Supersymmetric Grand Unification: Homework Problems

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1 Minor Issues.

1.1 Normalization.

SU(N) generators are defined such that

$$\text{Tr}\ (T_A T_B) = \frac{1}{2}\delta_{AB}. \tag{1}$$

We'll always normalize the gauge kinetic term such that we're tracing over SU(N) generators when we're talking about non-Abelian symmetries. So, for example,

$$\mathcal{S} = -\frac{1}{2g^2}\int d^4x \text{Tr}\ F^2. \tag{2}$$

1.2 Notation.

Small Greek indices starting with μ will denote an index in four dimensional space-time, while big Latin indices starting with M denote five dimensional space-time.

1.3 References.

For convenience, all references will also include the search string from SPIRES, to make things easier to find. For most group theory applications, Slansky's textbook[1] is a must have.

2 Lecture 1: SU(5) basics.

2.1 SU(5) Representations.

In the lecture, it was argued that a complete Standard Model generation can be fit in to the $\bar{\mathbf{5}} + \mathbf{10}(+1)$ representations of SU(5). In general, an element of the group SU(5) can be written as

$$U = e^{i\epsilon^A T_A} = \mathbb{1} + i\epsilon^A T_A + \mathcal{O}(\epsilon^2). \tag{3}$$

The fundamental representation of SU(5) transforms as follows:

$$\mathbf{5}'^{\alpha} = U^{\alpha}{}_{\beta}\mathbf{5}^{\beta} = \left\{\delta^{\alpha}{}_{\beta} + i\epsilon^A (T_A)^{\alpha}{}_{\beta} + \mathcal{O}(\epsilon^2)\right\}\mathbf{5}^{\beta}. \tag{4}$$

Using this, show how the two index anti-symmetric tensor ($\mathbf{10}^{\alpha\beta}$) transforms under infinitesimal gauge transformations. Also, show how the $\mathbf{5}^*(\equiv \bar{\mathbf{5}})$ (four index, totally anti-symmetric tensor) transforms.

[1] R. Slansky, Physics Reports **79**, 1-128 (1981). Find on SPIRES by with "FIND J PRPLC,79,1".

2.2 Proton Decay at Dimension 6.

In this exercise, we'll derive the current that couples to the **X** gauge bosons (transforming as $(\bar{3}, 2, +5/3)$ under the SM), which mediate proton decay at dimension 6. Qualitatively this is exactly the same as the Fermi theory. Starting with the terms in the Lagrangian which involve **X** gauge bosons, derive the current which is responsible for proton decay (at dimension 6). Integrate out the **X** bosons to get an effective (dimension 6) operator.

2.3 Higgsing SU(5).

In order to break SU(5) to the MSSM, we need to introduce some scalar GUT Higgs multiplets, which must transform in the adjoint (**24**) representation, and a scalar potential. (Why won't smaller representations work for this job?) Given that the Higgs scalar Σ has the following covariant derivative:

$$D_\mu \Sigma = \partial_\mu \Sigma + ig \left[V_\mu^A T_A, \Sigma \right], \tag{5}$$

show that the **X** gauge bosons obtain mass

$$m_{\mathbf{X}}^2 = \frac{25}{4} g^2 V^2. \tag{6}$$

Hint: Take $\langle \Sigma \rangle = V \ \mathrm{diag}(1, 1, 1, -3/2, -3/2)$, where $V \sim M_G$.

2.4 Flipped SU(5)

It turns out that there is *another* way to get an SU(5) theory out of SO(10), called flipped SU(5).[2] In this model, we take

$$\bar{\mathbf{5}} = \begin{pmatrix} u^c \\ u^c \\ u^c \\ -e^- \\ \nu \end{pmatrix}, \quad 1 = e^+. \tag{7}$$

The gauge group must be larger than SU(5), it should be $\mathrm{SU}(5) \times \mathrm{U}(1)_X$—why? Given the following X quantum numbers of the $\bar{\mathbf{5}}$ ($X = -3$) and **10** ($X = 1$), how do the fermions fit into the **10**? If $\tilde{Y}$ is the $U(1) \in SU(5)$, find the SM hypercharge Y in terms of $\tilde{Y}$ and X. How could symmetry breaking work in this model?

3 Lecture 2: Orbifold GUTs.

In the following exercises, we'll focus on the example of a 5-d orbifold GUT living on $S^1/\mathbb{Z}_2 \times \mathbb{Z}_2$. The theory that we will work with is a U(1) gauge theory, whose photon lives in the bulk. Most of the physics can be understood in this example, and the extension to the non-Abelian case is straightforward.

[2] S. Barr, Physics Letters **B**112, 219 (1982). FIND J PHLTA,B112,219

3.1 Kaluza-Klein Decomposition.

This is really just a fancy way to say "separation of variables", but to ensure that we get the proper expansions, we have to examine the way that the parity operations act on the various components of the gauge bosons. Recall that there are two operations which define our orbifold: a parity $\mathcal{P} : y \to -y$ and a translation $\mathcal{T} : y \to y + 2\pi R$. Given these two operations, we define P and T (the representation of the two operations on the fields) with $P' \equiv PT$. We demand that the action be invariant under $\mathcal{P}$ and $\mathcal{P}'$.

First, start with the action for a U(1) gauge theory in five dimensions:

$$\mathcal{S} = \frac{-1}{4g^2} \int d^5x F_{MN} F^{MN}. \tag{8}$$

Now decompose this action into it's pieces:

$$F_{MN} F^{MN} = F_{\mu\nu} F^{\mu\nu} + \dots \tag{9}$$

Using the transformation properties of ∂_5, and the fact that the action is invariant under $\mathcal{P}$ and $\mathcal{P}'$, deduce the transformation properties of A_5. Is there a requirement on the transformation properties of A_μ?

Take the following ansatz for the mode decomposition:

$$A_\mu(x, y) = \sum_{n=0}^{\infty} A_\mu^n(x) a_n(y), \tag{10}$$

and

$$A_5(x, y) = \sum_{n=0}^{\infty} A_\mu^n(x) b_n(y). \tag{11}$$

Using the various boundary conditions that we just derived, derive expressions for $a(y)$ and $b(y)$ in all of the relevant cases.

3.2 Massive Kaluza-Klein Gauge Bosons.

In this exercise, we want to evaluate the Lagrangian for the massive Kaluza-Klein gauge bosons. As you might expect, there is STILL no gauge invariant way to write down a mass term (even in five dimensions) for a gauge boson. But in order to get realistic phenomenology, the Kaluza-Klein modes *must* decouple from the spectrum somehow.

We just derived the exact form of A_μ and A_5. Now let $A_\mu(x, y)$ have $(++)$ boundary conditions, and take the appropriate boundary conditions for $A_5(x, y)$), and plug it into the gauge kinetic term in Equation (8). Find the low energy effective action by integrating over the fifth direction.

Find the mass of the KK modes. Where does the third helicity mode of the massive KK gauge bosons come from? Finally, put the theory in the unitary gauge, containing only physical degrees of freedom, i.e. show that the scalar degrees of freedom A_5 decouple with the appropriate choice of gauge.

3.3 A New Contribution to the Beta Functions

Between the compactification scale and the cutoff (where the UV completion takes over), new states enter the theory in loops. Consider the case of our simple U(1) theory[3], in which the photon couples to a tower of KK fermions. For simplicity, we'll just compactify on a circle—this means that the masses of the KK modes are given by $m_n = n/R$.

Show that the vacuum polarization is

$$\Pi_{\mu\nu}(p^2) = \sum_{n=0}^{\infty} -e^2 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr} \left\{ \frac{\gamma_\mu \left(k\cdot\gamma + m_n\right)\gamma_\nu \left((k+p)\cdot\gamma + m_n\right)}{\left(k^2 - m_n^2\right)\left((k+p)^2 - m_n^2\right)} \right\}. \tag{12}$$

Now use Ward identities and a Feynman x to get (in Euclidean space)

$$\Pi(p^2) = \frac{-8e^2}{3p^2} \sum_{n=0}^{\infty} \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^2 - x(1-x)p^2 + 2m_n^2}{\left[\ell^2 + p^2 x(1-x) + m_n^2\right]^2}. \tag{13}$$

Introduce a Schwinger parameter

$$\frac{1}{x^2} = \int_0^\infty dt\, t e^{-xt}, \tag{14}$$

and show that after some integrations, show that we're left with

$$\Pi(p^2) = \frac{e^2}{2\pi^2} \sum_{n=0}^{\infty} \int_0^1 dx\, x(1-x) \int_0^\infty \frac{dt}{t} e^{-t\left\{p^2 x(1-x) + m_n^2\right\}}, \tag{15}$$

$$\Rightarrow \Pi(0) = \frac{e^2}{12\pi^2} \sum_{n=0}^{\infty} \int_0^\infty \frac{dt}{t} e^{-tm_n^2}. \tag{16}$$

Given the definition of the Jacobi theta function

$$\theta_3(t) \equiv \sum_n e^{i\pi n^2 t}, \tag{17}$$

show that Equation (15) can be written as

$$\Pi(0) = \frac{e^2}{12\pi^2} \int_{M_{\mathrm{S}}^{-2}}^{M_{\mathrm{C}}^{-2}} \frac{dt}{t} \theta_3\left(\frac{it}{\pi R^2}\right), \tag{18}$$

where M_{S} is the UV cutoff. Note that the theta functions can be approximated in the limit where $t/R^2 << 1$:

$$\theta_3\left(\frac{it}{\pi R^2}\right) \cong \sqrt{\frac{\pi}{t}} R. \tag{19}$$

Use this to show that the beta functions get a power-law correction $\sim \frac{M_{\mathrm{S}}}{M_{\mathrm{C}}}$.

[3]See the appendices in K. Dienes, E. Dudas, and T. Gherghetta, Nuclear Physics **B**537, 47-108 (1999). FIND EPRINT HEP-PH/9806292

4 Lecture 3: Stringy Orbifold GUTs

4.1 A Terrible Prediction for Newton's Constant.

It is well known that grand unification doesn't seem to work in the simplest cases with the heterotic string—specifically, if we want unification, then the low energy data seems to be inconsistent with the relationship coming from the (weakly coupled) heterotic string. To derive this relationship[4], start with the (ten dimensional) effective action:

$$\mathcal{S} = -\int d^{10}x e^{-2\phi} \left\{ \frac{4}{\alpha'^4}\mathcal{R} + \frac{1}{\alpha'^3}\mathrm{Tr}\ F^2 + \dots \right\}. \tag{20}$$

Now compactify the theory on some six dimensional manifold (or orbifold) with volume V_6. Show that the relationship (up to threshold corrections) is given by

$$\frac{1}{8}\alpha' \alpha_{\text{GUT}} = G_N. \tag{21}$$

If we accept the typical running of the coupling constants (that is, no new states contribute to the beta functions), show that the value of the Planck mass is wrong by two-three orders of magnitude (depending on how you define α' in terms of M_{s}).

4.2 A 6d Orbifold GUT.

In this exercise, we want to calculate the massless spectrum in the *untwisted sector* of the 5d (or 6d) orbifold GUT described in Lecture 3, and defined by the mass equation, $\mathbf{P}^2 = 2$, and GSO projection conditions, $\mathbf{P}\cdot\mathbf{V}_3 - \mathbf{r}_i \cdot \mathbf{v} = \mathbb{Z}$ $(i = 1,2,3,4)$, with $\mathbf{V}_3 = 2\mathbf{V}$ and $\mathbf{P}\cdot\mathbf{W}_3 = \mathbb{Z}$ (which defines the massless states in the U_1, U_2, U_3 and gauge sectors, respectively). Show that the six dimensional spectrum is that of an SU(6) orbifold GUT, with $\mathcal{N} = 2$ SUSY. The $\mathcal{N} = 2$ SUSY chiral/vector multiplets in four dimensions fit in to $\mathcal{N} = 1$ hyper/vector multiplets in six dimensions. For concreteness, a 6d gauge boson has 4 transverse degrees of freedom, two of which turn into one complex scalar upon dimensional reduction from six to four dimensions. Likewise, the hypermultiplets in six dimensions contain two four dimensional chiral multiplets.

To complete this exercise, you should:

1. calculate how the SO(8) vectors/spinors transform— remember you're in six dimensions. This will let you know how to make states.

2. calculate the 6d gauge group. Which E_8 weights survive the projection conditions? You should find a total of 35 SU(6) gauge bosons.

[4] A very good example of this calculation can be found in E. Witten, Nuclear Physics **B**471, 135-158 (1996). FIND EPRINT HEP-TH/9602070

3. Are there any matter reps living in the untwisted sector? Given the representations of SU(6) (see Slansky), can you make an educated guess as to how these states transform?

4. BONUS: Can you see why you can't get a representation LARGER than the adjoint (at least in the untwisted sector) in these orbifold constructions?

Hint: find the E_8 weights satisfying the mass and GSO projection conditions. First do this for the gauge sector and determine the gauge group by identifying the simple roots and their Cartan matrix. However it is usually easy enough to just identify the roots in the Cartan-Weyl basis. Once the gauge group is identified one can determine the representation content of the other states using the highest weight and a table of highest weights a la Dynkin, eg. see Slansky. However, again it is sometimes easier to just identify the weights in the Cartan-Weyl basis.

4.3 Local GUTs in Orbifold Compactifications

As you've heard in the previous lectures, it is possible for the GUT at an orbifold fixed point to be larger than the MSSM gauge group, yet smaller than E_8. This is called a local GUT, and is a common feature in model building.

In the class of string models discussed in the Lecture (defined by Equations (9) in lecture 3), show that the local GUT at the T_1 fixed points at $x_6 = 0$ contains $SO(10)$. Identify the roots of $SO(10)$ as a subset of the E_8 roots. In addition, show that there is at least one spinor rep localized at that position. Use the mass equations given in lecture 3, equations (1) and (2).