1. Problems for J. Maldacena's lectures

1.1. Problems related to the AdS/CFT tutorial

a) Consider a U(N) Yang Mills theory and understand the power counting in 't Hooft arguments. In other words, show that an L loop Feynman diagram contribution to the vacuum energy that can be drawn on a genus h Riemann surface (after using the double line notation) contributes as $N^{2-2h}(g^2N)^{L-1}$. Show that to leading order in N $\langle Tr[M^k]Tr[M^l]\rangle \sim \langle Tr[M^k]\rangle\langle Tr[M^l]\rangle$.

b) Consider a massive particle in AdS with a mass m with $M_{Plank} \gg m \gg 1/R$, where R is the AdS radius of curvature. Compute the two point function of the associated operator by computing the classical action for the propagation of this particle between two points in the boundary that are separated by a boundary distance L. Compute the formula for the anomalous dimensions Δ. You can use the formula $\langle O(0)O(L)\rangle \sim e^{-m(\text{length})}$. Notice that the length of the geodesic will be infinite and you will have to regularize and renormalize the result. You can use coordinates $ds^2 = R^2\frac{dx^2+dz^2}{z^2}$, impose a cutoff at $z=\epsilon$ and then take $\epsilon \to 0$.

c) Suppose that you have a metric $ds^2 = w(z)^2(-dt^2+dx^2+dz^2)$. Suppose that the minimum of the warp factor is attained at z_0 with a value $w(z_0) > 0$. Compute the minimum energy for a particle of mass m that moves in this space. The energy is normalized as $E = i\frac{\partial}{\partial t}$.

1.2. Problem related to the lecture

a) Consider the $AdS_5 \times S^5$ geometry. Consider a massless geodesic moving along a great circle on S^5. These geodesics are associated to particles with large angular momentum J under and $SO(2) \subset SO(6)$ and with $\Delta = J$. Suppose that you are interested in operators with fixed $\Delta - J$ but large J. In addition, suppose that you scale $R \to \infty$, $J \to \infty$ keeping R^2/J fixed. Rescaling the charges in this fashion is associated to a rescaling of the coordinates in $AdS_5 \times S^5$. Show that in this limit the metric becomes that of a plane wave, $ds^2 = -2dx^+dx^- - r^2(dx^+)^2 + dr^2$, with $x^+ = t$, $x^- = R^2(t-\varphi)$ where t is global time and φ is the angle of S^5 that is shifted by J. Hint: Write the metric as

$$ds^2 = R^2[\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + \cos^2\theta d\varphi^2 + d\theta^2 + \sin^2\theta d\tilde{\Omega}_3^2]$$

and rescale r and θ so that Rr and $R\theta$ are finite.

b) Consider the action of a string moving in the plane wave background, ignoring the worldsheet fermions. Choose light cone gauge, with $x^+ = \tau$ and constant p_- density on the worldsheet. Show that the action reduces to that of free massive bosons. Derive a formula for the energy spectrum of the string. (You can ignore numerical factors).

c) Recalling the formula for the radius of AdS in term of the 't Hooft coupling $R^4 = \lambda \alpha'^2$, write the above formula in terms of gauge theory quantities. (You can ignore numerical factors, and you will need to use the explicit form of the rescaling of coordinates in a)).

d) Compare the expression in c) with the expression that you obtain for the 1-loop scaling dimensions for operators of the form $\sum_l e^{ipl} Tr[WZ^l WZ^{J-l}]$. Derive the quantization condition for p for large J. Hint: you write a Bethe equation where you will have e^{ipJ} on the left hand side and an a scattering phase $S(p,-p)$ in the right hand side. Assume that

$$\lim_{\lambda\to\infty} S(p_1 = \frac{\alpha_1}{\sqrt{\lambda}}, p_2 = \frac{\alpha_2}{\sqrt{\lambda}}) = 1$$

where α_i are fixed in the limit. With this assumption, derive the quantization condition for p in the plane wave (or BMN) limit

$$J \to \infty \; , \qquad \lambda \to \infty \; , \qquad pJ = \text{fixed} \; , \qquad \lambda / J^2 = \text{fixed} \tag{1.1}$$

Having fixed the momentum p, derive now the expression for $\Delta - J$ using the dispersion relation that we saw in the lecture and taking this limit (1.1).