Homework problems for Herman Verlinde's lectures

Problem 1.

One of the simplest non-orbifold singularities is the cone over $\mathbf{P}^1 \times \mathbf{P}^1$. In this problem you work out the quiver for D-branes at this singularity, and check some of its properties.

Let us denote coordinates on the first $\mathbf{P}^1$ as z^α, $\alpha = 1, 2$, and coordinates on the second $\mathbf{P}^1$ as $w^{\dot\beta}$, $\dot\beta = 1, 2$. The line bundles are of the form $\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(n, m)$. That is, $H^0(\mathcal{O}(n, m))$ is generated by polynomials $P(z, w)$ of total degree n in z and total degree m in w (assuming $n, m \geq 0$).

An exceptional collection of fractional branes is given by

$$\{E_1, E_2, E_3, E_4\} = \{\mathcal{O}(0, 0), \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1)\}. \tag{0.1}$$

Their Chern characters are

$$\begin{aligned} \mathrm{ch}(E_1) &= (1, 0, 0) \\ \mathrm{ch}(E_2) &= (1, H_1, 0) \\ \mathrm{ch}(E_3) &= (1, H_2, 0) \\ \mathrm{ch}(E_4) &= (1, H_1 + H_2, 1) \end{aligned} \tag{0.2}$$

Here H_1 is the 2-cycle class corresponding to the first $\mathbf{P}^1$, and H_2 is the 2-cycle class corresponding to the second $\mathbf{P}^1$. They satisfy the relations

$$H_1^2 = H_2^2 = 0, \qquad H_1 \cdot H_2 = 1. \tag{0.3}$$

The canonical class is $K = -2H_1 - 2H_2$.

(a) Use the Chern characters and the relative Euler character $\chi(E_i, E_j)$ to work out the quiver diagram for a single D3 brane. Check that (# arrows in = # arrows out) for each node. What type of additional fractional branes are allowed?

(b) Let us denote the generators of the cohomologies as follows:

$$\begin{aligned} H^0(E_1, E_2) &= A_\alpha z^\alpha \\ H^0(E_2, E_4) &= B_{\dot\beta} w^{\dot\beta} \\ H^0(E_1, E_3) &= C_{\dot\beta} w^{\dot\beta} \\ H^0(E_3, E_4) &= D_\alpha z^\alpha \\ H^0(E_1, E_4) &= E_{\alpha\dot\beta} z^\alpha w^{\dot\beta}. \end{aligned} \tag{0.4}$$

Use this to find the superpotential of the quiver in (a).

(c) Perform a Seiberg duality on node (2).

Aside: note that the new quiver has a Z_2 quantum symmetry, namely rotation by 90^0. Identifying the quiver by this symmetry yields the quiver for a D3 brane at the conifold singularity, discussed in Klebanov's lectures. This is no coincidence: in fact the cone over $\mathbf{P}^1 \times \mathbf{P}^1$ is a Z_2 orbifold of the conifold. To see this, define $N^{\alpha\dot\beta} = z^\alpha w^{\dot\beta}$, then we have $N^{1\dot 1}N^{2\dot 2} - N^{1\dot 2}N^{2\dot 1} = 0$ (with a little care, you also get the Z_2).

(d) Let us try to find the moduli space of the quiver. We can save ourselves a little time because the quiver happens to be toric.

Show that the solutions to the F-term equations of the quiver in part (b) can be parametrized by variables Z, W, P and H:

$$\begin{aligned} A_\gamma &= \epsilon_{\alpha\gamma} Z^\alpha \\ B_{\dot\delta} &= \epsilon_{\dot\beta\dot\delta} W^{\dot\beta}\, H \\ D_\gamma &= -\,\epsilon_{\alpha\gamma} Z^\alpha\, H \\ C_{\dot\delta} &= \epsilon_{\dot\beta\dot\delta} W^{\dot\beta} \\ E^{\alpha\dot\beta} &= P Z^\alpha W^{\dot\beta} \end{aligned} \tag{0.5}$$

We still need to quotient out by the four (complexified) $U(1)$'s. The sum of all the $U(1)$'s acts trivially on all the fields, so we can ignore it. We can use $U(1)_4$ to set $H = 1$. If we denote by q_i the charge operator for $U(1)_i$, then we can parameterize the remaining charges as $l_1 \equiv q_2 + q_4$ and $l_2 \equiv q_3 + q_4$. Check that the remaining fields carry the following charges:

	Z^1	Z^2	W^1	W^2	P
l_1	1	1	0	0	−2
l_2	0	0	1	1	−2

(0.6)

This is the well-known toric (or linear sigma model) description of the cone over $\mathbf{P}^1 \times \mathbf{P}^1$.

Problem 2.

Consider the unoriented quiver diagram in Figure 1. It consists of the MSSM and two extra $U(1)$'s. The notation is as follows. In unoriented quivers, the chiral fields need not be in the (fundamental,anti-fundamental) representation of two gauge groups. To indicate this, for each chiral field we draw *two* arrows on the corresponding edge on opposite ends. The arrow closest to a gauge group indicates whether the chiral field is in the fundamental (outgoing arrow) or anti-fundamental (incoming arrow). If the representations for some gauge group are real, we do not draw an arrow near that gauge group.

This quiver has three $U(1)$'s. Which ones are anomalous and which are not? Which linear combination is hypercharge? Do you recognize the other two $U(1)$'s?

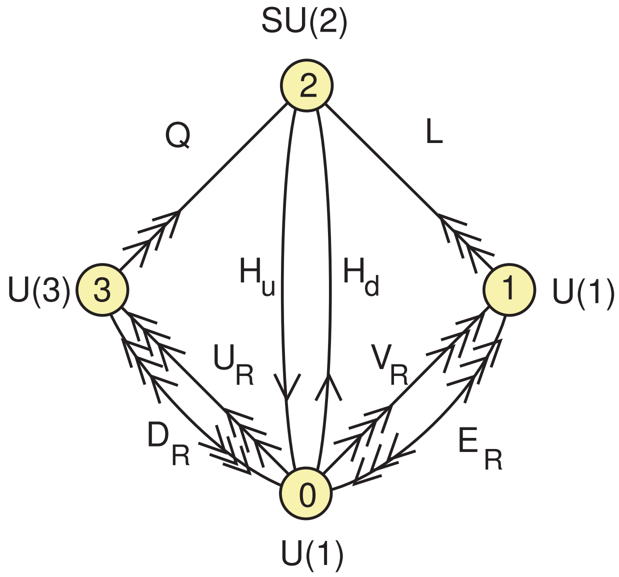

Figure 1: *An unoriented version of the quiver discussed in the lectures.*