Homework Problems

Problem 6.1

Consider the type IIA and type IIB superstring theories compactified on a circle so that the space-time is $M_{10} = \mathbb{R}^{8,1} \times S^1$, where $\mathbb{R}^{8,1}$ denotes nine-dimensional Minkowski space-time. Show that the spectrum of the type IIA theory for radius R agrees with the spectrum of the type IIB theory for radius $\widetilde{R} = \alpha'/R$.

Problem 6.6

Show that the Born–Infeld action (6.100) gives a finite classical self-energy for a charged point particle. Hint: show that the solution to the equations of motion with a point particle of charge e at the origin is given by

$$E_r = F_{rt} = \frac{e}{\sqrt{(r^4 + r_0^4)}}, \qquad r_0^2 = 2\pi\alpha' e.$$

Problem 6.13

Consider the static-gauge DBI action for a Dp-brane given in Eq. (6.126) that was discussed in Exercise 6.10.

(i) Find the action for a D3-brane in spherical coordinates (t, r, θ, ϕ) for the special case in which the only nonzero fields are $A_t(r)$ and one scalar $\Phi(r)$.

(ii) Obtain the equations of motion for $A_t(r)$ and $\Phi(r)$.

(iii) Find a solution of the equations of motion that corresponds to an electric charge at the origin, and deduce the profile of the string that is attached to the D3-brane. For what range of r are the DBI approximations justified?

Problem 6.15

Compute the minimum of the potential function in Eq. (6.141) when the N-dimensional representation of $SU(2)$ is irreducible. What is the minimum of the potential if the N-dimensional representation of $SU(2)$ is the sum of two irreducible representations? How does it compare to the previous result? Describe the fuzzy sphere configuration in this case.

PROBLEM 7.4

Generalize the analysis of Exercise 7.6 to the heterotic string. In particular, verify that the Wilson lines, together with the B and G fields, have the right number of parameters to describe the moduli space $\mathcal{M}^0_{16+n,n}$ in Eq. (7.121).

PROBLEM 7.9

Consider the Euclideanized world-sheet theory for a string coordinate X

$$S[X] = \frac{1}{\pi}\int_M \partial X \bar{\partial} X d^2 z.$$

Suppose that X is circular, so that $X \sim X + 2\pi R$ and that the world sheet M is a torus so that $z \sim z + 1 \sim z + \tau$. Define winding numbers W_1 and W_2 by

$$X(z+1, \bar{z}+1) = X(z, \bar{z}) + 2\pi R W_1,$$

$$X(z+\tau, \bar{z}+\bar{\tau}) = X(z, \bar{z}) + 2\pi R W_2.$$

(i) Find the classical solution $X_{\rm cl}$ with these winding numbers.

(ii) Evaluate the action $S_{\rm cl}(W_1, W_2) = S[X_{\rm cl}]$.

(iii) Recast the classical partition function

$$Z_{\rm cl} = \sum_{W_1, W_2} e^{-S_{\rm cl}(W_1, W_2)}$$

by performing a Poisson resummation. Is the result consistent with T-duality?

PROBLEM 7.10

Consider a Euclidean lattice generated by basis vectors e_i, $i = 1, \ldots, 8$, whose inner products $C_{ij} = e_i \cdot e_j$ are described by the following metric:

$$C = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

This is the Cartan matrix for the Lie group E_8.

(i) Find a set of basis vectors that gives this metric.

(ii) Prove that the lattice is even and self-dual. It is the E_8 lattice.

Problem **7.15**

(i) Compactifying the $E_8 \times E_8$ heterotic string on a six-torus to four dimensions leads to a theory with $\mathcal{N} = 4$ supersymmetry in four dimensions. Verify this statement and assemble the resulting massless spectrum into four-dimensional supermultiplets.

(ii) Repeat the analysis for the type IIA or type IIB superstring. What is the amount of supersymmetry in four dimensions in this case? What is the massless supermultiplet structure in this case?

Problem **8.1**

Derive the bosonic equations of motion of 11-dimensional supergravity.

Problem **8.2**

Show that a particular solution of the bosonic equations of motion of 11-dimensional supergravity, called the Freund–Rubin solution, is given by a product space-time geometry $AdS_4 \times S^7$ with

$$F_4 = M\varepsilon_4,$$

where ε_4 is the volume form on AdS_4, and M is a free parameter with the dimensions of mass.[1] AdS_4 denotes four-dimensional anti-de Sitter space, which is a maximally symmetric space of negative curvature, with Ricci tensor

$$R_{\mu\nu} = -(M_4)^2 g_{\mu\nu} \qquad \mu, \nu = 0, 1, 2, 3.$$

The seven-sphere has Ricci tensor

$$R_{ij} = (M_7)^2 g_{ij} \qquad i, j = 4, 5, \ldots, 10.$$

What are the masses M_4 and M_7 in terms of the mass parameter M?

[1] Actually, in the quantum theory it has to be an integer multiple of a basic unit.

Problem 8.6

Consider the type IIB bosonic supergravity action in ten dimensions given in Eq. (8.53). Setting $C_0 = 0$, perform the transformations $\Phi \to -\Phi$ and $g_{\mu\nu} \to e^{-\Phi} g_{\mu\nu}$. What theory do you obtain, and what does the result imply?

Problem 8.7

Verify that the actions in Eqs (8.73) and (8.81) map into one another under the transformations (8.88) and (8.89).