Lecture II: Integrability in Strings on Coset Spaces

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- Strings in Flat Space
- Strings on Coset Spaces
- Formalism: Currents, Action, Equations of Motion
- Lax Connection, Integrability
- Wilson Lines, Monodromy, Spectral Curve
- Properties of the Curve
- Moduli of the Curve, Cycles
- Ansätze
- Spectral Curves for various Models

Strings in Flat Space

Fields: $\vec{X} : \mathbb{R}^{1,1} \to \mathbb{R}^{D-1,1}$

2D World Sheet — D-dim Target Space

Periodicity: $\vec{X}(\tau, \sigma + 2\pi) = \vec{X}(\tau, \sigma)$

Action (Polyakov)

$$S \approx \int d\tau \int_0^{2\pi} d\sigma \; dX_\mu \wedge * dX^\mu$$

EOM: $d * d\vec{X} = 0$

General Solution of EOM (Fourier mode decomposition)

$$\vec{X}(\tau, \sigma) = \vec{X}_0 + \vec{p}\tau + \sum_{n \neq 0} \mathrm{Re}\, c_n \exp(i\sigma n + i\tau |n|)$$

Easily quantised: set of independent HO modes

String theory has also WS-metric γ_{ab} (in $*$)

Virasoro constraint: $(\partial_\pm \vec{X})^2 = 0$ from variation of γ_{ab}

$$\partial_\pm = \partial_\tau \pm \partial_\sigma$$

Constraints imposed on quantised system.

Strings on Coset Spaces

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eg. $AdS_5 \times S^5$. Non-Linear Sigma Models Non-Trivial models

$$\left.\begin{array}{l} S_{N-1} = SO(N)/SO(N-1) \\ AdS_{N-1} = SO(N-2,2)/SO(N-2,1) \text{ (universal cover)} \end{array}\right\} G/H$$

Field: $g(\tau,\sigma) \in G$ eg. $N \times N$ orthogonal matrix for S_{N-1}

Coset: $g(\tau,\sigma) \hat{=} g(\tau,\sigma) h(\tau,\sigma)$, $h \in H$ gauge transformation

Periodicity: $g(\tau,\sigma+2\pi) = \underline{g(\tau,\sigma)h(\tau,\sigma)} \hat{=} g(\tau,\sigma)$, $h \in H$

Currents (moving frame)

Full Current: $J = -g^{-1}dg \in \mathfrak{g}$ (algebra of G)

Decompose $J = B + P$, $B \in \mathfrak{h}$ gauge field (unphysical), $P \perp \mathfrak{h}$ momentum (physical)

(algebra of H)

Action $$S = \frac{\sqrt{\lambda}}{2\pi} \int dt \int_0^{2\pi} d\sigma \, \tfrac{1}{2} \mathrm{tr}(P \wedge * P)$$

EOM: $d*P = B \wedge *P + *P \wedge B$

Jacobi Identities: $dJ = J \wedge J \to dB = B \wedge B + P \wedge P$, $dP = B \wedge P + P \wedge B$

Currents (static frame)

lowercase $j = gJg^{-1}$, $J = g^{-1}jg$ easier!

EOM: $d*p = 0$ (p is Noether curr.)

Jacobi: $dj = -j \wedge j \to db = -b\wedge p - p \wedge b$, $dp = -2p\wedge p$

p is gauge invariant

String Theory $\to$ WS-metric

Virasoro constraints: $\mathrm{tr}(P_\pm)^2 = 0$

will be imposed later

Virasoro: $\mathrm{tr}(p_\pm)^2 = 0$

Action: $S = \frac{\sqrt{\lambda}}{2\pi} \int dt \int_0^{2\pi} d\sigma \, \frac{1}{2} \mathrm{tr}(p \wedge * p)$

Integrable Structures on the Worldsheet

Useful for extracting physical data from WS (diffeo-indep. data)

Let's construct a flat connection $D = d + a$, ie $D^2 = da + a \wedge a = 0$

Ansatz: $a = \alpha p + \beta * p$

Flatness: $D^2 = \alpha \underbrace{dp}_{= -2 p\wedge p \text{ (Jacobi)}} + \beta \underbrace{d*p}_{=0 \text{ by EOM}} - \alpha^2 p\wedge p \overbrace{- \alpha\beta\, p \wedge *p - \alpha\beta *p \wedge p}^{=0} - \beta^2 \overbrace{*p \wedge *p}^{= -p\wedge p}$

$$= \left(-2\alpha - \alpha^2 + \beta^2\right) p\wedge p \overset{!}{=} 0$$

Solution: $\alpha = \dfrac{2}{x^2-1}$, $\beta = \dfrac{2x}{x^2-1}$.

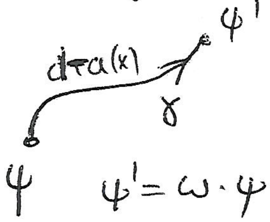
Lax connection: $a(x) = \dfrac{2}{x^2-1}\, p + \dfrac{2x}{x^2-1} *p$

Family of flat connections $(D(x))^2 = 0$ $\quad da(x) = a(x) \wedge a(x)$

Wilson Lines on the WS & Monodromy

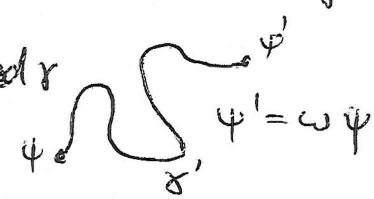
Parallel transport of connection $d - a(x)$ (fixed "spectral" par.) along curve γ on WS

Wilson Line: $\omega(\gamma, x) = P\exp \int_\gamma (-a(x))$



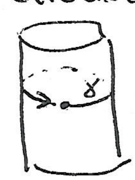
Connection $d - a(x)$ is flat $\Rightarrow$ ω indep. of deformations of γ

$\omega(\gamma, x) = \omega(\gamma', x)$ if γ' deformed γ

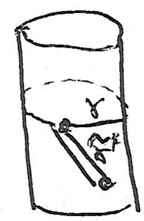


define Monodromy $\omega(x) = \omega(\gamma, x)$ where γ once around string

eg. $\omega(x) = P\exp \oint_0^{2\pi} (-a_\sigma(x))\, d\sigma$



$\omega(x)$ indep of γ, but not of endpoints $\gamma(0)$, $\gamma(1)$



$\omega'(x) = \omega(\tilde{\gamma}^{-1} \gamma \tilde{\gamma}, x) = \omega(\tilde{\gamma}, x)^{-1}\, \omega(x)\, \omega(\tilde{\gamma}, x)$

Similarity transformation $\omega'(x) \cong \omega(x)$ by $\omega(\tilde{\gamma}, x)$

Eigenvalues of monodromy indep. of WS!

$\omega(x) \cong \mathrm{diag}(\omega_1(x), \ldots, \omega_N(x))$

Eigenvalues $\omega_k(x)$ depend on spectral parameter x only.

Still a lot of data on string, but only conserved qts.

Have transformed $\underbrace{g(\tau, \sigma)}_{\text{string embedding}}$ to $\underbrace{\omega_k(x)}_{\text{conserved quantities}}$

Investigate properties of $\omega_k(x)$

Diagonalisation and Branch Points

$$\det(\lambda - \omega(x)) = (\lambda - \omega_1(x)) \ldots (\lambda - \omega_N(x))$$

$\{\omega_k\}$ form N Riemann sheets of a function $\omega(x)$

when two EV degenerate. consider 2×2 submatrix

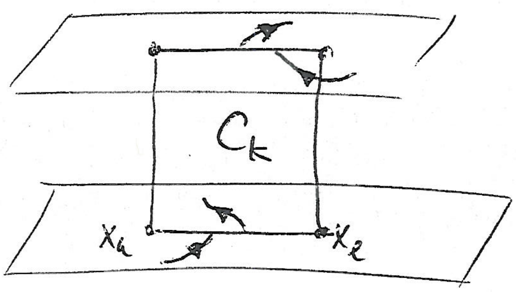
$$\omega(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \qquad \omega_{1,2}(x) = \frac{1}{2}(a+d) \pm \sqrt{\frac{1}{4}(a-d)^2 + bc}$$

Typically at degeneracy point x_k $\qquad \sim (x - x_k)$

$$\omega_{1,2}(x) = \omega(x_k) \pm \alpha_k \sqrt{x - x_k}$$

Two sheets join at degeneracy point x_k

Branch cuts originates from x_k (and ends at x_ℓ).



2 Riemann sheets

$\omega(x)$ has N sheets connected by branch cuts.

Singularities at $x = \pm 1$

$$a(x) = \frac{2}{x^2-1}\, p + \frac{2x}{x^2-1} * p$$

$$a_\sigma(x) \overset{x\to 1}{=} \frac{1}{x-1}\, p_\sigma + \frac{1}{x-1}\, p_\tau + O((x-1)^0) = \frac{1}{x-1}\, p_+ + \dots$$

Diagonalise $p_+ = S \tilde{p}_+ S^{-1}$.

Then

$$S^{-1}(\partial_\sigma + a_\sigma(x))S = \frac{1}{x-1}\, \tilde{p}_+ + O((x-1)^0)$$

$$\omega(x) \overset{x\to 1}{=} \exp\left(-\frac{1}{x-1} \int_0^{2\pi} \tilde{p}_+ \, d\sigma + O((x-1)^0)\right)$$

Exponential singularity!

Consider "quasi-momenta" $\omega_k(x) = \exp i q_k(x)$

Single poles in $q_k(x)$ at $x = \pm 1$

Residues linked by Virasoro $\mathrm{tr}(p_\pm)^2 = 0$

Moduli of Spectral Curves

$\omega(x)$ is single-valued on the curve

$q(x) = -i \log \omega(x)$ is single-valued modulo 2π.

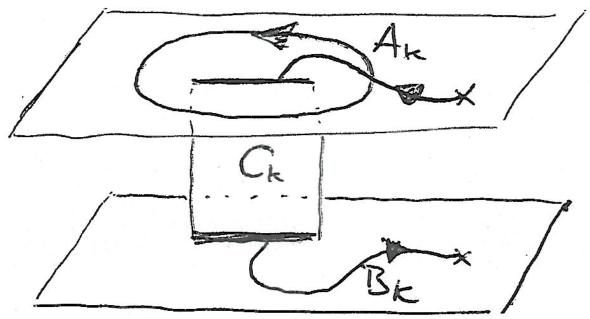
$q'(x) = -i\, \omega'(x)/\omega(x)$ is single-valued

Closed cycles $\quad \oint dq = \oint q'(x)\, dx \in 2\pi\mathbb{Z}$

Can arrange branch cuts s.t.

A-cycles vanish

$\oint_{A_k} dq = 0$



Cut #k

Moduli:

Mode numbers: $n_k = \frac{1}{2\pi} \oint_{B_k} dq \in \mathbb{Z}$ discrete!

Filling $\quad K_k = \frac{\sqrt{\lambda}}{8\pi^2 i} \oint_{A_k} (x + 1/x)\, dq \sim$ length of cut C_k

Compare to flat space

cut C_k corresponds to excitation of mode n_k with amplitude K_k

$\Rightarrow$ Structure qualitatively similar to flat space!

Global Charges at $x=\infty$

Expand at $x=\infty$

$$a(x) = \frac{2}{x} * p + \mathcal{O}(1/x^2)$$

Monodromy

$$\omega(x) = 1 - \frac{2}{x} \oint * p + \ldots$$

Noether charge

$$Q = \frac{\sqrt{\lambda}}{2\pi} \oint * p$$

$$\omega(x) = \exp\left(-\frac{4\pi}{\sqrt{\lambda}x} Q + \ldots\right)$$

Some more properties depending on G/H

Ansätze for Spectral Curves

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Consider $y(x) = (x^2-1)^2 q'(x)$ (to remove poles at $x=\pm 1$)

$y(x)$ has
- branchpoints $\sim \frac{1}{\sqrt{x-x_k}}$
- no branch points $\sim \sqrt{x-x_k}$
- no single poles or double poles.
- analytic otherwise

$\Rightarrow$ $y(x)$ is an algebraic curve if finite genus.

Finite genus - finite number of branch cuts, - "finite gap"

Ansatz for algebraic curve! $F_k(x)$ polynomials

$$F_N(x)\, y^N + F_{N-1}(x)\, y^{N-1} + F_{N-2}(x)\, y^{N-2} + \dots + F_0(x) = 0$$

Branch point $\frac{1}{\sqrt{x-x_k}}$ requires $F_N(x_k) = F_{N-1}(x_k) = 0$, $F_{N-2}(x_k) \neq 0$
$F_N'(x_k) \neq 0$ ~~$F_{N-1}'(x_k) \neq 0$~~

$F_N(x_k) \sim \prod (x - x_k)$, $F_{N-1}(x) = 0$ typically.

Restrict other coefficients of polynomials $F_k(x)$ by
- Absence of branch points $\sqrt{x-x_k}$ or poles
- Behaviour at $x = 0, \infty, \pm 1$
- A-cycles, mode numbers, fillings of cuts
- Further properties of sigma model on G/H.

Can fix all coefficients.

General Spectral Curves

VIk

Monodromy in N-dim representation of symmetry group

$\rightsquigarrow$ Spectral curve of degree N (N Riemann sheets)

Conjugation class of representation

$\rightsquigarrow$ symmetries in curve, eg. $x \to 1/x$, $x \to -x$

For $AdS_5 \times S^5$ superstrings 4+4-dim repr. of $PSU(2,2|4)$

4+4 sheets