Minimal String Theory

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IAS

Klebanov, Maldacena and N. S., hep-th/0309168

N.S. and Shih, hep-th/0312170

Kutasov, Okuyama, Park , N. S., Shih, hep-th/0406030

Motivation

- Simple (minimal) and tractable string theory

- Explore D-branes, nonperturbative phenomena

- Other formulations of the theory – matrix models, holography

Approach

Minimal String Theory =

(p,q) Minimal CFT + Liouville + Ghosts

Use worldsheet techniques to derive

- geometric description (similar to topological string theory)

- matrix model

Review of minimal CFT (BPZ)

Labelled by $p < q$ relatively prime

$$c = 1 - \frac{6(p-q)^2}{p\,q}$$

Finite set of Virasoro representations

$$\Delta(\mathcal{O}_{r,s}) = \frac{(rq-sp)^2-(p-q)^2}{4p\,q}$$

$$1 \leq r < p \;, \quad 1 \leq s < q \;, \quad sp < rq$$

Fusion rules

$$(r_1, s_1) \times (r_2, s_2) = \sum_{r,s} (r, s)$$

with a certain range of (r, s)

Review of Liouville theory

Worldsheet Lagrangian

$$(\partial\phi)^2 + \mu\, e^{2\,b\,\phi}$$

Will set in the second term (cosmological constant) $\mu = 1$

Central charge

$$c = 1 + 6Q^2$$
$$Q = b + \frac{1}{b}$$

Virasoro primaries

$$\Delta(e^{2\alpha\phi}) = -\left(\frac{Q}{2} - \alpha\right)^2 + \frac{Q^2}{4}$$

Degenerate representations labelled by integer $r, s \geq 1$

$$2\alpha_{r,s} = \frac{1}{b}(1-r) + b(1-s)$$

have special fusion rules and allow to solve the theory (Dorn, Otto, Zamolodchikov, Zamolodchikov, Teschner)

Minimal String Theory

Combine the minimal CFT ("matter") with Liouville and ghosts.

Total $c = 26$ sets $b^2 = \frac{p}{q}$

Simplest operators in the BRST cohomology are "tachyons"

$$\mathcal{T}_{r,s} = c\,\bar{c}\,\mathcal{O}_{r,s}e^{2\,\beta_{r,s}\,\phi}$$

$$\mathcal{T}_{r,s} = \mathcal{T}_{p-r,q-s}$$

$$2\beta_{r,s} = \frac{p+q-|rq-sp|}{\sqrt{p\,q}}$$

$$1 \leq r < p\ , \qquad 1 \leq s < q$$

Because of degenerate matter and Liouville representations, there are additional physical operators with other ghost numbers (Lian and Zuckerman).

Ground Ring

Special operators with ghost number zero

$$\widehat{\mathcal{O}}_{r,s} = \mathcal{L}_{r,s} \cdot \mathcal{O}_{r,s} e^{2\,\alpha_{r,s}\,\phi}$$
$$2\alpha_{r,s} = \frac{p+q-rq-sp}{\sqrt{p\,q}}$$
$$1 \leq r < p \;, \qquad 1 \leq s < q$$

with $\mathcal{L}_{r,s}$ a polynomial in ghosts, and Virasoro generators.

Using ghost number conservation they form a ring (modulo BRST commutators) (Witten).

Their multiplication is constrained by the fusion rules. This allows us to determine the ring relations (up to a few coefficients which are justified later)

In terms of

$$\widehat{\mathcal{O}}_{1,1} = 1$$
$$\widehat{\mathcal{O}}_{2,1} = 2X$$
$$\widehat{\mathcal{O}}_{1,2} = 2Y$$

we have

$$\widehat{\mathcal{O}}_{r,s} = U_{s-1}(X)U_{r-1}(Y)$$

$U_{s-1}(X)$ are Chebyshev polynomials

$$U_{s-1}(X = \cos\theta) = \frac{\sin s\,\theta}{\sin\theta}$$

Since $U_{s-1}(X = \cos\theta) = \frac{\sin s\,\theta}{\sin\theta}$ are $SU(2)$ characters, their products are the $SU(2)$ fusion rules (coefficients are zero or one)

The truncation to a finite number of elements is obtained by imposing the ring relations

$$U_{q-1}(X) = U_{p-1}(Y) = 0$$

(with only X present, this is familiar from the representation ring of $\widehat{SU(2)}$)

This guarantees that the ground ring multiplication is simple; i.e. all the co-efficients are zero or one!

In the traditional worldsheet analysis this would arise as a surprising cancellation between complicated expressions from the minimal CFT and Liouville

This interesting structure arises because $\mu \neq 0$

The Tachyon Module

By ghost number conservation,

$$\widehat{\mathcal{O}}_{r_1,s_1}\mathcal{T}_{r_2,s_2} = \sum_{r_3,s_3} \mathcal{T}_{r_3,s_3}$$

Therefore the tachyons are a module of the ring. In particular, using

$$\mathcal{T}_{r,s} = \widehat{\mathcal{O}}_{r,s}\mathcal{T}_{1,1}$$

the coefficients above are zero or one.

$\mathcal{T}_{r,s} = \mathcal{T}_{p-r,q-s}$ leads to a new relation in the module...

$$T_p(Y) = T_q(X)$$

with $T_p(Y)$ Chebyshev polynomials

$$T_p(Y = \cos\theta) = \cos p\,\theta$$

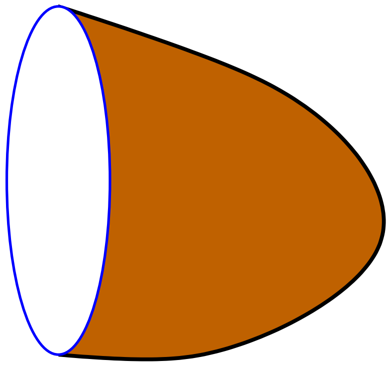
Using the ring and its module we easily derive some correlation functions; e.g.

$$\begin{aligned}
&\langle \mathcal{T}_{r_1,s_1}\mathcal{T}_{r_2,s_2}\mathcal{T}_{r_3,s_3}\rangle \\
&\qquad = \langle \widehat{\mathcal{O}}_{r_1,s_1}\widehat{\mathcal{O}}_{r_2,s_2}\widehat{\mathcal{O}}_{r_3,s_3}\mathcal{T}_{1,1}\mathcal{T}_{1,1}\mathcal{T}_{1,1}\rangle \\
&\qquad = N_{(r_1,s_1)(r_2,s_2)(r_3,s_3)}
\end{aligned}$$

This explains why the correlation functions are so simple: zero or one!

Review of Branes in Liouville

FZZT branes (Fateev, Zamolodchikov and Zamolodchikov, Teschner) – macroscopic loops in the worldsheet



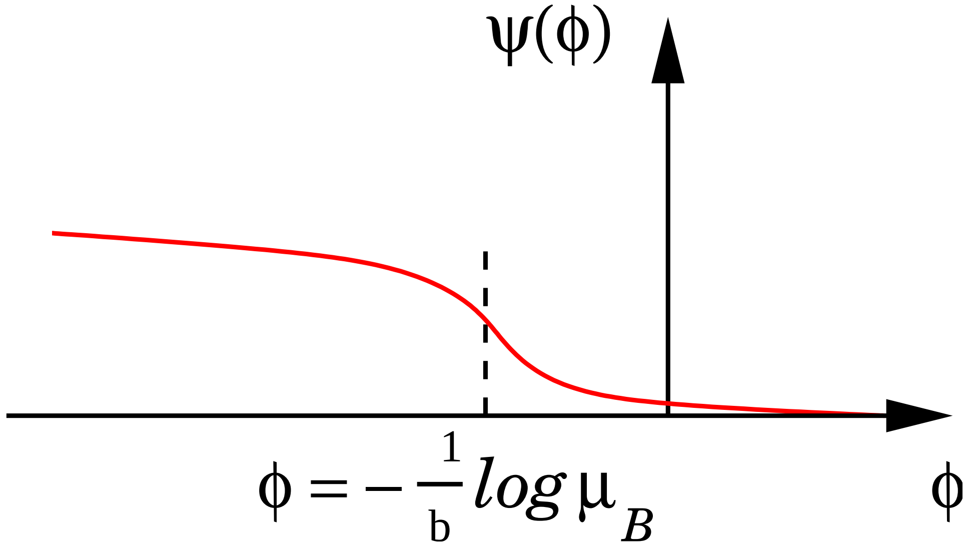
Labelled by the "boundary cosmological constant"

$$\delta\, S = \mu_B \oint e^{b\,\phi}$$

Minisuperspace wavefunction

$$\Psi(\phi) = \langle \phi | \mu_B \rangle = e^{-\mu_B e^{b\phi}}$$

The brane comes from infinity and dissolves at $\phi \approx -\frac{1}{b} \log \mu_B$.



In Cardy's formalism a brane is labelled by a representation in the open string channel

$$\mu_B = \cosh \pi \, b \, \sigma \qquad \longleftrightarrow \qquad \Delta = \frac{1}{4}\sigma^2 + \frac{Q^2}{4}$$

$$Q = b + \frac{1}{b}, \qquad c = 1 + 6Q^2$$

For the degenerate representations

$$\sigma = i\Big(\frac{m}{b} + nb\Big)$$

Subtracting the null vectors in the representation leads to the ZZ (Zamolodchikov and Zamolodchikov) branes

$$|m,n\rangle = |\sigma(m,n)\rangle - |\sigma(m,-n)\rangle$$

Same

$$\mu_B = (-1)^m \cos \pi\, n\, b^2$$

at $\sigma(m,\pm n)$ (Martinec).

These branes are localized in the strong coupling region $\phi \to +\infty$.

Branes in Minimal String Theory

FZZT branes – extended branes: Tensor a Liouville brane labelled by σ and a matter brane

ZZ branes – localized branes: Tensor a Liouville brane labelled by (m, n) and a matter brane

Simplification: the independent ZZ branes are

$$1 \leq m < p \ , \quad 1 \leq n < q \ , \quad np < mq$$

The ZZ branes are eigenstates of the ring elements

$$X|m,n\rangle = (-1)^m \cos\frac{\pi p\, n}{q}|m,n\rangle$$
$$Y|m,n\rangle = (-1)^n \cos\frac{\pi q\, m}{p}|m,n\rangle$$

$\Rightarrow$ a simple derivation of the ring relations.

Geometric Interpretation

The disk amplitude $Z(\mu_B)$ is not a single valued function of

$$x \equiv \mu_B = \cosh \pi b\, \sigma \ , \qquad b^2 = \frac{p}{q}$$

Instead, x and

$$y \equiv \partial_{\mu_B} Z(\mu_B) = \cosh \frac{\pi\sigma}{b}$$

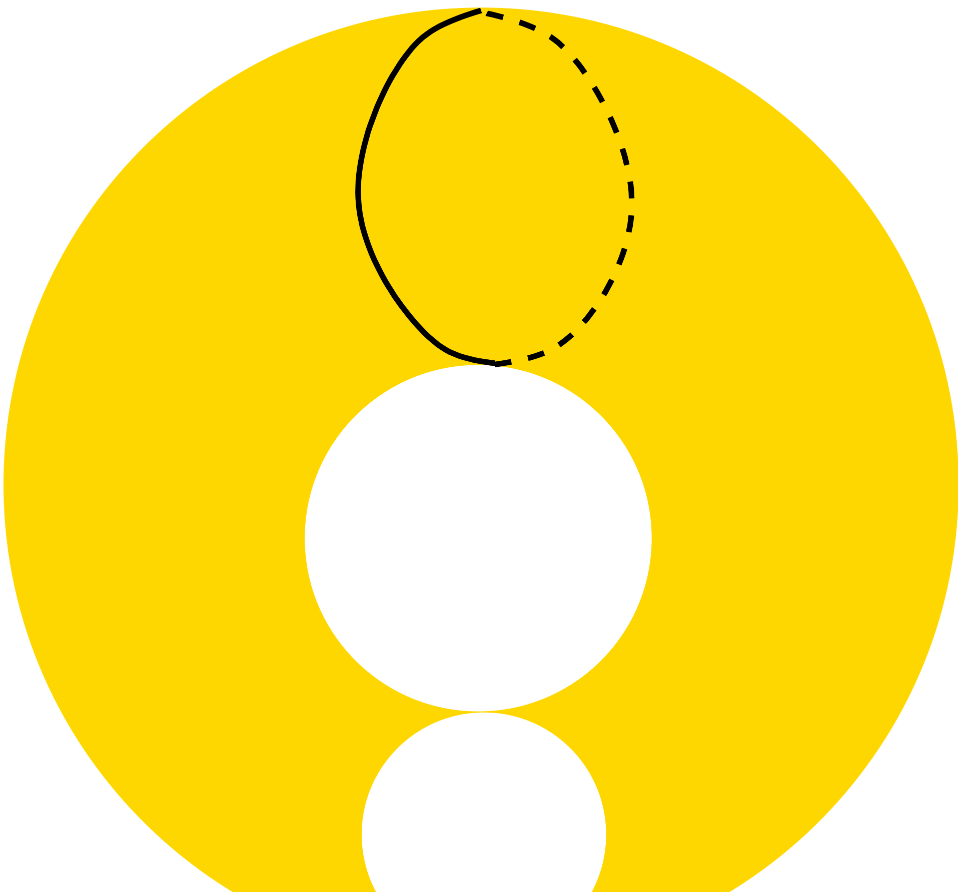
satisfy (recall, $T_p(y = \cos\theta) = \cos p\,\theta$)

$$T_p(y) = T_q(x)$$

Another perspective on the relation in the tachyon module

(Relation to earlier work of Kazakov, Kostov and collaborators)

This is a genus $\frac{(p-1)(q-1)}{2}$ Riemann surface $\mathcal{M}$ with $\frac{(p-1)(q-1)}{2}$ pinched A-cycles



Line integrals of $\omega \equiv y\,dx$ lead to branes:

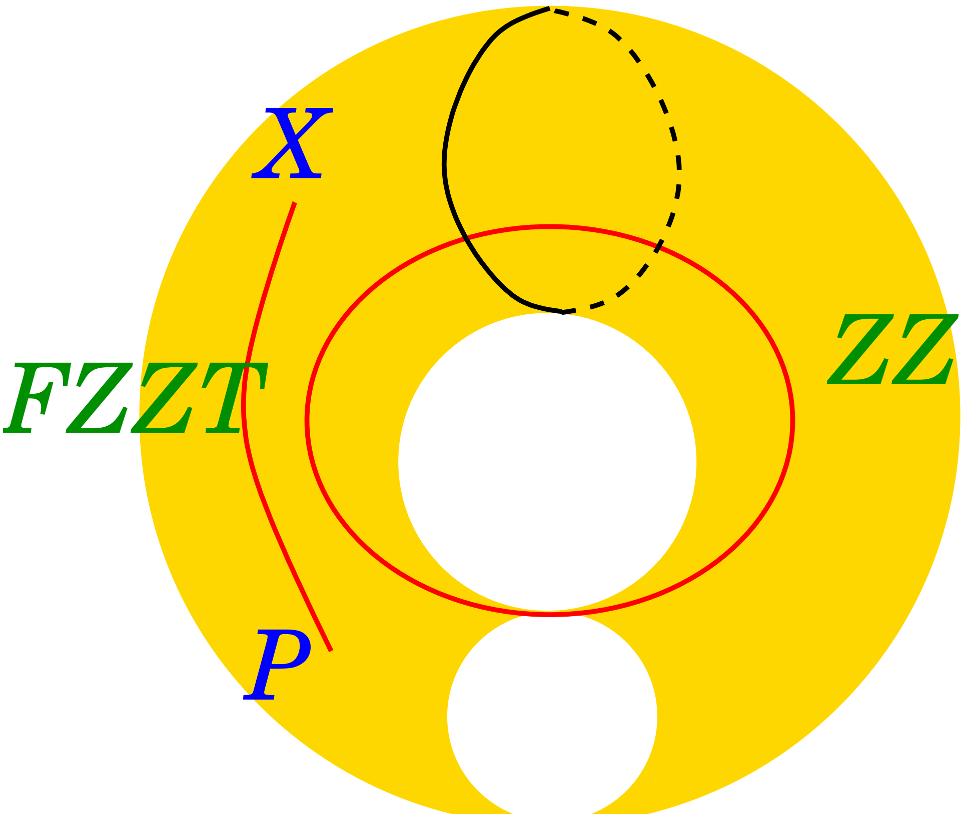
An FZZT brane is an open line integral

$$Z(x) = \int_P^x \omega$$

A ZZ brane is a difference between two FZZT branes. It turns out to pass through a singularity; *i.e.* it is an integral along a B-cycle

$$Z(m,n) = \oint_{B_{m,n}} \omega$$

FZZT and ZZ branes on the Riemann surface $\mathcal{M}$:



$(x_{m,n}, y_{m,n})$ at the singularities are the eigenvalues of the ring generators X and Y. Recall, the ZZ branes are eigenstates.

More explicitly, at the singularities we have

$$T_p(y) = T_q(x)$$
$$T_p'(y) = p\,U_{p-1}(y) = 0$$
$$T_q'(x) = q\,U_{q-1}(x) = 0$$

These are the ring relations and the relation in the tachyon module!

Deformations of $\mathcal{M}$

Closed string states $\longleftrightarrow$ singularity preserving deformations; $\oint_A \omega = 0$.

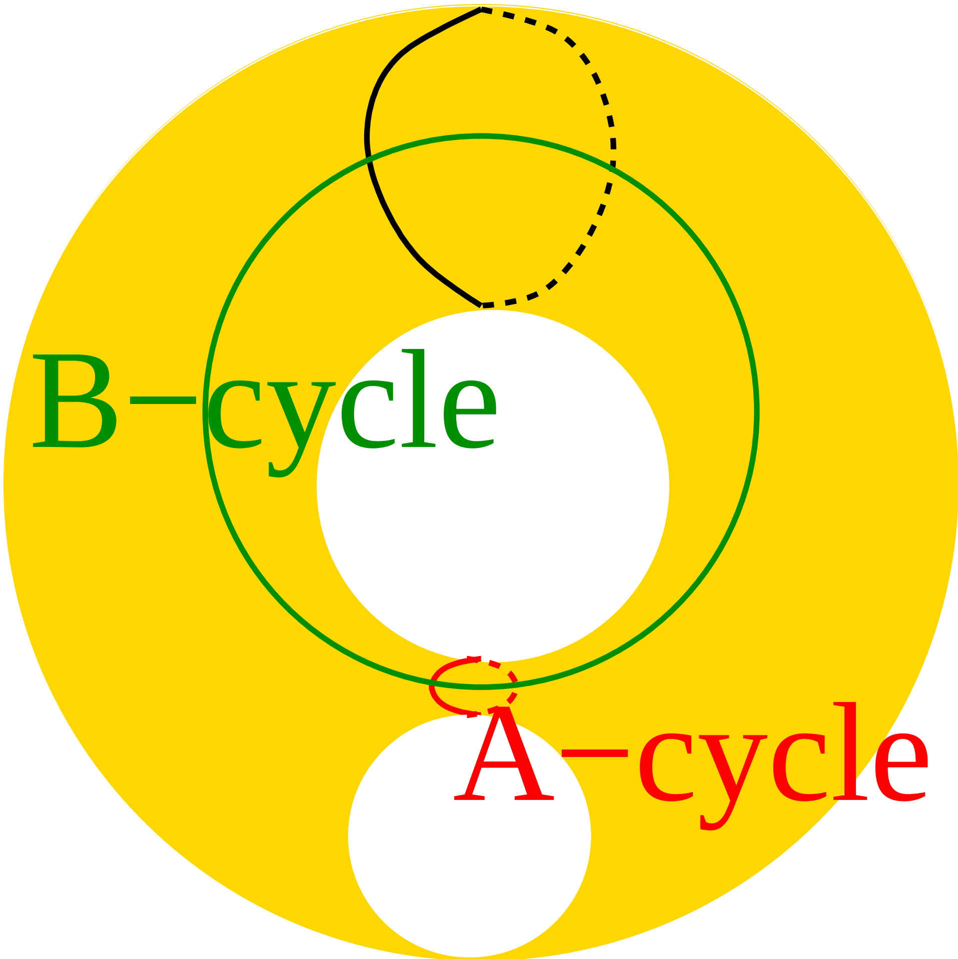
Here we find all the physical closed string states at all ghost numbers.

Adding $\mathcal{O}\left(\frac{1}{g_s}\right)$ ZZ branes $\longleftrightarrow$ open a pinched cycle (smooth out a singularity); $\oint_A \omega \neq 0$.

These lead to background tachyons with the "wrong" Liouville dressing $(\alpha \geq \frac{Q}{2})$; *i.e.* they diverge in the strong coupling region.

$\oint_B \omega$ creates ZZ branes. Their number is measured by the period of the conjugate A-cycle

$$\oint_A \omega = g_s N_{ZZ}$$



Matrix Model

Consider $(p = 2, q = 2l + 1)$, which corresponds to the one matrix model

Our surface is

$$2y^2 - 1 = T_q(x)$$

It has two copies of the complex x plane which are connected along a cut $(-\infty, -1)$ and l singularities (pinched cycles)

$$\left(x_n = \cos\frac{2\pi n}{q} \ , \ y_n = 0 \right) , \qquad n = 1, ..., l$$

Interpretation:

Discontinuity along the cut

$$\rho(x) = \mathrm{Im}\sqrt{2 + 2\,T_q(x)}$$

is the eigenvalue density.

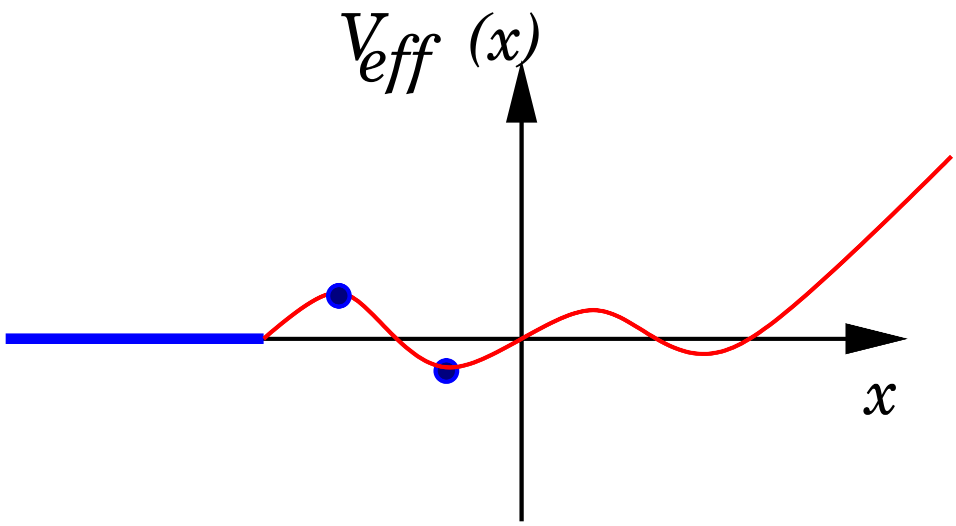
y is the force on an eigenvalue. $y = 0$ at the singularities.

The disk amplitude of FZZT brane

$$Z(x) = \int^x y\,dx' = -\frac{1}{2}V_{eff}(x)$$

is the effective potential of a probe eigenvalue.

ZZ brane: Eigenvalue at a stationary point of $V_{eff}(x)$ (where $y = 0$).



The ZZ branes decay (condense) and fill the Fermi sea

Matrix model $M \longleftrightarrow$ open strings between $N \to \infty$ condensed ZZ branes

FZZT brane in the matrix model

$$\left\langle \text{Tr}\,\frac{1}{x-M} \right\rangle \quad \longleftrightarrow \quad y$$

or after exponentiation

$$\langle \det(x-M) \rangle \quad \longleftrightarrow \quad e^{\int^x y\,dx}$$

Can write the FZZT brane as

$$\det(x-M) = \int d\psi^\dagger d\psi e^{\psi^\dagger (x-M)\psi}$$

ψ, $\psi^\dagger$ $\longleftrightarrow$ fermionic open strings between ZZ and FZZT branes.

Conclusions

- A "target space" Riemann surface $\mathcal{M}$ with a one form ω emerges as the moduli space of branes.

- $\mathcal{M}$ captures many of the properties of the minimal string: branes, observables, correlation functions, etc.

- Branes:
  - $\int^x \omega \longleftrightarrow$ creates extended branes
  - $\oint_B \omega \longleftrightarrow$ creates localized branes
  - $\oint_A \omega \longleftrightarrow$ measures # of localized branes

- Deformations of $\mathcal{M}$ $\longleftrightarrow$ observables $=$ closed strings
  - preserving $\oint_A \omega$ $\longleftrightarrow$ ordinary closed strings
  - changing $\oint_A \omega$ $\longleftrightarrow$ create localized branes, their background fields are "wrong branch" closed strings

- The ring relations control
  - the correlation functions
  - the defining equation of the surface
  - its singularities

This gives a worldsheet derivation of the matrix model, and adds a new perspective to the understanding that

the eigenvalues are associated with D-branes (Shenker, Polchinski, McGreevy, Verlinde, Klebanov, Maldacena, N.S., Martinec...)