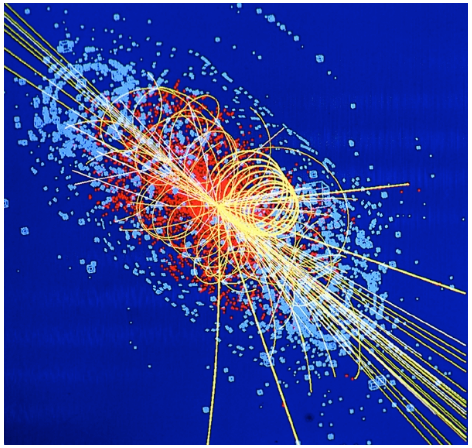
QCD at Colliders
Lecture 2

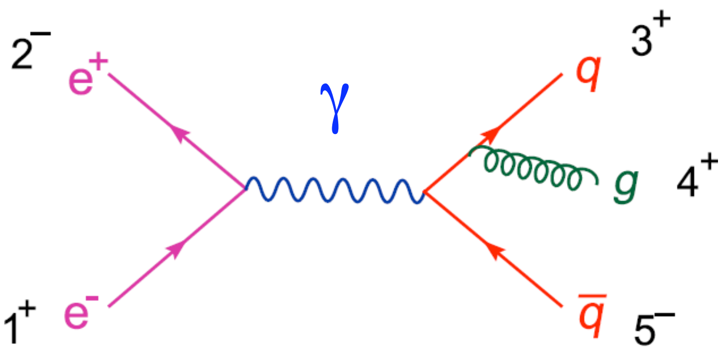


Lance Dixon, SLAC

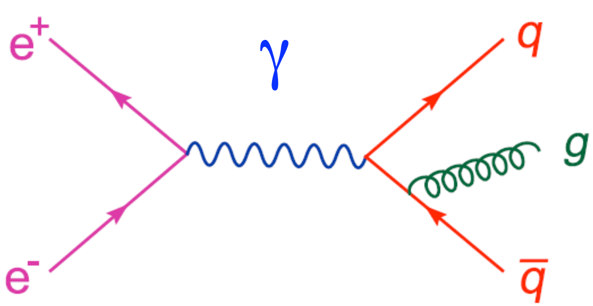
Prospects in Theoretical Particle Physics
Institute for Advanced Study, Princeton
July 21, 2005

Next most famous pair of Feynman diagrams

(to a higher-order QCD person)



$$\mathcal{A}_5 = 2ie^2 g Q_e Q_q (T^{a_4})_{i_3}^{\bar{\imath}_5} A_5$$



$$A_5 = \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)\not{\varepsilon}_4^+|3^-\rangle}{\sqrt{2}\, s_{34}} + \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)\not{\varepsilon}_4^+|5^+\rangle}{\sqrt{2}\, s_{45}}$$

Helicity formalism for massless vectors

$$(\varepsilon_i^+)_\mu = \varepsilon_\mu^+(k_i, q) = \frac{\langle i^+|\gamma_\mu|q^+\rangle}{\sqrt{2}\,\langle i\, q\rangle}$$

$$(\not\varepsilon_i^+)_{\alpha\dot\alpha} = \not\varepsilon_{\alpha\dot\alpha}^+(k_i, q) = \frac{\sqrt{2}\,\tilde\lambda_i^{\dot\alpha}\lambda_q^\alpha}{\langle i\, q\rangle}$$

reference vector q^μ is null, $q^2 = 0$

$\not q|q^\pm\rangle = 0$

obeys $\varepsilon_i^+ \cdot k_i = 0$ (required transversality)

$\varepsilon_i^+ \cdot q = 0$ (bonus)

under azimuthal rotation about k_i axis, helicity +1/2 $\tilde\lambda_i^{\dot\alpha} \to e^{i\phi/2}\tilde\lambda_i^{\dot\alpha}$

helicity -1/2 $\lambda_i^\alpha \to e^{-i\phi/2}\lambda_i^\alpha$

so $\not\varepsilon_i^+ \propto \frac{\tilde\lambda_i^{\dot\alpha}}{\lambda_i^\alpha} \to e^{i\phi}\ \not\varepsilon_i^+$ as required for helicity +1

$e^+e^- \to q g \bar{q}$ (cont.)

$$A_5 = \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)\,\not{\varepsilon}_4^+|3^-\rangle}{\sqrt{2}\, s_{34}} + \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)\,\not{\varepsilon}_4^+|5^+\rangle}{\sqrt{2}\, s_{45}}$$

$$= \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)|q^+\rangle[4\,3]}{s_{34}\langle 4\,5\rangle} + \frac{[1\,3]}{s_{12}} \frac{\langle 2^-|(\not{k}_4 + \not{k}_5)|4^-\rangle\langle q\,5\rangle}{s_{45}\langle 4\,5\rangle}$$

$$= \frac{\langle 2\,5\rangle}{s_{12}} \frac{\langle 1^+|(\not{k}_3 + \not{k}_4)|5^+\rangle[4\,3]}{s_{34}\langle 4\,5\rangle}$$

$$= -\frac{\langle 2\,5\rangle[1\,2]\langle 2\,5\rangle[4\,3]}{\langle 1\,2\rangle[2\,1]\langle 3\,4\rangle[4\,3]\langle 4\,5\rangle}$$

$$\boxed{A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle}}$$

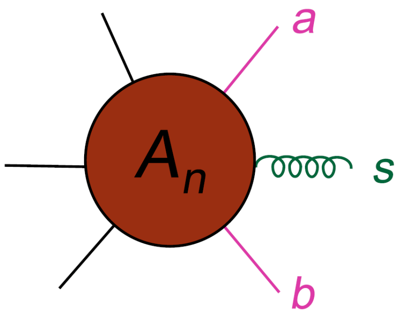
Choose $q = k_5$
to remove 2nd graph

Properties of $\mathcal{A}_5(e^+e^- \to q g \bar{q})$

1. Soft gluon behavior $\quad k_4 \to 0$

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} = \frac{\langle 3\,5\rangle}{\langle 3\,4\rangle\langle 4\,5\rangle} \times \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,5\rangle}$$

$$\to \mathcal{S}(3,4^+,5) \times A_4(1^+,2^-,3^+,5^-)$$



Universal "eikonal" factors
for emission of soft gluon *s*
between two hard partons *a* and *b*

$$\mathcal{S}(a,s^+,b) = \frac{\langle a\,b\rangle}{\langle a\,s\rangle\langle s\,b\rangle}$$

$$\mathcal{S}(a,s^-,b) = -\frac{[a\,b]}{[a\,s][s\,b]}$$

Soft emission is from the classical chromoelectric current:
independent of parton type (*q* *vs.* *g*) and helicity
-- only depends on momenta of *a*,*b*, and color charge

Properties of $\mathcal{A}_5(e^+e^- \to q g \bar{q})$ (cont.)

2. Collinear behavior

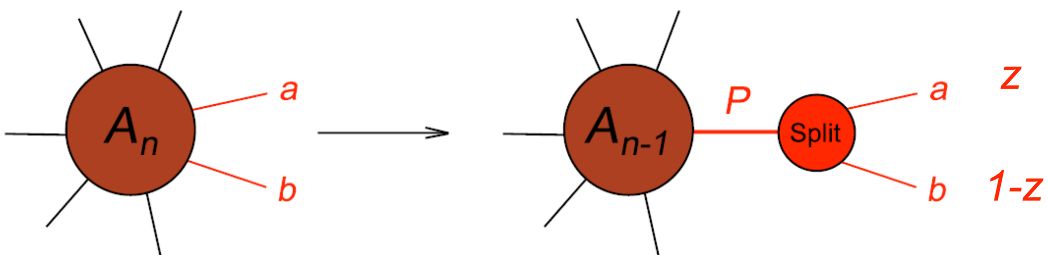
$k_3 \,||\, k_4$: $\quad k_3 = z\, k_P, \quad k_4 = (1-z)\, k_P$

$k_P \equiv k_3 + k_4, \quad k_P^2 \to 0$

$\lambda_3 \approx \sqrt{z}\lambda_P, \quad \lambda_4 \approx \sqrt{1-z}\lambda_P,$ etc.

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} \approx \frac{1}{\sqrt{1-z}\,\langle 3\,4\rangle} \times \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle P\,5\rangle}$$

$$\to \text{Split}_-(3_q^+, 4_g^+) \times A_4(1^+, 2^-, P^+, 5^-)$$



Time-like kinematics (fragmentation). Space-like (parton evolution) related by crossing

Universal collinear factors, or splitting amplitudes $\text{Split}_{-h_P}(a^{h_a}, b^{h_b})$ depend on parton type and helicity h

Collinear limits (cont.)

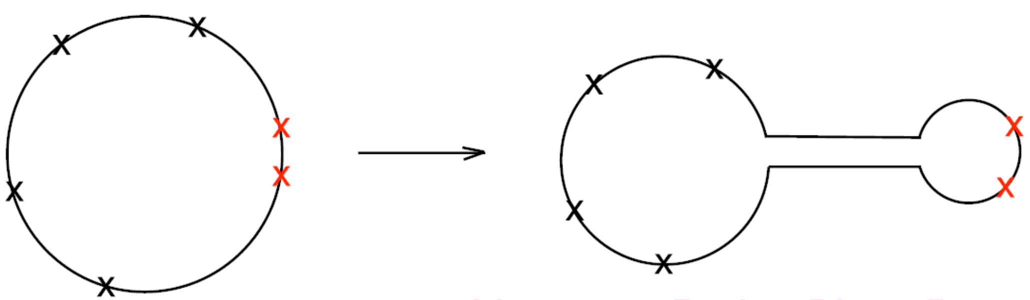
We found, from $k_3 \,||\, k_4$: $\qquad \mathrm{Split}_-(a_q^+, b_g^+) = \frac{1}{\sqrt{1-z}\,\langle a\, b\rangle}$

Similarly, from $k_4 \,||\, k_5$: $\qquad \mathrm{Split}_+(a_g^+, b_{\bar{q}}^-) = \frac{1-z}{\sqrt{z}\,\langle a\, b\rangle}$

$$\Downarrow$$

Applying C and P: $\qquad \mathrm{Split}_-(a_q^+, b_g^-) = -\frac{z}{\sqrt{1-z}\,[a\, b]}$

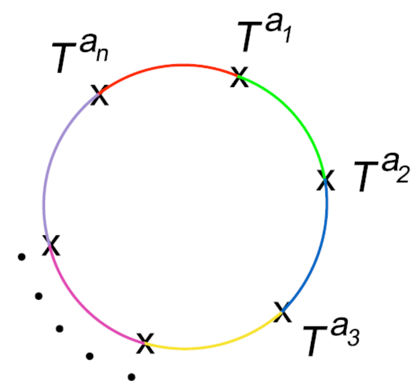
Universality can be argued various ways, including from factorization + operator product expansion in string theory:

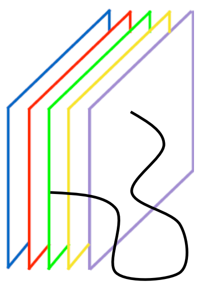


Mangano, Parke, Phys. Rept. 200, 301 (1991)

Aside: Strings and Color

For more complicated (e.g. pure-gluonic) QCD amplitudes, decompose into simpler color-ordered subamplitudes, using SU(N_c) generators in fundamental representation, T^a, also known as Chan-Paton factors.

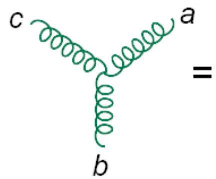




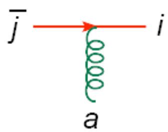
$$\mathcal{A}_n^{\text{tree}}(\{k_i, a_i, h_i\}) \;=\; g^{n-2}\mathrm{Tr}(T^{a_1}T^{a_2}\cdots T^{a_n})\, A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n}) \\ + \text{ non-cyclic perm's}$$

$A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n})$ only has cyclicly-adjacent collinear poles

Also can see from Feynman diagrams

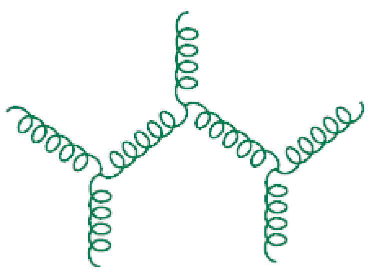
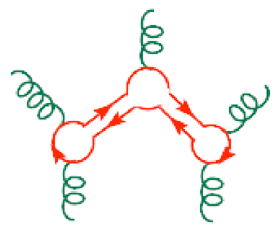
Insert 

$= f^{abc} = Tr([T^a, T^b], T^c) =$ 

where 

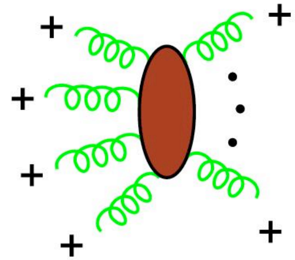
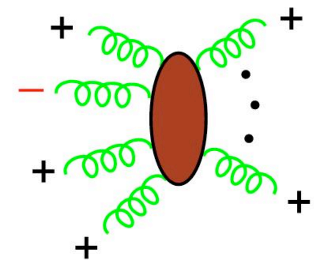
$= (T^a)_i^{\bar{j}}$

into the typical string of f^{abc} structure constants for a Feynman diagram:

 $=$  + *permutations*

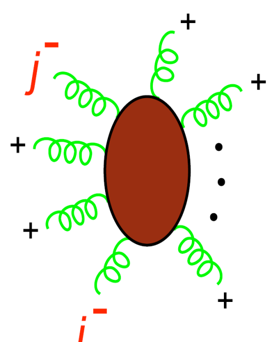
- Always single traces (at tree level)
- $A_n^{\text{tree}}(1^{h_1}, 2^{h_2}, \ldots, n^{h_n})$ comes from planar diagrams with cyclic ordering of external legs fixed to 1,2,…,n

Simplest pure-gluonic amplitudes

$$A_n^{\text{tree}}(1^{\pm}, 2^{+}, \ldots, n^{+}) \;=$$  $=$  $= \;\; 0$

Maximally helicity-violating (MHV) amplitudes:

$$A_n^{{\color{red}ij},\ \text{MHV}} = A_n^{\text{tree}}(1^{+}, 2^{+}, \ldots, {\color{red}i^{-}}, \ldots, {\color{red}j^{-}}, \ldots, n^{+})$$

$=$ 

$$= \frac{{\color{red}\langle i\, j\rangle}^4}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}$$

Parke-Taylor formula (1986)

Properties of MHV amplitudes

1. Verify soft limit

$k_s \to 0$

$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \cdots \langle a\, s\rangle\langle s\, b\rangle \cdots \langle n\, 1\rangle} = \frac{\langle a\, b\rangle}{\langle a\, s\rangle\langle s\, b\rangle} \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \cdots \langle a\, b\rangle \cdots \langle n\, 1\rangle}$$

$$\to \mathrm{Soft}(a, s^+, b) \times A_{n-1}^{ij,\ \mathrm{MHV}}$$

2. Extract gluonic collinear limits: $k_a \,||\, k_b \quad (b = a+1)$

$$\frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \cdots \langle a-1, a\rangle\langle a\, b\rangle\langle b, b+1\rangle \cdots \langle n\, 1\rangle} = \frac{1}{\sqrt{z(1-z)}\langle a\, b\rangle} \frac{\langle i\, j\rangle^4}{\langle 1\, 2\rangle \cdots \langle a-1, P\rangle\langle P, b+1\rangle \cdots \langle n\, 1\rangle}$$

$$\to \mathrm{Split}_-(a^+, b^+) \times A_{n-1}^{ij,\ \mathrm{MHV}}$$

So

$$\mathrm{Split}_-(a^+, b^+) = \frac{1}{\sqrt{z(1-z)}\,\langle a\, b\rangle}$$

and

$$\mathrm{Split}_+(a^-, b^+) = \frac{z^2}{\sqrt{z(1-z)}\,\langle a\, b\rangle}$$

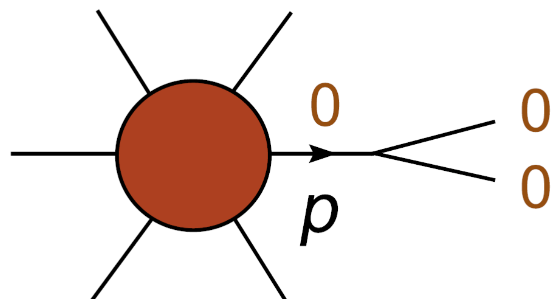
plus parity conjugates

$$\mathrm{Split}_+(a^+, b^-) = \frac{(1-z)^2}{\sqrt{z(1-z)}\,\langle a\, b\rangle}$$

Spinor Magic

Spinor products precisely capture
square-root + phase behavior in **collinear limit**.
Excellent variables for **helicity amplitudes**

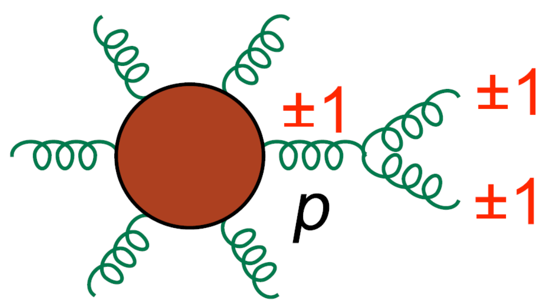
scalars



$$\sim \frac{1}{p^2} \sim \frac{1}{s_{ij}}$$

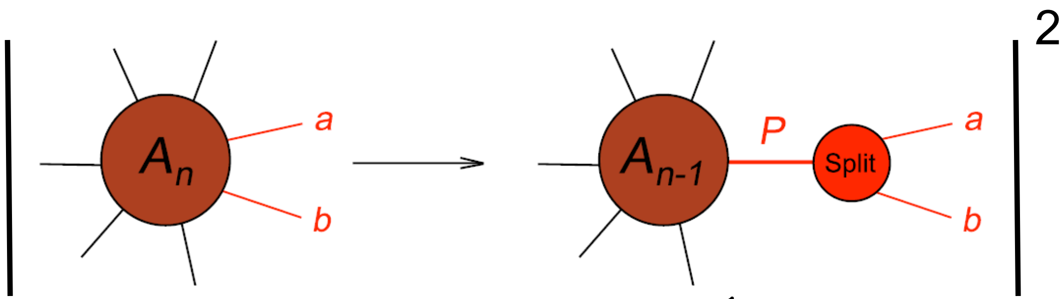
gauge theory

angular momentum mismatch



$$\sim \frac{e^{\pm i\phi}}{\sqrt{p^2}} \sim \frac{1}{\langle i\, j\rangle} \text{ or } \frac{1}{[i\, j]}$$

From splitting amplitudes to probabilities



$$d\sigma_n \;\sim\; d\sigma_{n-1} \times \frac{1}{s_{ab}} \times P(z)$$

$$P(z) \propto \sum_{h_P, h_a, h_b} \left|\text{Split}_{-h_P}(a^{h_a}, b^{h_b})\right|^2 s_{ab}$$

$q \to qg$:

$$P_{qq}(z) \;\propto\; C_F \left\{ \left|\frac{1}{\sqrt{1-z}}\right|^2 + \left|\frac{z}{\sqrt{1-z}}\right|^2 \right\}$$

$$C_F = \frac{N_c^2 - 1}{2N_c}$$

$$= \; C_F \frac{1+z^2}{1-z} \qquad z < 1$$

Note soft-gluon singularity as $z_g = 1 - z \;\to\; 0$

Endpoint behavior requires "plus" prescription

$$\Rightarrow \qquad \boxed{P_{qq}(z) \;=\; C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]}$$

$$\frac{1}{(1-z)_+} \equiv \frac{1}{1-z} \qquad \text{for } 0 \le z < 1, \qquad \text{and}$$

$$\int_0^1 dz\, \frac{f(z)}{(1-z)_+} = \int_0^1 dz\, \frac{f(z)-f(1)}{1-z} \qquad \text{for smooth } f$$

Use probabilistic interpretation (conservation of quark number) to deduce probability of no radiation – (3/2)δ(1-z) term:

$$0 = \int_0^1 dz\, \frac{P_{qq}(z)}{C_F} \;=\; \int_0^1 dz\left[\frac{1+z^2-2}{(1-z)} + \frac{3}{2}\delta(1-z)\right]$$

$$= \frac{3}{2} - \int_0^1 dz(1+z) = 0$$

Similarly for gluons

$g \to gg$:

$$P_{gg}(z) \propto C_A \left\{ \left| \frac{1}{\sqrt{z(1-z)}} \right|^2 + \left| \frac{z^2}{\sqrt{z(1-z)}} \right|^2 + \left| \frac{(1-z)^2}{\sqrt{z(1-z)}} \right|^2 \right\}$$

$$= C_A \frac{1 + z^4 + (1-z)^4}{z(1-z)} \qquad C_A = N_c$$

$$= 2C_A \left[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \right] \qquad z < 1$$

Again a soft-gluon singularity. Gluon number not conserved.
But momentum is. Momentum conservation mixes $g \to gg$ with

$g \to q\bar{q}$:

$$\boxed{P_{qg}(z) = T_R \left[z^2 + (1-z)^2 \right]} \qquad T_R = \frac{1}{2}$$

(can deduce, up to color factors, by taking
$e^+ \,||\, e^-$ in $\mathcal{A}_5(e^+e^- \to qg\bar{q})$)

Gluon splitting (cont.)

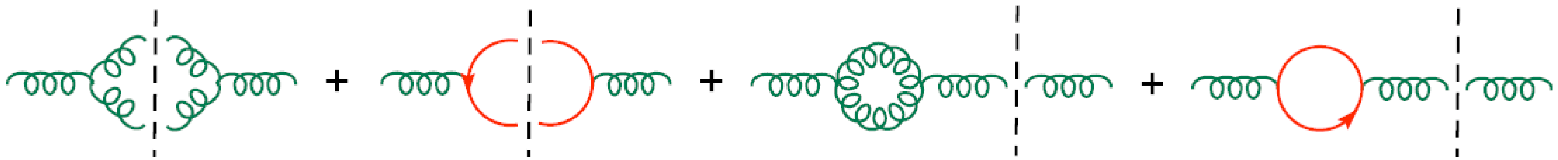
$g \to gg$: Applying momentum conservation,

$$\int_0^1 dz\, z \left[P_{gg}(z) + 2n_f P_{qg}(z)\right] = 0$$

gives

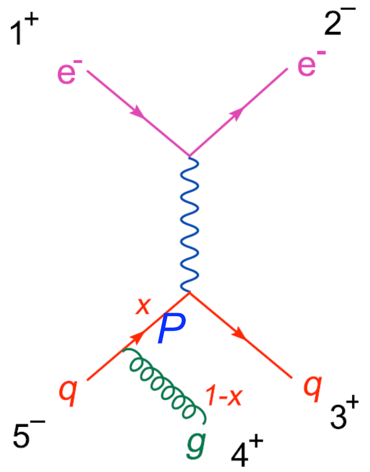
$$\boxed{\begin{aligned} P_{gg}(z) &= 2C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] + b_0\, \delta(1-z) \\ b_0 &= \frac{11C_A - 4n_f T_R}{6} \end{aligned}}$$

Amusing that first β-function coefficient enters,
since no loops were done, except implicitly via unitarity:



Space-like splitting

- The case relevant for parton evolution
- Related by crossing to time-like case
- Have to watch out for flux factor, however



$$q \to qg: \quad k_P = x\, k_5, \quad k_4 = (1-x)\, k_5$$

$$A_5 = \frac{\langle 2\,5\rangle^2}{\langle 1\,2\rangle\langle 3\,4\rangle\langle 4\,5\rangle} \approx \frac{\frac{1}{x}}{\sqrt{\frac{1-x}{x}}\,\langle 4\,5\rangle} \times \frac{\langle 2\,P\rangle^2}{\langle 1\,2\rangle\langle 3\,P\rangle}$$

$$= \frac{1}{\sqrt{x}}\frac{1}{\sqrt{1-x}\,\langle 4\,5\rangle} \times \frac{\langle 2\,P\rangle^2}{\langle 1\,2\rangle\langle 3\,P\rangle}$$

absorb into flux factor:
$$d\sigma_5 \propto \frac{1}{s_{15}}$$
$$d\sigma_4 \propto \frac{1}{s_{1P}} = \frac{1}{x\, s_{15}}$$

When dust settles, get exactly the same splitting kernels (at LO)

Physical consequences of time-like splitting

- Cannot predict perturbatively any observable which is sensitive to collinear splitting, such as the number of partons
- **Instead construct infrared-safe observables *O* which satisfy**:

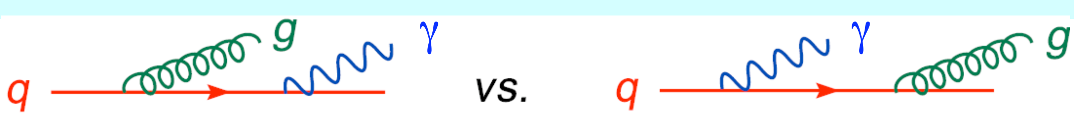
$$O_n(\ldots, k_s, \ldots) \;\rightarrow\; O_{n-1}(\ldots, \cancel{k_s}, \ldots) \qquad k_s \rightarrow 0$$

$$O_n(\ldots, k_a, k_b, \ldots) \;\rightarrow\; O_{n-1}(\ldots, k_P, \ldots) \qquad k_a \,||\, k_b$$

Then virtual corrections with 1 fewer parton can cancel real singularities

For example,

- jets (cluster or suitable cone)
- thrust $T = \max_{\hat{n}} \frac{\sum_{j=1}^{N} |\hat{n} \cdot \vec{k}_j|}{\sum_{j=1}^{N} |\vec{k}_j|}$ works because $\hat{n} \cdot (\vec{k}_a + \vec{k}_b) = \hat{n} \cdot \vec{k}_P$, $|\vec{k}_a| + |\vec{k}_b| = |\vec{k}_P|$
- ~ any kinematic distribution of *W, Z, Higgs*
  (photons tricky because they can come from fragmentation)

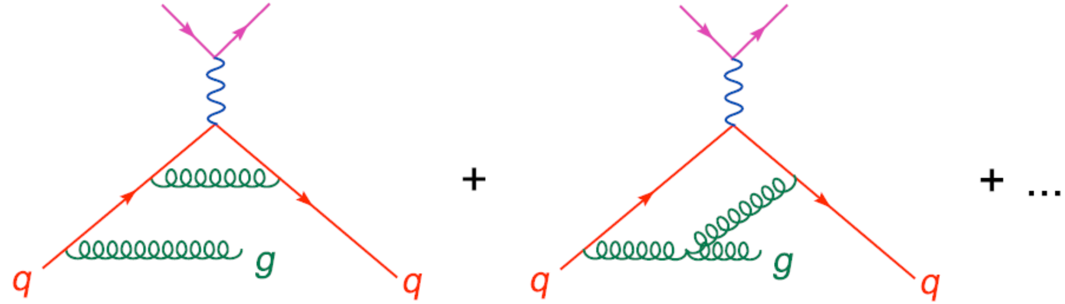


Physical consequences of space-like splitting

- All observables for processes with initial-state hadrons have collinear divergences associated with incoming partons
- Fortunately, factorization theorems show that divergences are universal and precisely correspond to "renormalization" of the parton distribution functions

Ellis, Georgi, Machachek, Politzer, Ross (1978)
Mueller (1981)
Collins, Soper, Sterman (1984)
…

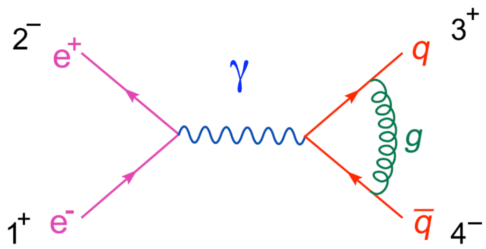
Factorization is not obvious beyond LO in arbitrary gauges

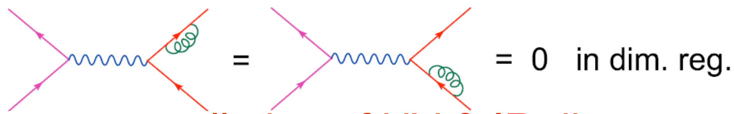


Most transparent in a physical gauge like light-cone gauge ($A^+ = 0$).

Virtual Corrections

Again examine the simplest process:





cancellation of UV & IR divergences!

overlap of soft & collinear IR divergences

$$\mathcal{A}_4^{1-\text{loop}} = \mathcal{A}_4^{\text{tree}} \frac{\alpha_s}{4\pi} \exp[\epsilon(\ln(4\pi) - \gamma_E)] \times 2C_F \left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} \left[-\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} - \frac{7}{2} - \frac{\delta_R}{2} + \frac{\pi^2}{12}\right]$$

for $2 - 2\epsilon\delta_R$ virtual-gluon helicity states

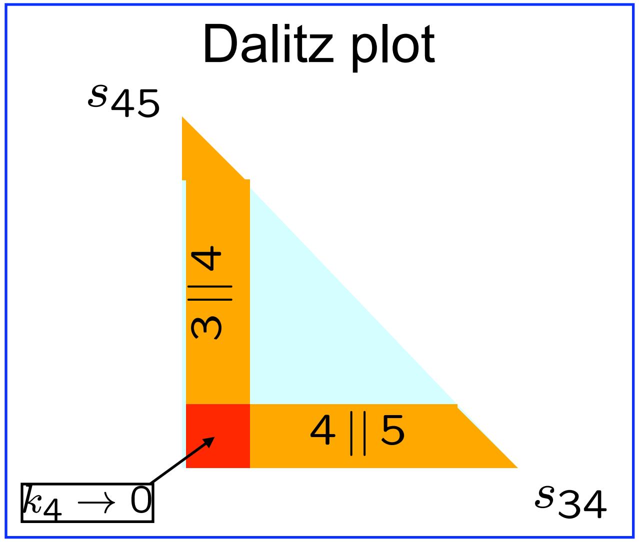
$\delta_R = 1$ for CDR & HV schemes; $\delta_R = 0$ for FDH $\approx$ $\overline{\text{DR}}$ scheme

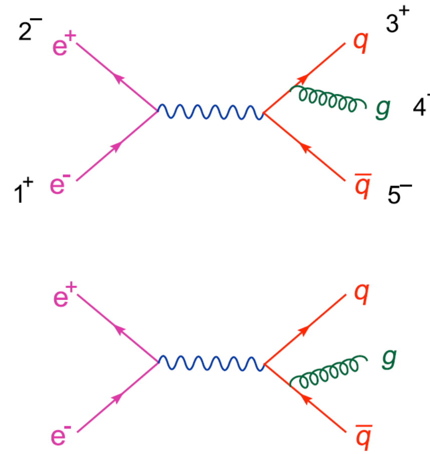
IR Cancellations in e$^+$e$^-$

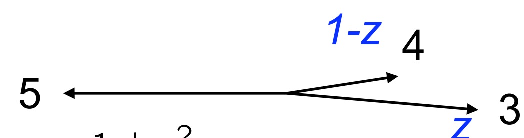
Or can use as variables:

- 1 invariant mass, say s_{34}
- CM polar angle for "decay" of (3,4) pair, cosθ, equivalent to z

$$s_{34} + s_{45} + s_{35} = s_{12}$$







$$\sigma_{\rm real}^{\rm tot} = \mu^{2\epsilon} \int d^{4-2\epsilon}{\rm LIPS}_{3,4,5}\, |\mathcal{A}_5|^2 \;\propto\; \left(\frac{\mu^2}{s_{12}}\right)^{\epsilon} |\mathcal{A}_4|^2 \frac{\alpha_s}{2\pi} \int_0 \frac{ds_{34}}{s_{34}^{1+\epsilon}} \int_0^1 dz (1-z)^{-\epsilon} C_F \frac{1+z^2}{1-z}$$

$$= |\mathcal{A}_4|^2 \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{s_{12}}\right)^{\epsilon} \left(-\frac{1}{\epsilon}\right) \int_0^1 dz \left[\frac{2}{(1-z)^{1+\epsilon}} - 1 - z\right]$$

$$= |\mathcal{A}_4|^2 \frac{\alpha_s}{2\pi} C_F \left(\frac{\mu^2}{s_{12}}\right)^{\epsilon} \left(-\frac{1}{\epsilon}\right) \left[-\frac{2}{\epsilon} - \frac{3}{2}\right]$$

$$= |\mathcal{A}_4|^2 \frac{\alpha_s}{\pi} C_F \left(\frac{\mu^2}{s_{12}}\right)^{\epsilon} \left[\frac{1}{\epsilon^2} + 2\frac{3}{4\epsilon} + \text{finite}\right]$$

after doubling z < 1 term to account for 4 || 5 region

IR Cancellations in e$^+$e$^-$ (cont.)

$$\sigma_{\text{real}}^{\text{tot}} = |\mathcal{A}_4|^2 \frac{\alpha_s}{\pi} C_F \left(\frac{\mu^2}{s_{12}}\right)^{\epsilon} \left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + \text{ finite}\right]$$

$$\sigma_{\text{virtual}}^{\text{tot}} = 2\,\text{Re}[\mathcal{A}_4^{1-\text{loop}} \mathcal{A}_4^*] = |\mathcal{A}_4|^2 \frac{\alpha_s}{\pi} C_F \left(\frac{\mu^2}{s_{12}}\right)^{\epsilon} \left[-\frac{1}{\epsilon^2} - \frac{3}{2\epsilon} + \text{ finite}\right]$$

Same cancellation takes place when phase-space integrals are weighted with any infrared-safe observable ***O***

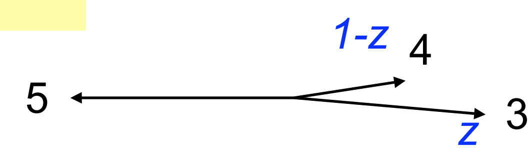
$$O_n(\ldots, k_s, \ldots) \to O_{n-1}(\ldots, \cancel{k_s}, \ldots) \qquad k_s \to 0$$

$$O_n(\ldots, k_a, k_b, \ldots) \to O_{n-1}(\ldots, k_P, \ldots) \qquad k_a \,||\, k_b$$

because ***O**$_{n-1}$* can be pulled outside of integrals over

For example, in 2-jet rate, once $s_{34} < y_{cut}$ s_{12},

parto $O_3(k_3, k_4, k_5) \to O_2(``k_3 + k_4", k_5) = 1$ ıer,



Next time

- How factorization at hadron collider works in practice: NLO corrections to Drell-Yan/W/Z production
- Why are corrections big? Life at the edge of phase space.