}} \cdot \hat{R}$!!

10d geom → $\boxed{\hat{R}^4 = 4\pi g_{YM}^2 N_c (\alpha')^2}$ ← 4d gauge thy

$g^2 N_c$ large.

SIMPLIFIED STRING DYNAMICS

Boost string to lightlike equatorial S^5 geodesic
($\rho = \theta = 0$; $t = \phi$)
$\omega, J \to \infty$; $\omega - J$ finite



SO(4) x SO(4) residual symm.

Boosted fluctuations see simpler metric:

$$x^+ = t\ ,\ x^- = \hat{R}^2(\phi - t)\ ,\ \rho = \frac{r}{\hat{R}}\ ,\ \theta = \frac{y}{\hat{R}}\ ,\ \hat{R} \to \infty$$

$$ds^2 \to -4\,dx^+ dx^- - (\vec{r}^{\,2} + \vec{y}^{\,2})(dx^+)^2 + d\vec{y}^{\,2} + d\vec{r}^{\,2} + O(1/\hat{R}^2)$$

X^A : $\delta\vec{r} \in AdS_5$, $\delta\vec{y} \in S^5$ $\oplus$ ψ^α (fermions)

Boosted l.c. kinematics:

$$P_+ = \omega - J \to \text{finite}\ ;\quad \hat{R}^2 P_- = J \to \infty$$

Fix l.c. gauge ($x^+ = \tau$, $\gamma^9\psi = \psi$, ξ^-) to get
<u>free</u> l.c. Hamiltonian:

$$P_+ = \frac{1}{2\pi\alpha'}\int_0^{2\pi\alpha' P_-} d\sigma \left\{ \frac{1}{2}\left(\dot{X}^{A\,2} + X'^{A\,2} - X^{A\,2}\right) + \psi^\dagger \Pi \psi - \frac{i}{2}\left(\psi\psi' + \psi^\dagger\psi'^\dagger\right) \right\}$$

$\Pi \sim \gamma^1\gamma^2\gamma^3\gamma^4$

(free, massive (!), superstring + (?)
→ boson mass ← curvature
fermion mass ← 5-form flux

BOOSTED STRING SPECTRUM

Generic mode expansion for fields x^A, ψ^α

$$x^A(\sigma,\tau) = \sum_{-\infty}^{\infty} \frac{e^{-ik_n\sigma}}{\sqrt{2\omega_n P_-}} \left(a_n^A e^{-i\omega_n\tau} + a_{-n}^{A\dagger} e^{i\omega_n\tau} \right)$$

mode index

$$k_n = \frac{n}{\alpha' P_-} = \frac{n\hat{R}^2}{\alpha' J} = n\sqrt{\frac{4\pi g^2 N_c}{J^2}} \quad \begin{cases} g^2 N_c \to \infty \\ \frac{g^2 N_c}{J^2} \text{ fixed} \end{cases}$$

$$\omega_n = \sqrt{1+k_n^2} \simeq 1 + n^2 \cdot \left(\frac{2\pi g^2 N_c}{J^2} \right) + \cdots$$

$SO(4) \times SO(4)$ classification of oscillators:

a_n^A : $(2,2)\times(1,1) + (1,1)\times(2,2)$ 8 bosons

b_n^α : $(2,1)\times(2,1) + (1,2)\times(1,2)$ 8 fermions

Leading light-cone Hamiltonian

$$\omega - J = \sum_n \sqrt{1+k_n^2} \left(\sum_A (a_n^A)^\dagger (a_n^A) + \sum_\alpha (b_n^\alpha)^\dagger b_n^\alpha \right)$$

String energy eigenstates (degenerate)

$$|J\rangle \; ; \; (a_n^A)^\dagger (a_{-n}^B)^\dagger |J\rangle, \; (b_n^\alpha)^\dagger (b_{-n}^\beta)^\dagger |J\rangle, \ldots$$

$$\omega - J = 0 \quad ; \quad = 2\sqrt{1+k_n^2} = 2 + \left(\frac{4\pi g^2 N_c}{J^2} \right) n^2 + \ldots$$

gd. state $\omega = J$; 256-member supermultiplet of string excited states

BMN CORRESPONDENCE TO GAUGE THEORY

String state $\Rightarrow$ Operator ; $\omega \to$ dim'n ; $J \to$ R-charge.

Boosted string $\Longrightarrow$ Operator $(D \to \infty, R \to \infty)$

Building blocks : $\phi^A = (\underbrace{\phi^1, \ldots, \phi^4}_{R=0} ; \underbrace{\phi^{5+i6}, \phi^{5-i6}}_{R=\pm 1})$ Z

$SO(6)_R \supset SO(4) \times U(1)_R \rightarrow R=0$ ϕ^a $a=1,2,3,4$

Build operators of large D, small $\Delta = D - R$:

(chiral primary) $\Delta = 0 \Rightarrow tr(Z^R) \sim |R\rangle$

$\Delta = 1 \Rightarrow tr(\phi^a Z^R) \sim (a_0^a)^\dagger |R\rangle$

(SO(4) irreps: $[a,b]$, (a,b)) $\Delta \doteq 2 \Rightarrow \{tr(\phi^a\phi^b Z^R), tr(\phi^a Z \phi^b Z^{R-1}),$
$\ldots (\sim R/2$ diff't ops$)\}$

$\sim (a_n^a)^\dagger (a_{-n}^b)^\dagger |R\rangle$?

Must diagonalize anomalous dimension on this basis; one-loop calculation is easy!

$$\Delta_n = 2 + \frac{4}{\pi} g^2 N_c \sin^2\left(\frac{n\pi}{R+c}\right) \qquad n = 0, 1, \ldots, R/2$$

C depends on SO(4) irrep

$$\to 2 + 4\pi \frac{g^2 N_c}{R^2} \cdot n^2 \cdot \left(1 - \frac{2c}{R}\right) \qquad \begin{cases} n \text{ fixed} \\ R \to \infty \\ \frac{g^2 N_c}{R^2} \text{ fixed} \end{cases}$$

Remarkable match to boosted string spectrum (fill out 256-d multiplet)

BEYOND THE LEADING LIGHT-CONE

Operator dimension formula motivates study of finite $R(J)$: expand in R^{-1} (J^{-1}).

String Theory Issues–

a) Keep curvature corrections to H_{lc}:

$$\omega - J = \sum a_r^+ a_r \omega_r + \frac{1}{J} \sum f_{rstu} a_r^+ a_s^+ a_t a_u$$

b) Treat interaction term by 1^{st}-order degenerate pert'n theory

Gauge Theory Issues –

a) Find one-loop dimension formula for all $\Delta \simeq 2$ operators, exact in R

b) Organize results in supermultiplets of full supergroup

Degeneracies of $R \to \infty$ limit are lifted on <u>both</u> sides. New test of AdS/CFT is to show that they are lifted in same way!

Quantization of interacting string in RR bkgd is terra incognita. We're testing that too

BEYOND ppWAVE: ZERO-MODE EXAMPLE

What happens for finite ℓ? For $n=0$ oscillators $\omega-\ell$ formula is exact 'as is'!

Zero mode of string is sugra field. Find energies by solving scalar Laplacian in $AdS_5\times S^5$

$$\Box\Phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{g}\, g^{\mu\nu}\frac{\partial\Phi}{\partial x^\nu}\right) = 0\ ;\ \Phi = e^{-i\omega t + i\ell\phi} F(x_\perp)$$

Choose convenient (exact) metric + separate vbles:

$$ds^2 = -dt^2(R^2+x^2) + d\phi^2(R^2-y^2) + dx^I\left(\delta_{IJ} - \frac{x^I x^J}{R^2+x^2}\right)dx^J + dy^I\left(\delta_{IJ} + \frac{y^I y^J}{R^2-y^2}\right)dy^J$$

$$\Box\Phi \Rightarrow \Bigg\{\underbrace{-\frac{\omega^2}{R^2+x^2} - \frac{\partial}{\partial x^I}\left(\delta_{IJ} + \frac{x^I x^J}{R^2}\right)\frac{\partial}{\partial x^J}}_{H_{AdS_5}} + \underbrace{\frac{\ell^2}{R^2-y^2} - \frac{\partial}{\partial y^I}\left(\delta_{IJ} - \frac{y^I y^J}{R^2}\right)\frac{\partial}{\partial y^J}}_{H_{S^5}}\Bigg\} F(x)G(y)$$

Poses separated eigenvalue problems w discrete solns
Amazingly, eigenvalue condition for ω^2 factns:

$$0 = \Big(\underbrace{\omega - \ell - \sum_{i=1}^4 (n_i+\tfrac12) - \sum_{i=1}^4 (m_i+\tfrac12)}\Big)\times\Big(\omega + \ell - \sum(n_i+\tfrac12) + \sum(m_i+\tfrac12)\Big)$$

But this is exactly the pp-wave hamiltonian for the zero modes. The $\{n_i\}$ are osc. occn. #s

<u>LESSON</u> String zero mode "frequency" is exactly integer whatever ℓ is. Will be important for interpretation of gauge theory correspondence

Gauge Theory Anomalous Dimensions: Useful Lowest-Order Results

SU(4) OPERATOR ANALYSIS

CATALOG SYM R-CHARGES ($SU(4) \supset SU(2)\times SU(2)\times U(1)_R$)

$R=1: \ \phi^{\begin{smallmatrix}1\\2\end{smallmatrix}}$ $\quad R=0: \ \phi^{\begin{smallmatrix}1\\3\end{smallmatrix}}, \phi^{\begin{smallmatrix}1\\4\end{smallmatrix}}, \phi^{\begin{smallmatrix}2\\3\end{smallmatrix}}, \phi^{\begin{smallmatrix}2\\4\end{smallmatrix}}$ $\quad R=-1: \ \phi^{\begin{smallmatrix}3\\4\end{smallmatrix}}$

$R=\frac{1}{2}: \ \chi^{\boxed{1}} \ \chi^{\boxed{2}} \ \bar{\chi}^{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} \ \bar{\chi}^{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}$ $\quad R=-\frac{1}{2}: \ \chi^{\boxed{3}} \ \chi^{\boxed{4}} \ \bar{\chi}^{\begin{smallmatrix}1\\3\\4\end{smallmatrix}} \ \bar{\chi}^{\begin{smallmatrix}2\\3\\4\end{smallmatrix}}$

<u>SCALAR</u> OPERATORS W. DIMENSION J <u>AND</u> $R_{MAX} \geq J-2$:

$$\underset{(R_{MAX}=J)}{\mathrm{tr}\left((\phi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}})^J\right)}, \quad \underset{(R_{MAX}=J-2)}{\mathrm{tr}\left(\chi^{\square}\phi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}\cdot p}\sigma_2\chi^{\square}\phi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}\cdot(J-3-p)}\right)}, \quad \underset{(R_{MAX}=J-2)}{\mathrm{tr}\left(\bar{\chi}^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\phi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}\cdot p}\sigma_2\bar{\chi}^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\phi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}(J-3-p)}\right)}$$

IRREP MULTIPLICITIES (SINGLE TRACE, CYCLICITY ISSUE)

$$\mathrm{tr}\left(\phi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}\cdot J}\right) \to 1\times \overbrace{\begin{smallmatrix}\square\square\cdots\square\\\square\square\cdots\square\end{smallmatrix}}^{J} + \left(\tfrac{J-1}{2}\right)\times\overbrace{\begin{smallmatrix}\square\square\cdots\square\\\square\square\cdots\square\end{smallmatrix}}^{J-2} + \quad \text{(even vs. odd J)}$$

$$\left(\tfrac{J-1}{2}\right)\times\underbrace{\begin{smallmatrix}\square\square\cdots\square\square\square\\\square\square\cdots\square\end{smallmatrix}}_{J-1} + \left(\tfrac{J-1}{2}\right)\times\underbrace{\begin{smallmatrix}\square\square\cdots\square\\\square\square\cdots\square\\\square\end{smallmatrix}}_{J-1} + \left(\tfrac{J-3}{2}\right)\underbrace{\begin{smallmatrix}\square\square\cdots\square\square\square\\\square\square\cdots\square\\\square\end{smallmatrix}}_{J} + \cdots$$

$$\mathrm{tr}\left(\chi^{\square}\sigma_2\phi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}\cdot p}\chi^{\square}\phi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}\cdot(J-3-p)}\right) \to \left(\tfrac{J-3}{2}\right)\times\underbrace{\begin{smallmatrix}\square\square\cdots\square\\\square\square\cdots\square\end{smallmatrix}}_{J-2} + \left(\tfrac{J-1}{2}\right)\times\underbrace{\begin{smallmatrix}\square\square\cdots\square\square\square\\\square\square\cdots\square\end{smallmatrix}}_{J-1} + \cdots$$

$$\mathrm{tr}\left(\bar{\chi}^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\sigma_2\phi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}\cdot p}\bar{\chi}^{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\phi^{\begin{smallmatrix}\square\\\square\end{smallmatrix}\cdot(J-3-p)}\right) \to \left(\tfrac{J-3}{2}\right)\times\underbrace{\begin{smallmatrix}\square\square\cdots\square\\\square\square\cdots\square\end{smallmatrix}}_{J-2} + \left(\tfrac{J-1}{2}\right)\times\underbrace{\begin{smallmatrix}\square\square\cdots\square\\\square\square\cdots\square\\\square\end{smallmatrix}}_{J-1} + \cdots$$

We list only irreps containing $J-R \geq 2$ members. Representation structure governs mixing - SU(4) is exact.

More About Operator Dimensions (BMN)

Conformal invariance ⇒ operators have well-def'd dimns

$$\langle O_r(x)\, O_s(0)\rangle \sim C_{rs} / (x^2)^{d_r}$$

↳ if $C_{rs} \neq 0$, $d_r = d_s$

How does this work perturbatively? ($d = n + O(g^2)$)

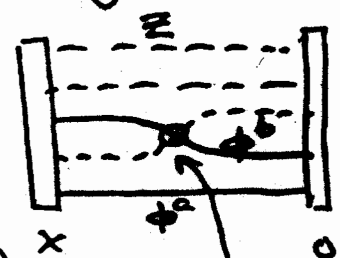
$$\langle O_r(x)\, O_s(0)\rangle \sim \frac{1}{(x^2)^n}\left[C_{rs} + 2\delta_{rs}\log(x^2) + \dots\right]$$

labels operators that can "mix"

$O(g^2)$ anom-dim matrix → must diagonalize

Diagrams giving rise to $\log(x^2)$ terms exist:

$$\langle O_1(x)\, O_2(0)\rangle \sim$$



$$\sim \frac{g^2 N_c \log x^2}{(x^2)^{\ell+2}}$$

O_1: $\mathrm{tr}(\phi^a \phi^b Z^\ell)$ O_2: $\mathrm{tr}(\phi^a Z \phi^b Z^{\ell-1})$

$\mathrm{tr}([\phi^a, \phi^b]^2)$ intn

The problem: diagonalize anom dim matrix on finite dim'l space of mixing. Kind of lattice Laplacian problem (lattice mom!)

$$\delta_n \sim \frac{g^2 N_c}{4} \sin^2\left(\frac{n\pi}{\ell}\right) \qquad n = \cancel{0}, 1, 2, \dots, \ell/2$$

← BPS state

$$\sim \left(\frac{g^2 N_c}{\ell^2}\right) \cdot n^2 \quad \text{for } n \text{ fixed}, \ \ell \text{ large}$$

can be small

"lattice" momentum matches string mode number

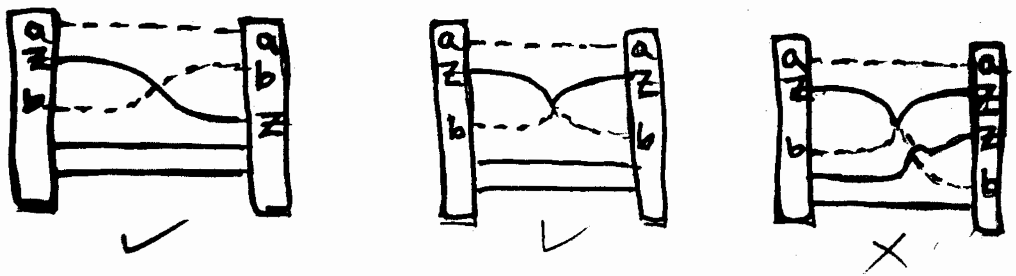
Compare to expansion of pp wave energies!

DIAGONALIZE ANOM. DIM'N: LATTICE LAPLACIAN APPROACH

SYM INTERACTIONS MIX OPERATORS IN SU(4) IRREPS

$$\{O_i^J\} = \{(\phi^a\phi^b Z^J), (\phi^a Z \phi^b Z^{J-1}), \dots, (\phi^a Z^J \phi^b)\}$$

Basis of J+1 operators, $D=J+2$, $R=J$, mixes as follows:



drop nonleading in N_c^{-1}

Net effect is $(J+1)\times(J+1)$ matrix

$$d_J^{(1)} = C \begin{bmatrix} -3 & 2 & 0 & \cdots & 1 \\ 2 & -4 & 2 & 0 & 0 \\ 0 & 2 & -4 & 2 & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & & 2 \; -4 & 2 \\ 1 & 0 & \cdots & 0 \; 2 & -3 \end{bmatrix}$$

Lattice Laplacian with b.c.'s

Now diagonalize –

$$\delta_n^{J+} = 8C \sin^2\left(\frac{n\pi}{J+1}\right) \qquad \begin{cases} n = \cancel{0}, 1, \dots, n_{max} \\ X_{ni}^{J+} = \frac{2}{\sqrt{J+1}} \cos\left[\frac{2\pi n}{J+1}(i-\tfrac{1}{2})\right] \end{cases}$$

$i = 1, \dots, J+1$

$$\delta_n^{J-} = 8C \sin^2\left(\frac{n\pi}{J+2}\right) \qquad \begin{cases} n = 1, \dots, n_{max} \\ X_{ni}^{J-} = \frac{2}{\sqrt{J+2}} \sin\left[\frac{2\pi n}{J}\, i\right] \end{cases}$$

Extended Supersymmetry Considerations

We need anomalous dimensions for *all* multiplets containing operators with $\Delta_0 = 2$. Extended superconformal symmetry implies that conformal primary operators are organized into multiplets obtained from a lowest-dimension primary $\mathcal{O}_D$ of dimension D by anticommutation with the supercharges Q_i^α (i is an $SL(2,C)$ Lorentz spinor index and α is an $SU(4)$ index). We first deal with the case where $\mathcal{O}_D$ is a spacetime scalar (of dimension D and R-charge R).

There are 16 supercharges of which 8 are raising operators: $\delta D = \delta R = \frac{1}{2}$. There are $2^8 = 256$ operators we can reach by 'raising' the lowest one. The operators at level L, obtained by acting with L supercharges, have the same dimension and R-charge (and the same $\Delta = D - R$ as the ground level).

Level	0	1	2	3	4	5	6	7	8
Multiplicity	1	8	28	56	70	56	28	8	1
Dimension	D	$D+1/2$	$D+1$	$D+3/2$	$D+2$	$D+5/2$	$D+3$	$D+7/2$	$D+4$
$R-$charge	R	$R+1/2$	$R+1$	$R+3/2$	$R+2$	$R+5/2$	$R+3$	$R+7/2$	$R+4$

Table 1: R-charge content of a supermultiplet

Two Impurity Supermultiplet Results

The one-loop results already stated determine everything if we can identify the level L of the corresponding operators. See the comprehensive analysis of Beisert (hep-th/0211032)

L	R	$SU(4)$ Irreps	Operator	$\Delta-2$	Multiplicity
0	R_0	$(0,R_0,0)$	$\Sigma_A \,\mathrm{tr}\left(\phi^A Z^p \phi^A Z^{R_0-p}\right)$	$\frac{g_{YM}^2 N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0)+3})$	$n=1,.,\frac{R_0+1}{2}$
2	R_0+1	$(0,R_0,2)+c.c.$	$\mathrm{tr}\left(\phi^{[i} Z^p \phi^{j]} Z^{R_0+1-p}\right)$	$\frac{g_{YM}^2 N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0+1)+2})$	$n=1,.,\frac{R_0+1}{2}$
4	R_0+2	$(2,R_0,2)$	$\mathrm{tr}\left(\phi^{(i} Z^p \phi^{j)} Z^{R_0+2-p}\right)$	$\frac{g_{YM}^2 N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0+2)+1})$	$n=1,.,\frac{R_0+1}{2}$
4	R_0+2	$(0,R_0+2,0)\times 2$	$\mathrm{tr}\left(\chi^{[\alpha} Z^p \chi^{\beta]} Z^{R_0+1-p}\right)$	$\frac{g_{YM}^2 N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0+2)+1})$	$n=1,.,\frac{R_0+1}{2}$
6	R_0+3	$(0,R_0+2,2)+c.c.$	$\mathrm{tr}\left(\chi^{(\alpha} Z^p \chi^{\beta)} Z^{R_0+2-p}\right)$	$\frac{g_{YM}^2 N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0+3)+0})$	$n=1,.,\frac{R_0+1}{2}$
8	R_0+4	$(0,R_0,0)$	$\mathrm{tr}\left(\nabla_\mu Z Z^p \nabla^\mu Z Z^{R_0+2-p}\right)$	$\frac{g_{YM}^2 N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0+4)-1})$	$n=1,.,\frac{R_0+1}{2}$

Table 2: Dimensions and multiplicities of spacetime scalar operators

L	R	Operator	$\Delta-2$	$\Delta-2\rightarrow$
2	R_0+1	$\mathrm{tr}\left(\phi^i Z^p \nabla_\mu Z Z^{R_0-p}\right)+\ldots$	$\frac{g_{YM}^2 N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0+1)+2})$	$\frac{g_{YM}^2 N_c}{R_0^2}n^2(1-\frac{4}{R_0})$
4	R_0+2	$\mathrm{tr}\left(\phi^i Z^p \nabla_\mu Z Z^{R_0+1-p}\right)$	$\frac{g_{YM}^2 N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0+2)+1})$	$\frac{g_{YM}^2 N_c}{R_0^2}n^2(1-\frac{2}{R_0})$
4	R_0+2	$\mathrm{tr}\left(\nabla_{(\mu} Z Z^p \nabla_{\nu)} Z Z^{R_0-p}\right)$	$\frac{g_{YM}^2 N_c}{\pi^2}\sin^2(\frac{n\pi}{(R_0+2)+1})$	$\frac{g_{YM}^2 N_c}{R_0^2}n^2(1-\frac{2}{R_0})$
6	R_0+3	$\mathrm{tr}\left(\phi^i Z^p \nabla_\mu Z Z^{R_0+2-p}\right)+\ldots$	$\frac{g_{YM}^2 N_c}{\pi^2}\sin^2(\frac{n\pi}{R_0+3})$	$\frac{g_{YM}^2 N_c}{R_0^2}n^2(1-\frac{0}{R_0})$
6	R_0+3	$\mathrm{tr}\left(\nabla_{[\mu} Z Z^p \nabla_{\nu]} Z Z^{R_0+1-p}\right)$	$\frac{g_{YM}^2 N_c}{\pi^2}\sin^2(\frac{n\pi}{R_0+3})$	$\frac{g_{YM}^2 N_c}{R_0^2}n^2(1-\frac{0}{R_0})$

Table 3: Anomalous dimensions of some non-scalar operators

General formula for dimensions of operators of R-charge R at level L:

$$\Delta_n^{R,L} = 2+\frac{g_{YM}^2 N_c}{\pi^2}\sin^2\left(\frac{n\pi}{R+3-L/2}\right)\rightarrow 2+\frac{g_{YM}^2 N_c}{R^2}\,n^2\left(1-\frac{6-L}{R}+O(R^{-2})\right)\ .$$

Gauge theory prediction for the way in which worldsheet interactions lift the degeneracy of the two-impurity string multiplet.

Quantizing the String: the Gory Details

GREEN-SCHWARZ SUPERSTRING ON $AdS_5 \times S^5$

Need classical action with full PSU(2,2|4) sgroup.

Coset target space: $AdS_5 \times S^5 \simeq SO(4,2) \times SO(6) / SO(4,1) \times SO(5)$

Stabilizer group manifest; rest nonlinearly realized.

$$\text{Action} = \int d\sigma \Big(\underbrace{\frac{h^{ab}}{2} L_a^\mu L_b^\nu \eta_{\mu\nu}}_{\text{kinetic term}} - \underbrace{2i\epsilon^{ab} \int_0^1 dt\, L_{at}^\mu \bar{\theta}^I \Gamma^\nu L_{bt}^I \eta_{\mu\nu}}_{\text{WZ term}} \Big)$$

$L_a^\mu = L_{at}^\mu \big|_{t=1}$

h^{ab}: worldsheet metric $\quad \theta^I$: 2 10d MW spinors

Γ^μ: SO(9,1) gammas $\quad \eta_{\mu\nu}$: SO(9,1) Minkowski

one-forms
$$\begin{cases} L_{at}^\mu = e_\rho^\mu \partial_a x^\rho - i\bar{\theta}^I \Gamma^\mu t^2 D_a \theta^I + \text{nonlinear} \\ L_{bt}^J = t D_b \theta^J + \text{nonlin} \qquad D_a\theta^I = \partial_a \theta^I + \dots \end{cases}$$

(x^ρ, θ^I): coset parameters, e_ρ^μ: $AdS^5 \times S^5$ vielbein

(Metsaev + Tseytlin; Kallosh + Rajaraman)

Formally correct but horrendously complex!

Step 1: expand e_ρ^μ in $1/\hat{R}$

Step 2: implement l.c. gauge fixing

Step 3: identify canonical variables + construct the Hamiltonian

Expansion, Reduction, Quantization

- $$ds^2 = 2dx^+dx^- + d\bar{r}^2 + d\bar{y}^2 - (\bar{r}^2+\bar{y}^2)(dx^+)^2$$
$$+\frac{1}{R^2}\Big[-2\bar{y}^2dx^+dx^- + \tfrac{1}{2}(y^4-r^4)(dx^+)^2 + (dx^-)^2 + \tfrac{1}{2}\bar{r}^2d\bar{r}^2 - \tfrac{1}{2}\bar{y}^2d\bar{y}^2\Big] + \dots$$

  - Expanded action has quartic interactions

- Light-cone gauge-fixing

exact $\left\{\begin{array}{lll}\Gamma_{11}\theta^I = \theta^I & \Gamma^0\Gamma^9\theta^I = \theta^I & \Rightarrow \quad \Psi \in \underline{8}_s \\ \text{MW} & \text{gauge fix} & \text{complex, 8 comp'ts}\end{array}\right.$

exact $\{x^+ = \tau \Rightarrow$ use x^- eom $+ T_{ab} = 0$ to eliminate x^-, h^{ab} ; $h^{ab} \neq (-1,1)$ at $O(1/R^2)$:
h^{ab}, x^- become fns of x^A, Ψ

- Canonical momentum subtlety

$$x^A, \dot{x}^A \rightarrow x^A, p^A = \frac{\delta\mathcal{L}}{\delta\dot{x}^A}$$

$$p_\Psi = \frac{\delta\mathcal{L}}{\delta\dot{\Psi}}, \quad p_{\Psi^+} = \frac{\delta\mathcal{L}}{\delta\dot{\Psi}^+} = O(1/R^2) \neq 0$$

Nontrivial P.B. of constraints modifies ACR's of unconstrained $(\Psi^\alpha, p_\Psi^{\alpha})$
Eliminate by field redefinition.

Light Cone Hamiltonian

$$H_{lc} = \frac{1}{2\pi\alpha'}\int_0^{2\pi\alpha' P_-} d\sigma\,[\mathcal{H}_{pp}+\mathcal{H}_{BB}+\mathcal{H}_{FF}+\mathcal{H}_{BF}]$$

$$\begin{aligned}
\mathcal{H}_{\rm pp} &= \frac{1}{2}\left[(p_A)^2+(\acute{x}^A)^2+(x^A)^2\right]+\frac{i}{2}(\psi\psi'-\rho\rho')+i\rho\Pi\psi\\
\mathcal{H}_{\rm BB} &= \frac{1}{R^2}\Big\{\frac{1}{4}\left[-y^2\left(p_z^2+z'^2+2y'^2\right)+z^2\left(p_y^2+y'^2+2z'^2\right)\right]\\
&\quad +\frac{1}{8}\left[(x^A)^2\right]^2-\frac{1}{8}\left\{\left[(p_A)^2\right]^2+2(p_A)^2(x'^A)^2+\left[(x'^A)^2\right]^2\right\}\\
&\quad +\frac{1}{2}\left(x'^A p_A\right)^2\Big\}\\
\mathcal{H}_{\rm FF} &= -\frac{1}{4R^2}\Big\{\left[(\psi'\psi)+(\rho\rho')\right](\rho\Pi\psi)-\frac{1}{2}\left[(\psi'\psi)-(\rho'\rho)\right]^2\\
&\quad -\frac{1}{2}\left[(\psi\rho')-(\psi'\rho)\right]^2+[\frac{1}{12}(\psi\gamma^{jk}\rho)(\rho\gamma^{jk}\Pi\rho')\\
&\quad -\frac{1}{48}(\psi\gamma^{jk}\psi-\rho\gamma^{jk}\rho)(\rho'\gamma^{jk}\Pi\psi-\rho\gamma^{jk}\Pi\psi')-(j,k\rightleftharpoons j',k')]\Big\}
\end{aligned}$$

$$\Pi = \gamma^1\gamma^2\gamma^3\gamma^4=\gamma^5\gamma^6\gamma^7\gamma^8$$

$\boxed{\textbf{Light Cone Hamiltonian}}$

$$
\begin{aligned}
\mathcal{H}_{\rm BF} \;=\; & \frac{1}{R^2}\Big\{-\frac{i}{4}\left[(p_A)^2+({x'}^A)^2\right]+(y^2-z^2)\Big]\,(\psi\psi'-\rho\rho') \\
& -\frac{1}{2}(p_A {x'}^A)(\rho\psi'+\psi\rho')-\frac{i}{2}\left(p_k^2+{y'}^2-z^2\right)\rho\Pi\psi \\
& +\frac{i}{4}(z'_j z_k)\left(\psi\gamma^{jk}\psi-\rho\gamma^{jk}\rho\right) \\
& -\frac{i}{4}(y'_{j'} y_{k'})\left(\psi\gamma^{j'k'}\psi-\rho\gamma^{j'k'}\rho\right) \\
& -\frac{i}{8}(z'_k y_{k'}+z_k y'_{k'})\left(\psi\gamma^{kk'}\psi-\rho\gamma^{kk'}\rho\right) \\
& +\frac{1}{4}(p_k y_{k'}+z_k p_{k'})\psi\gamma^{kk'}\rho \\
& +\frac{1}{4}(p_j z'_k)\left(\psi\gamma^{jk}\Pi\psi+\rho\gamma^{jk}\Pi\rho\right) \\
& -\frac{1}{4}(p_{j'} y'_{k'})\left(\psi\gamma^{j'k'}\Pi\psi+\rho\gamma^{j'k'}\Pi\rho\right) \\
& -\frac{1}{4}(p_k y'_{k'}+z'_k p_{k'})\left(\psi\gamma^{kk'}\Pi\psi+\rho\gamma^{kk'}\Pi\rho\right) \\
& -\frac{i}{2}(p_k p_{k'}-z'_k y'_{k'})\psi\gamma^{kk'}\Pi\rho\Big\}
\end{aligned}
$$

Perturbation Theory Logic

As $J \to \infty$, $H \to H_{\rm pp}$. The $8+8$ oscillators for mode n each contribute $\omega_n = \sqrt{1+n^2\lambda'}$. Limiting spectrum very degenerate. Simplest example: the 256 dimensional 'two-impurity' subspace:

$$
\begin{array}{lll}
a_n^{A\dagger} a_{-n}^{B\dagger} \left|J\right\rangle & b_n^{\alpha\dagger} b_{-n}^{\beta\dagger} \left|J\right\rangle & \text{(spacetime bosons)} \\
a_n^{A\dagger} b_{-n}^{\beta\dagger} \left|J\right\rangle & b_n^{\alpha\dagger} a_{-n}^{B\dagger} \left|J\right\rangle & \text{(spacetime fermions)}
\end{array}
$$

Degeneracy is broken at the first order in J^{-1} by the perturbing Hamiltonian $H_{\rm int} = H_{\rm BB} + H_{\rm FF} + H_{\rm BF}$. To do first-order perturbation theory (adequate for $O(J^{-1})$), diagonalize the explicit 256×256 perturbation matrix

$(H)_{\rm int}$	$a_n^{A\dagger} a_{-n}^{B\dagger} \left\|J\right\rangle$	$b_n^{\alpha\dagger} b_{-n}^{\beta\dagger} \left\|J\right\rangle$	$a_n^{A\dagger} b_{-n}^{\alpha\dagger} \left\|J\right\rangle$	$a_{-n}^{A\dagger} b_n^{\alpha\dagger} \left\|J\right\rangle$
$\left\langle J\right\| a_n^A a_{-n}^B$	$H_{\rm BB}$	$H_{\rm BF}$	0	0
$\left\langle J\right\| b_n^\alpha b_{-n}^\beta$	$H_{\rm BF}$	$H_{\rm FF}$	0	0
$\left\langle J\right\| a_n^A b_{-n}^\alpha$	0	0	$H_{\rm BF}$	$H_{\rm BF}$
$\left\langle J\right\| a_{-n}^A b_n^\alpha$	0	0	$H_{\rm BF}$	$H_{\rm BF}$

Table 1: Complete Hamiltonian in the space of two-impurity string states

Matrix elements computed by expanding the fields in $H_{\rm BB}$, $H_{\rm FF}$ and $H_{\rm BF}$ in creation/annihilation operators and evaluating. Algebraically tedious and requires symbolic manipulation programs, but results are quite simple.

Two-Impurity Matrix Elements (Bosons)

The full matrix elements, exact in λ', are not too awful. They will be needed when we try to make contact with gauge theory at higher loop order.

$$\left\langle J \middle| a_n^A a_{-n}^B \left(\mathcal{H}_{\rm BB}\right) a_{-n}^{C\dagger} a_n^{D\dagger} \middle| J \right\rangle = -2n^2\lambda' \frac{\delta^{AD}\delta^{BC}}{J} + \frac{n^2\lambda'}{J(1+n^2\lambda')}\left[\delta^{ab}\delta^{cd} + \delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}\right]$$
$$-\frac{n^2\lambda'}{J(1+n^2\lambda')}\left[\delta^{a'b'}\delta^{c'd'} + \delta^{a'd'}\delta^{b'c'} - \delta^{a'c'}\delta^{b'd'}\right]$$

$SO(4)$ indices $a,b,c,d \in 1,\ldots,4$ indicate that A,B,C,D are chosen from the first $SO(4)$, and $a',b',c',d' \in 5,\ldots,8$ indicate the second $SO(4)$. $\mathcal{H}_{\rm BB}$ does not mix states built out of oscillators from different $SO(4)$ subgroups.

$$\begin{aligned}\left\langle J \middle| b_n^\alpha b_{-n}^\beta \left(\mathcal{H}_{\rm FF}\right) b_{-n}^{\gamma\dagger} b_n^{\delta\dagger} \middle| J \right\rangle \;&=\; -2n^2\lambda' \frac{\delta^{\alpha\delta}\delta^{\beta\gamma}}{J} \\ &+\frac{n^2\lambda'}{24J(1+n^2\lambda')}\left[(\gamma^{ij})^{\alpha\delta}(\gamma^{ij})^{\beta\gamma} + (\gamma^{ij})^{\alpha\beta}(\gamma^{ij})^{\gamma\delta} - (\gamma^{ij})^{\alpha\gamma}(\gamma^{ij})^{\beta\delta}\right] \\ &-\frac{n^2\lambda'}{24J(1+n^2\lambda')}\left[(\gamma^{i'j'})^{\alpha\delta}(\gamma^{i'j'})^{\beta\gamma} + (\gamma^{i'j'})^{\alpha\beta}(\gamma^{i'j'})^{\gamma\delta} - (\gamma^{i'j'})^{\alpha\gamma}(\gamma^{i'j'})^{\beta\delta}\right]\end{aligned}$$

The index labeling is similar to the bosonic case. Distinguish Π parity projections: $\Pi\hat{\psi} = \hat{\psi}$ (irrep $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$) vs $\Pi\tilde{\psi} = -\tilde{\psi}$ (irrep $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$). Selection rule this time is $++ \to ++$, $-- \to --$ ($++ \to --$ vanishes).

$$\begin{aligned}\left\langle J \middle| b_n^\alpha b_{-n}^\beta \left(\mathcal{H}_{\rm BF}\right) a_{-n}^{A\dagger} a_n^{B\dagger} \middle| J \right\rangle \;&=\; \frac{n^2\lambda'}{2J(1+n^2\lambda')}\Big\{\sqrt{1+n^2\lambda'}\Big[\left(\gamma^{ab'}\right)^{\alpha\beta} - \left(\gamma^{a'b}\right)^{\alpha\beta}\Big] \\ &+ n\sqrt{\lambda'}\left[\left(\gamma^{a'b'}\right)^{\alpha\beta} - \left(\gamma^{ab}\right)^{\alpha\beta} + \left(\delta^{ab} - \delta^{a'b'}\right)\delta^{\alpha\beta}\right]\Big\}\end{aligned}$$

This mixes everything with everything in general. At $O(\lambda')$, it annihilates $++$ or $--$ bifermions and also bosons built out of just one $SO(4)$. These special selection rules simplify the first-order analysis.

Two-Impurity Matrix Elements (Fermions)

Mustn't forget the 128 spacetime fermions. They are mixed by matrix elements of $\mathcal{H}_{\rm BF}$ taken between string states of the general form $b_n^{\alpha\dagger} a_{-n}^{A\dagger} \left|J\right\rangle$. There are two independent types of spacetime fermion mixing matrix elements:

$$
\begin{aligned}
\left\langle J \middle| b_n^{\alpha} a_{-n}^{A} \left(\mathcal{H}_{\rm BF}\right) b_n^{\beta\dagger} a_{-n}^{B\dagger} \middle| J \right\rangle \;&=\; +\frac{n^2\lambda'}{2J(1+n^2\lambda')}\Big\{ \left(\gamma^{ab}\right)^{\alpha\beta} \\
&- \left(\gamma^{a'b'}\right)^{\alpha\beta} - (3+4n^2\lambda')\delta^{ab}\delta^{\alpha\beta} - (5+4n^2\lambda')\delta^{a'b'}\delta^{\alpha\beta} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\left\langle J \middle| b_n^{\alpha} a_{-n}^{A} \left(\mathcal{H}_{\rm BF}\right) b_{-n}^{\beta\dagger} a_{n}^{B\dagger} \middle| J \right\rangle \;&=\; \frac{n^2\lambda'}{2J\sqrt{1+n^2\lambda'}}\Big\{ \left(\gamma^{ab}\right)^{\alpha\beta} - \left(\gamma^{a'b'}\right)^{\alpha\beta} \\
&- \frac{n\lambda'^{1/2}}{\sqrt{1+n^2\lambda'}} \left[\left(\gamma^{ab'}\right)^{\alpha\beta} - \left(\gamma^{a'b}\right)^{\alpha\beta} \right] - \delta^{\alpha\beta}\left(\delta^{ab} - \delta^{a'b'}\right) \Big\}
\end{aligned}
$$

Normal-Ordering Issues

We normal-ordered the Hamiltonian to obtain above results. Strictly speaking, certain m.e.s contain normal-ordering ambiguity functions:

$$
\begin{aligned}
\left\langle J \middle| a_n^{A} a_{-n}^{B} \left(\mathcal{H}_{\rm BB}\right) a_{-n}^{C\dagger} a_{n}^{D\dagger} \middle| J \right\rangle \;&=\; N_{\rm BB}(n^2\lambda')\frac{\delta^{AD}\delta^{BC}}{J} + \dots \\
\left\langle J \middle| b_n^{\alpha} b_{-n}^{\beta} \left(\mathcal{H}_{\rm FF}\right) b_{-n}^{\gamma\dagger} b_{n}^{\delta\dagger} \middle| J \right\rangle \;&=\; N_{\rm FF}(n^2\lambda')\frac{\delta^{\alpha\delta}\delta^{\beta\gamma}}{J} + \dots \\
\left\langle J \middle| b_n^{\alpha} a_{-n}^{A} \left(\mathcal{H}_{\rm BF}\right) b_{n}^{\beta\dagger} a_{-n}^{B\dagger} \middle| J \right\rangle \;&=\; N_{\rm BF}(n^2\lambda')\frac{\delta^{AB}\delta^{\alpha\beta}}{J} + \dots \\
N_{\rm BF}(n^2\lambda') \;&=\; N_{\rm BB}(n^2\lambda') + N_{\rm FF}(n^2\lambda')
\end{aligned}
$$

The normal-ordering constants affect the string spectrum. In order to maintain the level spacing uniformity required by extended supersymmetry, we *must* set $N_{\rm BB} = N_{\rm FF} = N_{\rm BF} = 0$. Exercise in using global symmetry to fix undetermined finite renormalizations.

First-Order Perturbation Matrix

Expand fields in creation/annihilation operators, evaluate m.e.s between degenerate states:

$$a_n^{A+} a_{-n}^{B+} |J\rangle,\ b_n^{\alpha+} b_{-n}^{\beta+} |J\rangle,\ a_n^{A+} b_{-n}^{\beta+} |J\rangle,\ b_n^{\alpha+} a_{-n}^{\beta+} |J\rangle$$

$$H_{pp} = 2\sqrt{1+k_n^2} \doteq 2 + k_n^2 = 2 + \lambda' n^2 \quad \left(\lambda' = \frac{4\pi g^2 N}{J^2}\right)$$

Bosonic sector pert'n matrix (expand in $k_n^2 = \lambda' n^2$):

$$\langle J | a_n^A a_{-n}^B H_{BB} a_{-n}^{C+} a_n^{D+} | J\rangle = \frac{\lambda' n^2}{J} \Big\{ -2 n_{BB} \delta^{AD} \delta^{BC}$$

$$\underbrace{[\delta^{ad}\delta^{bc} + \delta^{ab}\delta^{cd} - \delta^{ac}\delta^{bd}]}_{(AdS^5)} - \underbrace{[\delta^{a'd'}\delta^{b'c'} + \delta^{a'b'}\delta^{c'd'} - \delta^{a'c'}\delta^{b'd'}]}_{(S^5)} \Big\}$$

$$\langle J | b_n^\alpha b_{-n}^\beta H_{FF} b_{-n}^{\gamma+} b_n^{\delta+} | J\rangle = \frac{\lambda' n^2}{J} \Big\{ -2 n_{FF} \delta^{\alpha\delta} \delta^{\beta\gamma}$$

$$+ \left[(\gamma^{ij})^{\alpha\delta} (\gamma^{ij})^{\beta\gamma} + (\gamma^{ij})^{\alpha\beta} (\gamma^{ij})^{\gamma\delta} - (\gamma^{ij})^{\alpha\gamma} (\gamma^{ij})^{\beta\delta} \right] - \left[ij \to i'j' \right] \Big\}$$

$$\langle J | b_n^\alpha b_{-n}^\beta H_{BF} a_{-n}^{A+} a_n^{B+} | J\rangle = \frac{\lambda' n^2}{2J} \left[(\gamma^{ab})^{\alpha\beta} - (\gamma^{a'b'})^{\alpha\beta} \right]$$

$$\begin{array}{c c} & \begin{array}{cc} a^+a^+ & b^+b^+ \end{array} \\ \begin{array}{c} a^+a^+ \\ b^+b^+ \end{array} & \left[\begin{array}{c|c} H_{BB} & H_{BF} \\ \hline H_{BF} & H_{FF} \end{array}\right] \end{array}$$

128 × 128 matrix to diagonalize

Not as bad as it looks!

COMMENTS ON THE MATRIX

- We have accounted for operator ordering ambiguities – adjust ~~to fit the~~ spectrum
- Matrix elements vanish for $n=0$:
  "zero-mode" energies get <u>no</u> correction.
  This is how you build $SO(6)$ multiplets –
  $$\{ |J\rangle,\ a_0^A |J-1\rangle,\ a_0^A a_0^B |J-2\rangle, \dots \}$$
  $$\hookrightarrow \delta D \equiv 1,\ \delta J = 0$$
- Useful $SO(4)\times SO(4)$ split of a's, b's:
  $$a^\dagger \rightarrow (4,1)\oplus(1,4) : (a_n^a)^\dagger,\ (a_n^{a'})^\dagger$$
  $$b^\dagger \rightarrow \underset{\Pi=+}{(2,1)\times(2,1)} \oplus \underset{\Pi=-}{(1,2)\times(1,2)} : (b_n^+)^\dagger,\ (b_n^-)^\dagger$$
  H_{BF} annihilates $(a_n^a)^\dagger (a_n^b)^\dagger,\ (b_n^+)^\dagger (b_n^+)^\dagger$ etc.
  Selection rules simplify diagonalization
- General form of the eigenvalues:
  $$E(n,J) = 2 + \lambda' n^2 \left(1 + \frac{\Lambda}{J} + \dots\right)$$
  $$\Delta(n,R) = 2 + \frac{4\pi g^2 N_c}{R^2}\left(1 + \frac{\bar{\Lambda}}{R} + \dots\right)$$
  Must show $\Lambda_{\text{string}} = \bar{\Lambda}_{\text{gauge}}$.

SYNOPSIS OF RESULTS

SO(4) analysis partially diagonalizes H_{int}:

$(4,1)\times(4,1)$: $(a_n^a)^\dagger (a_{-n}^b)^\dagger |J\rangle$ — killed by H_{BF},

$(1,4)\times(1,4)$: $(a_n^{a'})^\dagger (a_{-n}^{b'})^\dagger |J\rangle$ — unmixed by H_{BB}

Find eigenvalues of H_{BB} on 2 16x16 blocks:

$SO(4)\times SO(4)'$	Λ_{BB}	Operator	$\bar{\Lambda}_{gauge}$
(1,1)	$-3n_{BB}-3$	$tr(\phi^A Z^p \phi^A Z^{R-p})$	-6
(1,9)	$-3n_{BB}+1$	$tr(\phi^{(i} Z^p \phi^{j)} Z^{R-p})$	-2
(1,6)	$-3n_{BB}-1$	$tr(\phi^{[i} Z^p \phi^{j]} Z^{R-p})$	-4
(1,1)	$-n_{BB}+3$	$tr(\nabla_\mu Z Z^p \nabla^\mu Z Z^{R-2-p})$	2
(9,1)	$-n_{BB}-1$	$tr(\nabla_\mu Z^p \nabla_\nu Z Z^{R+2-p})$	-2
(6,1)	$-n_{BB}+1$	$tr([\nabla_\mu Z^p \nabla_\nu] Z Z^{R+2-p})$	0

Perfect match for these 32 states if $n_{BB}=1$.

Complete census yields (for $n_{BB}=n_{FF}=1$)

Mult	1	28	70	28	1
Λ_{string}	-6	-4	-2	0	2

Matches $\Lambda_{gauge} = L-6$ $(L=0,2,4,6,8)$

(and the representations match)

Synopsis of Results

$\mathcal{H}_{\rm FF}$ is closed on subspaces spanned by pair of $(\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2})$ $(\Pi=+)$ or pair of $(\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1})$ $(\Pi=-)$ fermionic oscillators. Diagonalize by direct projection.

$SO(4)_{AdS}\times SO(4)_{S^5}$	$\Lambda_{\rm FF}$
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-2
$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1})$	0
$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})$	-4
$(\mathbf{3},\mathbf{1};\mathbf{3},\mathbf{1})$	-2

$SO(4)_{AdS}\times SO(4)_{S^5}$	$\Lambda_{\rm FF}$
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-2
$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	0
$(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	-4
$(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{3})$	-2

Table 2: Energy shifts of states created by $++$ or $--$ fermions

Spacetime boson two-impurity states *not* annihilated by $\mathcal{H}_{\rm BF}$: 64-dimensional space spanned by pairs of bosonic creation operators from different $SO(4)$ subgroups and pairs of fermionic creation operators of opposite Π-parity.

$SO(4)_{AdS}\times SO(4)_{S^5}$	$\Lambda_{\rm BF}$
$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	-4
$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})\times 2$	-2
$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	0

Table 3: String eigenstates in the subspace for which $\mathcal{H}_{\rm BF}$ has non-zero matrix elements

Level	0	2	4	6	8
Mult.	1	28	70	28	1
$\Lambda_{\rm Bose}$	-6	-4	-2	0	2

Level	1	3	5	7
Mult.	8	56	56	8
$\Lambda_{\rm Fermi}$	-5	-3	-1	1

Table 4: First-order energy shift summary: two-impurity string multiplet

Gauge Theory Comparisons

The one-loop formula for operator dimensions takes the generic form

$$\Delta_n^{\mathcal{R}} = 2 + \frac{g_{YM}^2 N_c}{\mathcal{R}^2} n^2 \left(1 + \frac{\bar{\Lambda}}{\mathcal{R}} + \mathcal{O}(\mathcal{R}^{-2})\right)$$

Gauge theory calculations of $\bar{\Lambda}$ for two-impurity single-trace operators give

Operator	$SO(4)_{AdS} \times SO(4)_{S^5}$	$\bar{\Lambda}$
$\Sigma_A \,\mathrm{tr}\left(\phi^A Z^p \phi^A Z^{R-p}\right)$	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-6
$\mathrm{tr}\left(\phi^{(i} Z^p \phi^{j)} Z^{R-p}\right)$	$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{3})$	-2
$\mathrm{tr}\left(\phi^{[i} Z^p \phi^{j]} Z^{R-p}\right)$	$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$	-4
$\mathrm{tr}\left(\nabla_\mu Z Z^p \nabla^\mu Z Z^{R-2-p}\right)$	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	2
$\mathrm{tr}\left(\nabla_{(\mu} Z Z^p \nabla_{\nu)} Z Z^{R-2-p}\right)$	$(\mathbf{3},\mathbf{3};\mathbf{1},\mathbf{1})$	-2
$\mathrm{tr}\left(\nabla_{[\mu} Z Z^p \nabla_{\nu]} Z Z^{R-2-p}\right)$	$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$	0

Operator	$SO(4)_{AdS} \times SO(4)_{S^5}$	$\bar{\Lambda}$
$\mathrm{tr}\left(\chi^{[\alpha} Z^p \chi^{\beta]} Z^{R-1-p}\right)$	$(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$	-2
$\mathrm{tr}\left(\chi^{(\alpha} Z^p \chi^{\beta)} Z^{R-1-p}\right)$	$(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1})$	0
$\mathrm{tr}\left(\chi[\sigma_\mu, \tilde{\sigma}_\nu] Z^p \chi Z^{R-1-p}\right)$	$(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})$	-4

Operator	$SO(4)_{AdS} \times SO(4)_{S^5}$	$\bar{\Lambda}$
$\mathrm{tr}\left(\phi^i Z^p \nabla_\mu Z Z^{R-1-p}\right) + \dots$	$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	-4
$\mathrm{tr}\left(\phi^i Z^p \nabla_\mu Z Z^{R-1-p}\right)$	$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	-2
$\mathrm{tr}\left(\phi^i Z^p \nabla_\mu Z Z^{R-1-p}\right) + \dots$	$(\mathbf{2},\mathbf{2};\mathbf{2},\mathbf{2})$	0

Level	0	1	2	3	4	5	6	7	8
Multiplicity	1	8	28	56	70	56	28	8	1
$\bar{\Lambda}$	-6	-5	-4	-3	-2	-1	0	1	2

Table 5: Anomalous dimensions of two-impurity operators

Going To All Orders

String theory analysis is exact in λ': important for comparison with higher-order gauge theory operator dimensions. But diagonalization is harder: general $\mathcal{H}_{\rm BF}$ mixes bosons in same $SO(4)$ with bi-fermions of same Π-parity (basis mixing). Simplify by projecting on specific irreps and diagonaliziing a lower-dimension problem.

Ex: $(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})$ appears four times ($L=0,4,8$) yielding 4×4 problem:

	$a^{\dagger a}a^{\dagger a}\|J\rangle$	$a^{\dagger a'}a^{\dagger a'}\|J\rangle$	$\hat{b}^{\dagger\alpha}\hat{b}^{\dagger\alpha}\|J\rangle$	$\tilde{b}^{\dagger\alpha}\tilde{b}^{\dagger\alpha}\|J\rangle$
$\langle J\| a^a a^a$	$\mathcal{H}_{\rm BB}$	$\mathcal{H}_{\rm BB}$	$\mathcal{H}_{\rm BF}$	$\mathcal{H}_{\rm BF}$
$\langle J\| a^{a'} a^{a'}$	$\mathcal{H}_{\rm BB}$	$\mathcal{H}_{\rm BB}$	$\mathcal{H}_{\rm BF}$	$\mathcal{H}_{\rm BF}$
$\langle J\| \hat{b}^\alpha\hat{b}^\alpha$	$\mathcal{H}_{\rm BF}$	$\mathcal{H}_{\rm BF}$	$\mathcal{H}_{\rm FF}$	$\mathcal{H}_{\rm FF}$
$\langle J\| \tilde{b}^\alpha\tilde{b}^\alpha$	$\mathcal{H}_{\rm BF}$	$\mathcal{H}_{\rm BF}$	$\mathcal{H}_{\rm FF}$	$\mathcal{H}_{\rm FF}$

Table 6: Singlet projection at finite λ'

It is simple to project the general expressions for matrix elements of $\mathcal{H}_{\rm BB}$, etc., onto singlet states and so obtain the matrix as an explicit function of λ', n. The matrix can be exactly diagonalized and yields the following energies:

$$\begin{aligned}
E_0(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2+\frac{4}{\sqrt{1+n^2\lambda'}}\right] + \mathcal{O}(1/J^2)\\
E_4(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{2n^2\lambda'}{J} + \mathcal{O}(1/J^2)\\
E_8(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2-\frac{4}{\sqrt{1+n^2\lambda'}}\right] + \mathcal{O}(1/J^2)\ .
\end{aligned}$$

The subscript $L=0,4,8$ indicates the supermultiplet level to which the eigenvalue connects in the weak coupling limit. The middle eigenvalue (L=4) is doubly degenerate, as it was in the one-loop limit.

Going To All Orders

Antisymmetric bosonic and antisymmetric bi-fermionic states lie in $(\mathbf{1},\mathbf{1};\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{3})$ or $(\mathbf{3},\mathbf{1};\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1},\mathbf{1})$. This gives two 2×2 matrices that mix states at $L=2,6$. The 2×2 diagonalization is trivial to do and give the same result for both:

$$
\begin{aligned}
E_2(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2+\frac{2}{\sqrt{1+n^2\lambda'}}\right] + \mathcal{O}(1/J^2)\\
E_6(n,J) &= 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2-\frac{2}{\sqrt{1+n^2\lambda'}}\right] + \mathcal{O}(1/J^2)
\end{aligned}
$$

Similar diagonalizations for the remaining irreps, but no new eigenvalues
Similar story for spacetime fermions at $L=1,3,5,7$ levels.

Entire two-impurity spectrum (to leading order in $1/J$ and all orders in λ') can be written concisely:

$$
E_L(n,J) = 2\sqrt{1+\lambda' n^2} - \frac{n^2\lambda'}{J}\left[2+\frac{(4-L)}{\sqrt{1+n^2\lambda'}}\right] + O(1/J^2)\ ,
$$

The degeneracies and irrep content are identical to one loop in λ'. This expression can be rewritten, correct to order J^{-2}, as follows:

$$
E_L(n,J) \simeq 2\sqrt{1+\frac{\lambda n^2}{(J-L/2)^2}} - \frac{n^2\lambda}{(J-L/2)^3}\left[2+\frac{4}{\sqrt{1+\lambda n^2/(J-L/2)^2}}\right]\ .
$$

Joint dependence on J and L as needed for extended supersymmetry: nontrivial functional check on the correctness of our quantization.

In order to make contact with gauge theory we expand $E_L(n,J)$ in λ', obtaining

$$
\begin{aligned}
E_L(n,J) \approx &\left[2+\lambda' n^2 - \frac{1}{4}(\lambda' n^2)^2 + \frac{1}{8}(\lambda' n^2)^3 + \ldots\right]\\
&+\frac{1}{J}\left[n^2\lambda'(L-6) + (n^2\lambda')^2\left(\frac{4-L}{2}\right) + (n^2\lambda')^3\left(\frac{3L-12}{8}\right) + \ldots\right]\ .
\end{aligned}
$$