SM (Collider) HW I

1. Our discussion of the strong interactions at high energies begins with the quark/parton model. In this approximation we treat hadrons as composite states of quarks, but we ignore the details of how they are confined to the hadrons. We need only know that we can describe hadrons in terms of their quark (and eventually gluon) content and that quarks produced in short distance, large momentum transfer interactions will eventually evolve into hadrons with probability 1 (and do it on a distance scale small compared to the size of detectors). We also assume that these short distance interactions of quarks (and leptons) are accurately described by the Lagrangians of $SU(3)_{QCD}$ x $SU(2)_L$ x $U(1)_Y$. One of the simplest applications is to the annihilation process of electrons and positrons at high energy. As an introduction we want to calculate the total cross section for $e^+e^- \to$ hadrons in the naive quark/parton model where the $e^+e^- \to q\bar{q}$ cross section (for a specific quark antiquark pair) is (assumed to be) identical to that for $e^+e^- \to \mu^+\mu^-$ except for the change in the electric charge. Evaluate your result at Q = 2, 7 and 15 GeV where Q is the total e^+e^- center of mass energy. Compare your results to data noted below (see also the PDG Reviews) and comment on how it works. Compare both for the naive parton model and including color. [HINT: Remember to sum over all appropriate quantum numbers.]

Note: The size of these cross section is given by

$$\frac{4\pi\alpha^2}{3\left(1\ \text{GeV}\right)^2} \simeq 0.87\times 10^{-4}\ \text{mb.}$$

A typical way to present the data on electron-positron annihilation to hadrons is in terms of the ratio

$$R \equiv \frac{\sigma\left(e^+e^- \to \text{hadrons}\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)} \simeq \sum_q e_q^2,$$

which scales out the bulk of the variation with Q^2.

The data suggest

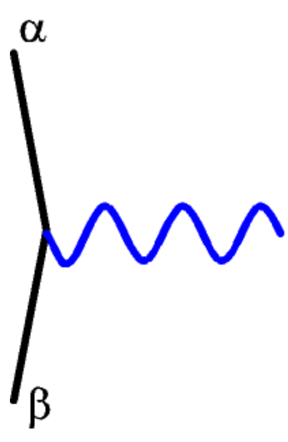
Energy (GeV)	R_{data}
2	~ 1.6
7	~ 3.3
15	~ 3.7

As a more challenging exercise try one of the following two.

2. As a first exercise let us recall how to find the cross section for (LO) elastic $e\mu$ scattering. The Feynman rules (in my notation) for the relevant propagators and vertices are.

Fermion $$\frac{\mathbf{i}}{(\gamma^\mu \mathbf{q}_\mu - \mathbf{m})} = \frac{\mathbf{i}(\gamma^\mu \mathbf{q}_\mu + \mathbf{m})}{\mathbf{q}^2 - \mathbf{m}^2}$$

Photon  $$\frac{-\mathbf{i}\mathbf{g}^{\mu\nu}}{\mathbf{q}^2}$$



$-\mathbf{ie}\gamma^\mu{}_{\alpha\beta}$

a) With this starting point evaluated the appropriate spin averaged matrix element squared, $\left|\bar{\mathcal{M}}_{e^-\mu^- \to e^-\mu^-}\right|^2$, in terms of invariant quantities using the properties of the spinors and standard trace identities, *e.g.*,

$$\left(\not{p}-m\right)u\left(s,p\right)=0:\bar{u}\left(s,p\right)\left(\not{p}-m\right)=0 \text{ (fermions)},$$
$$\left(\not{p}+m\right)\mathrm{v}\left(s,p\right)=0:\bar{\mathrm{v}}\left(s,p\right)\left(\not{p}+m\right)=0 \text{ (antifermions)},$$
$$\sum_s u\left(s,p\right)\bar{u}\left(s,p\right)=\not{p}+m,\ \sum_s \mathrm{v}\left(s,p\right)\bar{\mathrm{v}}\left(s,p\right)=\not{p}-m,$$

yields

$$L_{\mu\nu}\equiv\frac{1}{2}\sum_{\mathrm{s_a},s_b}\bar{u}\left(s_b,p_b\right)\gamma_\mu u\left(s_a,p_a\right)\bar{u}\left(s_a,p_a\right)\gamma_\nu u\left(s_b,p_b\right)$$
$$=\frac{1}{2}\mathrm{Tr}\left[\left(\not{p}_b+m\right)\gamma_\mu\left(\not{p}_a+m\right)\gamma_\nu\right].$$
$$=2\left[p_{a\mu}p_{b\nu}+p_{a\nu}p_{b\mu}-g_{\mu\nu}\left(p_a\cdot p_b-m^2\right)\right],$$
$$\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right]=4\left[g^{\alpha\beta}g^{\mu\nu}-g^{\alpha\mu}g^{\beta\nu}+g^{\alpha\nu}g^{\beta\mu}\right]$$
$$\mathrm{Tr}\left[\gamma^\alpha\gamma^\beta\right]=4\left[g^{\alpha\beta}\right].$$

b) Find the corresponding invariant cross section, differential in the momentum transfer t, and the angular differential cross section evaluated in the "laboratory" frame (the rest frame of the muon) (and compare to the expressions in the lecture for electron-proton).

$$\frac{d\sigma}{dt}=\frac{\left|\bar{\mathcal{M}}_{e^-\mu^-\to e^-\mu^-}\right|^2}{64\pi s p_{CM}^2}=\frac{\left|\bar{\mathcal{M}}_{e^-\mu^-\to e^-\mu^-}\right|^2}{16\pi\left(s-\left(m_e+m_\mu\right)^2\right)\left(s-\left(m_e-m_\mu\right)^2\right)}$$
$$\xrightarrow[s\gg m^2]{}\frac{\left|\bar{\mathcal{M}}_{e^-\mu^-\to e^-\mu^-}\right|^2}{16\pi s^2}$$

$$\frac{dt}{d\cos\theta_L}=\frac{2E^2}{\left(1+\frac{E}{m_\mu}\left(1-\cos\theta_L\right)\right)^2}=2E'^2,$$

$$\frac{d\sigma}{d\Omega_L}=\frac{\left|\bar{\mathcal{M}}_{e^-\mu^-\to e^-\mu^-}\right|^2}{64\pi^2 m_\mu^2}\left(\frac{E'}{E}\right)^2=\frac{\left|\bar{\mathcal{M}}_{e^-\mu^-\to e^-\mu^-}\right|^2}{64\pi^2 m_\mu^2}\left(\frac{1}{1+2\frac{E}{m_\mu}\sin^2\theta_L/2}\right)^2.$$

3. Now we want to consider the elastic and inelastic scattering of an electron from a proton. Unlike the electron, we know that the proton is not an elementary particle. It has internal structure (the quarks and gluons) and the coupling to the photon will be more complicated than the simple electron vertex of the Feynman rules. However, we can use general Lorentz and symmetry considerations and the fact that the proton wave function must satisfy the Dirac equation to write the *elastic* coupling in terms of just 3 functions (form factors).

a) Use general Lorentz considerations and the Dirac equation to verify that the most general parity-conserving form of the electromagnetic current of the proton is

$$J^\mu \sim \bar{u}(p')\left[\Gamma_1(q^2)\gamma^\mu+\Gamma_2(q^2)i\sigma^{\mu\nu}q_\nu+\Gamma_3(q^2)q^\mu\right]u(p),$$

where $q^\mu = p'^\mu - p^\mu$ is the momentum of the photon ($q^2 \neq 0$, in general) and, as usual, $\sigma^{\mu\nu}=\frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]$.

b) What are the consequences of current conservation, $\partial^\mu J_\mu = 0$?

Now we want to consider the general form of *inelastic* electron-proton scattering, e p $\to$ e X, where a sum and integration over all possible physical states in X is implied. The connection to our previous study of the exclusive amplitude comes from the optical theorem, which relates the square of an amplitude for an inclusive process (e.g., a + b $\to$ X), appropriately summed and integrated over the inclusive final state

(X), and the imaginary part of the corresponding forward elastic amplitude (e.g., a + b $\rightarrow$ a + b). We should think of $W_{\mu\nu}$ (defined below) as the imaginary part of the forward scattering amplitude of a (virtual) photon and a proton.

The general the spin averaged matrix element squared can be written in the form (assuming that only photon exchange contributes)

$$\left|\bar{\mathcal{M}}\right|^2 = \frac{e^4}{\left(q^2\right)^2} L^{\mu\nu} W_{\mu\nu}$$

where q^μ is the photon 4-momentum. This definition arose from the nonrelativistic limit for the proton (appropriate for earlier e p scattering experiments) and a factor of m_p (the proton mass) was pulled out of the $W_{\mu\nu}$ compared to the usual relativistic expectations, i.e., the dimension of $W_{\mu\nu}$ is 1/GeV. The tensor $L_{\mu\nu}$ describes the coupling to the electron and is given as an example in part a) above. With these definitions, the total (virtual) photon – proton cross section (in the lab frame) looks like

$$\sigma_\lambda\left(\gamma^* p\right) = \frac{4\pi^2\alpha}{q_{\text{lab}}} \varepsilon^\mu\left(\lambda\right)\varepsilon^{*\nu}\left(\lambda\right) W_{\mu\nu},$$

where the ε's are the photon polarizations (helicity λ). Note the flux factor lacks the expected factor of m_p (downstairs) due to the choice in the normalization of *W*. The corresponding definition of the inclusive e p scattering cross section (*i.e.*, put back in the electron – photon vertex in $L_{\mu\nu}$ and the photon propagators) is

$$d\sigma = \frac{\pi}{E}\left|\bar{\mathcal{M}}\right|^2 \frac{d^3 p'}{\left(2\pi\right)^3 2E'} = \frac{\pi}{E}\frac{e^4}{Q^4} L^{\mu\nu} W_{\mu\nu} \frac{d^3 p'}{\left(2\pi\right)^3 2E'}.$$

c) Use general considerations of Lorentz, parity and current conservation issues (again) to verify that the hadronic part can be written in terms of just *two* invariant functions, W_1 and W_2, of the invariants q^2 and $\nu = q \cdot p / m$. Here p^μ is the proton 4-momentum and *m* its mass. The following form defines the two functions. You will also need to know something about the symmetry properties of $L^{\mu\nu}$.

$$W_{\mu\nu} = -W_1\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{W_2}{m^2}\left(p_\mu - q\cdot p\frac{q_\mu}{q^2}\right)\left(p_\nu - q\cdot p\frac{q_\nu}{q^2}\right).$$

d) Verify that the resulting cross section has the following general form, both in invariant notation and in the laboratory frame (*i.e.*, rest frame of the proton). We define θ_L to be the laboratory angle of the scattered electron, E' to be its energy, E to be the incident electron energy and $Q^2 = -q^2$ (ignoring lepton masses).

$$\frac{d\sigma}{dE'd\Omega_{Lab}} = \frac{4\alpha^2 E'^2}{Q^4}\left[2W_1\left(Q^2,\nu\right)\sin^2\left(\frac{\theta_L}{2}\right) + W_2\left(Q^2,\nu\right)\cos^2\left(\frac{\theta_L}{2}\right)\right],$$

$$\frac{d\sigma}{dQ^2 d\nu} = \frac{4\pi\alpha^2}{Q^4}\frac{E'}{E}\left[2W_1\left(Q^2,\nu\right)\sin^2\left(\frac{\theta_L}{2}\right) + W_2\left(Q^2,\nu\right)\cos^2\left(\frac{\theta_L}{2}\right)\right].$$

4. We want to make use of the 4-D cross section for $e^+e^- \to q\bar{q}g$ presented in the Lecture

$$\frac{d\sigma}{dx_1 dx_2} = \sigma_0 \frac{\alpha_s}{2\pi} C_F \frac{x_1^2 + x_2^2}{\left(1-x_1\right)\left(1-x_2\right)},$$

where we have defined the dimensionless variables

$$x_i = \frac{2E_i}{\sqrt{q^2}} = \frac{2p_i\cdot q}{q^2} \Rightarrow x_i \geq 0 \Rightarrow \sum_{i=1}^{3} x_i = 2,$$

$$1-\cos\theta_{ij} = \frac{p_i\cdot p_j}{E_i E_j} = \frac{\left(q-p_k\right)^2}{2E_i E_j} = \frac{2\left(1-x_k\right)}{x_i x_j}$$

with i = 1 = quark, i = 2 = anti-quark, i= 3 = gluon. The lowest order cross section (with just a quark and anti-quark in the final state is given by

$$\sigma_0 = \frac{4\pi\alpha^2}{3s} \cdot \sum_{\text{color}} \cdot \sum_f e_f^2 = \frac{4\pi\alpha^2}{3s} \cdot 3\sum_f e_f^2 .$$

If you have time, you should verify, by explicit calculation, these cross sections using the Feynman rules noted in the lectures.

a) As suggested in the lecture, it is straightforward to evaluate the Thrust distribution at this order in perturbation theory (from real emission). Recall that the quantity Thrust is defined by

$$T_3\left(p_q, p_{\bar{q}}, p_g\right) \equiv \frac{\max_{\hat{u}} \sum_{i=1,3} \left|\vec{p}_i \cdot \hat{u}\right|}{\sum_{i=1,3} \left|\vec{p}_i\right|},$$

with the NLO (next-to-leading order) Thrust distribution defined by

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = \frac{1}{\sigma_0} \iint dx_1 dx_2 \frac{d\sigma}{dx_1 dx_2} \delta\left(T - T_3\left(x_i\right)\right).$$

The LO and virtual results contribute only at $T = 1$. Here we consider only the 3-body contribution. Use the definition of Thrust to verify that for any allowed configuration of the quark, anti-quark and gluon, the value of the thrust corresponds to the maximum x value,

$$T_3\left(p_q, p_{\bar{q}}, p_g\right) = \max\left[x_1, x_2, x_3\right],$$

and that

$$\frac{2}{3} \le T_3 \le 1.$$

HINT: Recall that $\sum_i \vec{p}_i = 0$.

b) Next focus on the configurations where the quark has the highest energy, $x_1 > x_2, x_3$. Verify that this region of phase space corresponds to

$$T = x_1 \geq x_2 \geq 2(1-T),$$

and makes the following contribution to the Thrust distribution,

$$\left.\frac{1}{\sigma}\frac{d\sigma}{dT}\right|_{x_1>x_2,x_3} = \frac{\alpha_s C_F}{2\pi}\frac{1}{1-T}\left\{\left(T^2+1\right)\ln\left(\frac{2T-1}{1-T}\right)+\left(\frac{3}{2}T^2-7T+4\right)\right\}.$$

c) Next consider the contribution from the configurations where the gluon is the most energetic parton,

$$T = x_3 = 2 - x_1 - x_2 \geq x_1, x_2.$$

Verify that this region of phase space corresponds to

$$x_1 = 2 - T - x_2, T \geq x_2 \geq 2(1-T),$$

and makes the following contribution to the Thrust distribution,

$$\left.\frac{1}{\sigma}\frac{d\sigma}{dT}\right|_{x_3>x_1,x_2} = \frac{\alpha_s C_F}{2\pi}\frac{2}{T}\left\{\left(T^2-2T+2\right)\ln\left(\frac{2T-1}{1-T}\right)+T\left(2-3T\right)\right\}.$$

d) Pull the 3 pieces (x_1, x_2 or x_3 as maximum) together to verify that the complete order α_s Thrust (away from 1) distribution is given by

$$\frac{1}{\sigma}\frac{d\sigma}{dT} = \frac{\alpha_s C_F}{2\pi}\left\{\frac{6T^2-6T+4}{T(1-T)}\ln\left(\frac{2T-1}{1-T}\right)-\frac{3(2-T)(3T-2)}{1-T}\right\}.$$

Note that this expression verifies the result stated in class for the leading behavior in the limit $T \to 1$,

$$\frac{1}{\sigma}\frac{d\sigma}{dT}\xrightarrow[T\to 1]{}\frac{\alpha_s C_F}{2\pi}\left\{\frac{4}{(1-T)}\ln\left(\frac{1}{1-T}\right)\right\}.$$

5. Now we want to try one calculation in 4-2ε dimensions. Part of the challenge is to calculate the matrix element in the continued dimensions and to work out the changes in phase space. Here we will just accept the integral noted in the lecture and simply verify that the resulting contribution to the cross section is finite (for $\varepsilon < 0$). In fact, we will focus on just the terms that are singular as $\varepsilon \to 0$, although you are encouraged to think about obtaining the finite bits. We are told that

$$\sigma^{q\bar{q}g}(\varepsilon)=\sigma_0 H(\varepsilon)\int dx_1 dx_2\,\frac{C_F\alpha_s(\mu)}{2\pi}\times\left[\frac{(1-\varepsilon)\left(x_1^2+x_2^2\right)+2\varepsilon\left(x_1+x_2-1\right)}{(1-x_1)^{1+\varepsilon}(1-x_2)^{1+\varepsilon}\left(x_1+x_2-1\right)^{\varepsilon}}-2\varepsilon\right],$$

where the function H(ε) expresses the ε dependence of the overall factor – the Born cross section in 4-2ε dimensions. By explicit calculation verify that

$$\sigma^{q\bar{q}g}(\varepsilon)=\sigma_0\frac{C_F\alpha_s(\mu)}{2\pi}H(\varepsilon)\left[\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\mathcal{O}\left(\varepsilon^0\right)\right].$$

HINTS: Knowledge of the (old) Beta function,

$$\int_0^1 dx\,x^{\alpha-1}(1-x)^{\beta-1}=B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

and the associated Gamma function is (very) useful.