Addendum To Tutorial Of E. Witten

I want to explain a few points that I didn't have time for at the end of the tutorial. (But see also the paper hep-th/9505186 for more.) I am writing this note because it would take too much time to explain all these details in the next lecture. I am probably going to go into more detail here than most students would find relevant, but having gotten this far I would like to tidy up a few loose ends.

In computing the partition function of free electrodynamics, the lattice sum comes out to be

$$\sum_{x\in\Lambda} q^{\frac{1}{4}(-(x,x)+(x,\star x)}\bar{q}^{\frac{1}{4}(x,x)+(x,\star x)}$$

The lattice Λ is $H^2(M,\mathbf{Z})$ modulo torsion, that is, it is the second cohomology group of the four-manifold M, modulo torsion. This type of lattice sum, as I mentioned, also arises as the partition function in genus 1 of a toroidally compactified string theory. (As such it is called the Narain theta function; in mathematics, it is attributed to C. L. Siegel.) In the sum, $q=\exp(2\pi i\tau)$ where $\tau=\theta/2\pi+4\pi i/g^2$. Also (x,x) is the intersection pairing, i.e. $(x,x)=\int_M x\cup x$, where $\cup$ is the cup product (if you think of x as a differential form then you can use the wedge product). Finally, to compute $(x,\star x)$, we represent x by a harmonic two-form of the right periods (which I called $F_0(x)/2\pi$ in the lecture), and apply to it the Hodge $\star$ operator (the duality operator, for physicists), and then $(x,\star x)=\int_M x\wedge\star x$. I wrote this formula using the wedge product, since here it is most natural to think in terms of differential forms.

In general, a Narain-Siegel lattice sum is modular under $\tau\to -1/\tau$ if the lattice Λ is unimodular (or self-dual). That is true for any M by virtue of Poincaré duality. It is modular under $\tau\to\tau+2$ if Λ is any integral lattice – again true here since the intersection form on M is integral. (Easy exercise: check using the fact that (x,x) takes integer values that the lattice sum above is invariant under $\tau\to\tau+2$.) We get modular properties under $\tau\to\tau+1$ if Λ is even (i.e. if (x,x) only takes even values) – which is true in the present context precisely if M is a spin manifold.

For a function Z to have modular properties under a modular transformation $\tau\to\tau'=(a\tau+b)/(c\tau+d)$ means that $Z(\tau')=Z(\tau)(c\tau+d)^\alpha(c\bar{\tau}+d)^\beta$, for some α and β, which are the holomorphic and antiholomorphic modular weights. For example, $\mathrm{Im}\,\tau$ is invariant under $\tau\to\tau+1$, and $\mathrm{Im}(-1/\tau)=\mathrm{Im}(\tau)/\tau\bar{\tau}$, so $\mathrm{Im}(\tau)$ is modular of weights $(-1,-1)$. The Narain-Seigel theta function of a lattice whose signature is (b_+,b_-) is modular of weights $(\frac{1}{2}b_+,\frac{1}{2}b_-)$. As explained in the paper I've referred to, upon combining these facts, one sees that the partition function of free abelian gauge theory on a four-manifold, with the minimal regularization in which we just set $B_1 - B_0$ to zero, is modular of weights $\frac{1}{4}(\chi - \sigma, \chi + \sigma)$, with χ and σ the Euler characteristic and the signature.

I didn't give a satisfactory answer to questions that were asked at the end of the lecture about whether the free abelian gauge theory is really a conformal theory, and precisely how to formulate and use that. First of all, it is easy to see that it is true if we are on flat $\mathbf{R}^4$. We need to show that the stress tensor $T_{\mu\nu}$ is traceless. It suffices to show that its matrix elements among initial and final states are traceless. The initial and final states are Fock states of free photons. In computing matrix elements of $T_{\mu\nu}$, there are no loop diagrams (and there are only very simple tree diagrams, in which the stress tensor scatters one photon, or emits or absorbs a pair). In tree diagrams there is no way to get an anomaly, so $T_{\mu\nu}$ is traceless, since this is so in the classical theory.

There is one small imprecision in the last paragraph. In the free theory on flat space, though there are no interaction vertices, one can draw a disconnected loop diagram on $\mathbf{R}^4$ with a single insertion of T (and no other vertices or insertions) representing a contribution to the vacuum expectation value of T. It is quartically divergent, so one can worry about it. However, by Poincaré invariance, its value, with any regularization, is a constant multiple of $g_{\mu\nu}$ (the metric tensor of Minkowski space). We simply subtract this multiple from T, and thereby obtain a new stress tensor, also conserved but now traceless.

We also, of course, could consider matrix elements of products of local operators, for example $\langle T_{\mu\nu}(x)T_{\alpha\beta}(y)\rangle$ if we want to. Then we will run into more complicated diagrams. However, we don't need to consider these questions to decide if T is traceless. For this it suffices to verify that all matrix elements of T among Fock space states, which give a basis of the Hilbert space, are traceless.

Now I get to a point where my answer to one of the questions was misleading. What happens if we formulate the theory on a curved four-manifold M? We still only get tree diagrams if we consider matrix elements of T among states with initial and final photons (a question that would make sense if M has Lorentz signature and suitable asymptotic behavior). However, now, when we compute the one-loop diagram for the expectation value of T, it need not just be a constant multiple of $g_{\mu\nu}$ but can be more complicated. Hence, there is no simple argument against a possible trace anomaly, and as shown by Duff et. al. in the 1970's, there actually is a trace anomaly. The trace of the stress tensor is a multiple of the polynomial in the Riemann tensor whose integral is the Euler characteristic.

It is still true that because the theory is free, there are no higher loop diagrams that might contribute to the trace of the stress tensor.

Since this trace of the stress tensor comes from a one-loop diagram, it is independent of the coupling constant, that is independent of $\mathrm{Im}\,\tau$. (Of course, it is even more trivially independent of $\mathrm{Re}\,\tau$, the theta angle.) So when we are discussing the τ dependence, we can assume conformal invariance. So in particular, when we were trying to determine the power of $\mathrm{Im}\,\tau$, this power is conformally invariant. Any ambiguity (i.e. difference in the results obtained with different regularizations) is the integral of a local density constructed from the metric, as I explained in the lecture, and also must be conformally invariant. So the ambiguity in the power of $\mathrm{Im}\,\tau$ is of the general form $(\mathrm{Im}\,\tau)^{a\chi+b\sigma}$, where χ and σ are the Euler characteristic and the signature, and a and b are constants. Moreover, if our regularization preserves parity, then b must vanish, so the ambiguity is just $(\mathrm{Im}\,\tau)^{a\chi}$ for some constant χ.

Just for fun, what happens if we consider $\mathcal{N}=4$ super Yang-Mills theory with non-abelian gauge group G, instead of the free $U(1)$ theory of the lecture? It is still true, but a lot less trivial, that the theory on flat $\mathbf{R}^4$ is conformally invariant. When we formulate it on a general four-manifold, we still get a one-loop trace anomaly, proportional again to the density whose integral is χ. Since the $\mathcal{N}=4$ theory is non-free, there appear to be opportunities for all kinds of higher contributions to the trace anomaly, but I believe that in fact (because of supersymmetry and holomorphy) there are no higher corrections. Hence, just as in the abelian theory, the trace anomaly is independent of τ.

Hence, when we discuss the τ-dependence of the theory on a curved manifold, we can assume conformal invariance (up to a factor independent of τ). However, as the theory is non-free, its τ-dependence is more complicated; there are contributions in all orders in $1/\mathrm{Im}\,\tau$ (unless we specialize, for example, by a topological twist, to a situation in which they are absent), and there are even instanton contributions that depend on $\mathrm{Re}\,\tau$. Two different regularizations of $\mathcal{N}=4$ super-Yang-Mills theory will in general, give partition functions that differ by a factor $\exp(a(\tau,\bar{\tau})\chi+b(\tau,\bar{\tau})\sigma)$, where a and b are functions (a is even under parity and b is odd). S-duality says that the partition function of $\mathcal{N}=4$ super Yang-Mills on a four-manifold has modular properties, but the details depend on the regularization, because of the appearance of the a and b functions.