Supersymmetric Grand Unification: Solutions

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1 Lecture 1: SU(5) basics.

1.1 SU(5) Representations.

First, we calculate how the $\bar{\mathbf{5}}$ transforms. We could do this by taking the totally antisymmetric combination of four $\mathbf{5}$s, or we can simply observe that

$$(\mathbf{5}^\alpha)^* \equiv (\bar{\mathbf{5}})_\alpha . \tag{1}$$

Then

$$(\mathbf{5}')^* = (U\mathbf{5})^* = (U)^* (\bar{\mathbf{5}}) \tag{2}$$

Finally, note that

$$(U)^* = \left(\mathbb{1} + i\omega_A T^A + \cdots\right)^* = \mathbb{1} - i\omega_A \left(T^A\right)^T + \cdots . \tag{3}$$

So the generators of the anti-fundamental representation are $-\left(T^A\right)^T$.

The index structure of a 10 is given by

$$\mathbf{10}^{\alpha\beta} = \frac{1}{\sqrt{2}}\left[\mathbf{5}^\alpha\mathbf{5}'^\beta - \mathbf{5}^\beta\mathbf{5}'^\alpha\right]. \tag{4}$$

Now, given that the transformation law of the $\mathbf{5}$ is

$$\mathbf{5}'^\alpha = U^\alpha{}_\beta \mathbf{5}^\beta = \left\{\delta^\alpha{}_\beta + i\omega^A (T_A)^\alpha{}_\beta + \mathcal{O}(\omega^2)\right\}\mathbf{5}^\beta, \tag{5}$$

we find :

$$\begin{aligned}
\mathbf{10}^{\alpha\beta} \to \mathbf{10}'^{\alpha\beta} &= \left(\delta^\alpha{}_\gamma + i\omega^A (T_A)^\alpha{}_\gamma\right)\left(\delta^\beta{}_\delta + i\omega^A (T_A)^\beta{}_\delta\right)\mathbf{10}^{\gamma\delta}, \\
&= \left\{\delta^\alpha{}_\gamma\delta^\beta{}_\delta + i\omega^A\left[\delta^\alpha{}_\gamma (T_A)^\beta{}_\delta + \delta^\beta{}_\delta (T_A)^\alpha{}_\gamma\right]\right\}\mathbf{10}^{\gamma\delta}.
\end{aligned} \tag{6}$$

Therefore

$$\mathbf{10}'^{\alpha\beta} \equiv \left\{\delta^\alpha{}_\gamma\delta^\beta{}_\delta + i\omega^A T^{\alpha\beta}_{A\gamma\delta}\right\}\mathbf{10}^{\gamma\delta} \tag{7}$$

where

$$T^{\alpha\beta}_{A\gamma\sigma} \equiv \left[\delta^\alpha{}_\gamma (T_A)^\beta{}_\delta + \delta^\beta{}_\delta (T_A)^\alpha{}_\gamma\right] \tag{8}$$

are the generators acting on the $\mathbf{10}$.

1.2 Proton Decay at Dimension 6.

In order to get the terms in the Lagrangian which involve $\mathbf{X}$ and $\mathbf{Y}$ gauge bosons, we will use the decomposition of the relevant representations under the SM gauge group. So[1]

$$\begin{aligned}
\bar{\mathbf{5}} &\to (\bar{\mathbf{3}},1)_{-2/3} + (1,\mathbf{2})_{-1}, &(9)\\
\mathbf{10} &\to (\mathbf{3},\mathbf{2})_{1/3} + (\bar{\mathbf{3}},1)_{-4/3} + (1,1)_2, &(10)\\
\mathbf{24} &\to (\mathbf{8},1)_0 + (1,\mathbf{3})_0 + (1,1)_0 + (\mathbf{3},\mathbf{2})_{-5/3} + (\bar{\mathbf{3}},\mathbf{2})_{5/3}. &(11)
\end{aligned}$$

[1] If you're following along in Slansky's book, see page 96, Table 30.

Now using Young's Tableaux, or just looking up the answer in Slansky's Physics Reports article, the only gauge invariant couplings that involve the adjoint representation are

$$\mathcal{L} \supset \bar{\mathbf{5}}\ \mathbf{24}\ \mathbf{5} + \overline{\mathbf{10}}\ \mathbf{24}\ \mathbf{10} \tag{12}$$

Then we can decompose these terms into their SM representations, and pick off the interesting terms. From

$$\bar{\mathbf{5}}\ \mathbf{24}\ \mathbf{5} \to \left[(\bar{\mathbf{3}},1)_{2/3} + (1,\mathbf{2})_{-1}\right] \times \left[\cdots + (\mathbf{3},\bar{\mathbf{2}})_{-5/3} + (\bar{\mathbf{3}},\mathbf{2})_{5/3}\right] \times \left[(\mathbf{3},1)_{-2/3} + (1,\mathbf{2})_{1}\right] \tag{13}$$

we see that the only gauge invariant combination is

$$(\bar{\mathbf{3}},1)_{2/3} \times (\mathbf{3},\bar{\mathbf{2}})_{-5/3} \times (1,\mathbf{2})_{1} + \text{h.c.} \tag{14}$$

Likewise, the $\overline{\mathbf{10}}\ \mathbf{24}\ \mathbf{10}$ term gives us

$$(\bar{\mathbf{3}},1)_{-4/3} \times (\bar{\mathbf{3}},\mathbf{2})_{5/3} \times (\bar{\mathbf{3}},\mathbf{2})_{-1/3} + (\mathbf{3},\mathbf{2})_{1/3} \times (\bar{\mathbf{3}},\mathbf{2})_{5/3} \times (1,1)_{-2} + \text{h.c.} \tag{15}$$

The other way to see this is to write things out in terms of matrices. The fermion reps look like:

$$\bar{\mathbf{5}} = \begin{pmatrix} d_1^c \\ d_1^c \\ d_1^c \\ -e \\ \nu \end{pmatrix}, \quad \mathbf{10} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & u_1 & d_1 \\ -u_3^c & 0 & u_1^c & u_2 & d_2 \\ u_2^c & -u_1^c & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^c \\ -d_1 & -d_2 & -d_3 & -e^c & 0 \end{pmatrix} \tag{16}$$

The factor of $\frac{1}{\sqrt{2}}$ in front of the $\mathbf{10}$ gives us a canonically normalized kinetic term. The adjoint looks like

$$\mathbf{24} = \begin{pmatrix} & & & \mathbf{X}_1 & \mathbf{Y}_1 \\ & & & \mathbf{X}_2 & \mathbf{Y}_2 \\ & & & \mathbf{X}_3 & \mathbf{Y}_3 \\ \mathbf{X}_1^\dagger & \mathbf{X}_2^\dagger & \mathbf{X}_3^\dagger & & \\ \mathbf{Y}_1^\dagger & \mathbf{Y}_2^\dagger & \mathbf{Y}_3^\dagger & & \end{pmatrix}, \tag{17}$$

where we have only written down the $\mathbf{X}$ and $\mathbf{Y}$ part.

From Equation (14), we have

$$(\bar{\mathbf{3}},1)_{2/3} \times (\mathbf{3},\bar{\mathbf{2}})_{-5/3} \times (1,\mathbf{2})_{1} \sim -e^* \mathbf{X}_i^\mu d_i^c - \nu^* \mathbf{Y}_i^\mu d_i^c. \tag{18}$$

The other terms come from Equation (15), but we have to be careful. We need to make sure that we antisymmetrize over all of the color indices in the $(\bar{\mathbf{3}},1)_{-4/3} \times (\bar{\mathbf{3}},\mathbf{2})_{5/3} \times (\bar{\mathbf{3}},\mathbf{2})_{-1/3}$ case:

$$(\bar{\mathbf{3}},1)_{-4/3} \times (\bar{\mathbf{3}},\mathbf{2})_{5/3} \times (\bar{\mathbf{3}},\mathbf{2})_{-1/3} \sim \epsilon^{ijk} \left[u_i^c \mathbf{X}_j^{\mu\dagger} d_k^* + u_i^c \mathbf{Y}_j^{\mu\dagger} u_k^* \right]. \tag{19}$$

Finally from $(\mathbf{3},\mathbf{2})_{1/3} \times (\bar{\mathbf{3}},\mathbf{2})_{5/3} \times (1,1)_{-2}$ we have

$$(\mathbf{3},\mathbf{2})_{1/3} \times (\bar{\mathbf{3}},\mathbf{2})_{5/3} \times (1,1)_{-2} \sim (e^c)^* \mathbf{Y}_i^{\mu\dagger} u_i + (e^c)^* \mathbf{X}_i^{\mu\dagger} d_i \tag{20}$$

Now it's easy to write down the charged currents to which the bosons couple. Note that i is an SU(3) index, and α is an SU(2) index:

$$\begin{aligned} J^\mu_{i\alpha} &= -(l_\alpha)^* \bar{\sigma}^\mu d^c_i + \epsilon_{ijk}\epsilon_{\alpha\beta}(u^c_j)^*\bar{\sigma}^\mu q_{k\beta} + (q_{i\alpha})^*\bar{\sigma}^\mu e^c + \text{h.c.}, \\ \Rightarrow J^\mu &= -(l)^*\bar{\sigma}^\mu d^c + (u^c)^*\bar{\sigma}^\mu q + (q)^*\bar{\sigma}^\mu e^c + \text{h.c.}, \end{aligned} \tag{21}$$

where color and SU(2) indices are understood in the last line.

Now that we have the "charged current" that the SU(5) bosons couple to, we can write down the effective lagrangian:

$$\mathcal{L}_{cc} = \frac{g_{\text{GUT}}}{\sqrt{2}} \mathbf{X}_\mu J^\mu + h.c.. \tag{22}$$

Any (tree level) process that we'll be interested in will involve the exchange of an **X** or **Y** gauge boson. As in the Fermi theory, we will make the substitution for the propagator

$$\frac{-1}{p^2 - M_X^2} = \frac{1}{M_{\mathbf{X}}^2}\frac{1}{1-\frac{p^2}{M_{\mathbf{X}}^2}} = \frac{1}{M_{\mathbf{X}}^2}\left\{1 + \mathcal{O}\left(\frac{p^2}{M_{\mathbf{X}}^2}\right)\right\}. \tag{23}$$

Integrating out the X boson leaves us with

$$\mathcal{L}_{eff} = \frac{g^2_{\text{GUT}}}{2M^2_{\mathbf{X}}} J^\mu J^*_\mu. \tag{24}$$

1.3 Higgsing SU(5).

The adjoint representation is the smallest representation which will work for higgsing SU(5), however, larger representations can be used (and MUST be used to get realistic models). The reason that we need *at least* a **24** is easy to see—once we assigned fermions to the **10** and $\bar{\mathbf{5}}$, we defined a charge operator. Specifically, there is no hypercharge neutral component of the $\bar{\mathbf{5}}$ or the **10**, so giving any of the components of those representations a VEV breaks hypercharge.

To calculate the gauge boson masses, we need to write down a lagrangian that tells us how they interact with the higgs:

$$\mathcal{L}_{\text{higgs}} = \text{ Tr } \left[(D_\mu\Sigma)^\dagger D^\mu\Sigma\right] + V(\Sigma^\dagger\Sigma). \tag{25}$$

Since we're only really interested in the mass terms, we can see that

$$\begin{aligned} \mathcal{L}_{\text{higgs}} &= \text{stuff} + g^2 A^A_\mu A^{\mu B} \text{ Tr } \left\{[T_A,\Sigma] \, . \, ([T_B,\Sigma])^\dagger\right\}, \\ &= \text{stuff} - g^2 A^A_\mu A^{\mu B} \text{ Tr } \left\{[T_A,\Sigma] \, . \, [T_B,\Sigma]\right\}. \end{aligned} \tag{26}$$

where we made the last replacement because $\Sigma^\dagger = \Sigma, T_B^\dagger = T_B$. In order to calculate the trace, we'll need to know the form of at least one of the SU(5) generators. Luckily, two of them are listed in the notes, Equation (31). We'll take

$$T_A = \frac{1}{2}\begin{pmatrix} & & & 1 & 0 \\ & & & 0 & 0 \\ & & & 0 & 0 \\ 1 & 0 & 0 & & \\ 0 & 0 & 0 & & \end{pmatrix}, \tag{27}$$

where the factor of $\frac{1}{2}$ is to ensure that the generators are properly normalized. Then

$$(\Sigma.T_A - T_A.\Sigma)^2 = -V^2 \ \text{diag}(\frac{25}{16}, 0, 0, \frac{25}{16}, 0). \tag{28}$$

This means

$$-g^2 A^A_\mu A^{\mu B} \ \text{Tr} \ \{[T_A, \Sigma] . [T_B, \Sigma]\} = \frac{25}{8} g^2 V^2 \delta_{AB} A^A_\mu A^{\mu B}, \tag{29}$$

which, for a canonically normalized gauge field, means that we have mass

$$m_A^2 = \frac{25}{4} g^2 V^2. \tag{30}$$

1.4 Flipped SU(5)

The easiest way to see that the gauge group of the flipped SU(5) "GUT" (or $\widetilde{\mathrm{SU}(5)}$, for short) is to check the decomposition of the GUT representations under the SM gauge group. Slansky tells us

$$\bar{\mathbf{5}} \rightarrow (\bar{\mathbf{3}}, 1)_{2/3} + (1, \mathbf{2})_{-1}. \tag{31}$$

So it seems pretty natural to just identify the U(1) generator which lives in SU(5) with the hypercharge generator. In the case of $\widetilde{\mathrm{SU}(5)}$, we can no longer make this identification, because the hypercharge of d^c is not 2/3. This means that the actual hypercharge must be (at least) a linear combination of two U(1) generators—one of which is embedded in SU(5) and one of which is not.

To calculate the definition of hypercharge in terms of the $\mathrm{U}(1)_X$ and $\mathrm{U}(1)_{\tilde{Y}}$, we first note that the standard definition of electric charge is $Q = T_3 + Y/2$. The new definition, in terms of X and $\tilde{Y}$ quantum numbers is

$$Q = T_3 + a\tilde{Y} + bX. \tag{32}$$

Now we can calculate the charge of the electron:

$$Q = -\frac{1}{2} + a(-1) + b(-3) = -1 \Rightarrow a + 3b = \frac{1}{2}, \tag{33}$$

and of the u^c:

$$Q = 0 + a\left(\frac{2}{3}\right) + b(-3) \Rightarrow 2a - 9b = -2. \tag{34}$$

This gives us

$$\Rightarrow Q = T_3 - \frac{1}{10}\tilde{Y} + \frac{1}{5}X. \tag{35}$$

Comparing to the familiar definition of hypercharge, we find

$$Y = -\frac{1}{5}\tilde{Y} + \frac{2}{5}X. \tag{36}$$

Once we have this definition of hypercharge, it's relatively easy to assign the fermions to the **10**. So, for example, we know that all of the components of the **10** have $X = 1$. The q states still live in the same place (as expected), because (for the up quark, for example)

$$Q = \frac{1}{2} - \frac{1}{10}\frac{1}{3} + \frac{1}{5}(+1) = \frac{2}{3}. \tag{37}$$

The d^c quarks "flip" places with the u^c quarks:

$$Q = 0 - \frac{1}{10}\left(-\frac{4}{3}\right) + \frac{1}{5}(+1) = \frac{1}{3}. \tag{38}$$

The singlet has

$$Q = 0 - \frac{1}{10}(+2) + \frac{1}{5}(+1) = 0. \tag{39}$$

This means that the $\widetilde{\mathrm{SU}(5)}$ model *requires* a right handed neutrino! (This is not really surprising, if you consider that the $\widetilde{\mathrm{SU}(5)}$ is just another embedding of SU(5) into SO(10).) The only state that is left is the anti-electron, which lives in the singlet, which must have $X = 5$:

$$Q = 0 - \frac{1}{10}(0) + \frac{1}{5}(+5) = 1. \tag{40}$$

Finally, we can look at symmetry breaking in this model. In fact, we don't need the adjoint representation any more—the reason is easy to see. Because the positron and the right handed neutrino have "flipped" places in the **10** and the 1, the **10** now has a hypercharge-neutral component, which can get a VEV. This means that we can take

$$\langle \Sigma \rangle = V \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{41}$$

This model still suffers some of the fermion mass predictions which plague minimal SU(5), but it was (and is) one of the easiest ways to get the MSSM out of string theory.

2 Lecture 2: Orbifold GUTs.

Note that the five dimensional metric is given as $g_{MN} = \mathrm{diag}(+, -, -, -, -)$.

2.1 Kaluza-Klein Decomposition.

First, the five dimensional action of a pure gauge U(1) theory should look like:

$$\mathcal{S} = \frac{-1}{4g^2}\int d^5x F_{MN}F^{MN}. \tag{42}$$

As usual, the equations of motion can be found by integrating by parts:

$$\begin{aligned} \mathcal{S} &= \frac{-1}{4g^2}\int d^5x \left(\partial_M A_N - \partial_N A_M\right)\left(\partial^M A^N - \partial^N A^M\right), \\ \Rightarrow &= \frac{1}{4g^2}\int d^5x \left\{A_N \partial_M \left(\partial^M A^N - \partial^N A^M\right) - A_M \partial_N \left(\partial^M A^N - \partial^N A^M\right)\right\}, \end{aligned} \tag{43}$$

which (after varying the action, as usual) gives the familiar Yang-Mills equations of motion:

$$\partial_M F^{MN} = 0. \tag{44}$$

This gives us a five dimensional equation of motion for the gauge field A:

$$\partial_M \partial^M A_N = 0. \tag{45}$$

We should look at how the five dimensional action decomposes. This will tell us how the different components of A_M behave under the parity and translation operations. The five dimensional action decomposes as

$$\mathcal{S} = \frac{-1}{4g^2}\int d^5x F_{MN}F^{MN} = \frac{-1}{4g^2}\int d^5x \left\{F_{\mu\nu}F^{\mu\nu} + F_{\mu 5}F^{\mu 5} + F_{5\nu}F^{5\nu} + F_{55}F^{55}\right\}. \tag{46}$$

The second two terms are identical and the last term vanishes— recall that F_{MN} is anti-symmetric. We are left with, then

$$\mathcal{S} = \frac{-1}{4g^2}\int d^5x \left\{F_{\mu\nu}F^{\mu\nu} + 2F_{\mu 5}F^{\mu 5}\right\}. \tag{47}$$

The first term says that $A_\mu(x,y)$ can have any of the four boundary conditions. While the first term gives no requirements on the transformation properties of $A_\mu(x,y)$ and $A_5(x,y)$, the second term does.

$$F_{\mu 5}F^{\mu 5} = \partial_\mu A_5(\partial^\mu A^5 - \partial^5 A^\mu) + \partial_5 A_\mu(\partial^5 A^\mu - \partial^\mu A^5). \tag{48}$$

Notice that under the parity transformation $\mathcal{P}$, $y \to -y$, $\partial_\mu \to -\partial_\mu$ and $\partial_5 \to -\partial_5$. Under the translation transformation $\mathcal{T}$, $y \to y + 2\pi R$, $\partial_\mu \to \partial_\mu$ and $\partial_5 \to \partial_5$. By construction, we require that our action be invariant under $P \equiv \mathcal{P}$ and $P' \equiv \mathcal{PT}$, thus it must be that under P we have $A_\mu(x,-y) \to A_\mu(x,y)$ and $A_5(x,-y) \to -A_5(x,y)$. Similarly, under the translation $\mathcal{T}$, $A_\mu(x, y+2\pi R) \to A_\mu(x,y)$ and $A_5(x, y+2\pi R) \to A_5(x,y)$. Thus the $A_\mu(x,y)$ component of $A_M(x,y)$ *always has* $(++)$ *boundary conditions* and the $A_5(x,y)$

component of $A_M(x,y)$ *always has* $(--)$ *boundary conditions*. This means that the field $A_5(x,y)$ *does not have a zero mode*, while the field $A_\mu(x,y)$ always does.

This all suggests that we use the following Kaluza-Klein mode expansion:

$$\begin{aligned} A_\mu(x,y) &= \sum_{n=0}^{\infty} A_\mu^{(n)}(x) a_{(n)}(y), \\ A_5(x,y) &= \sum_{n=1}^{\infty} A_5^{(n)}(x) b_{(n)}(y) \end{aligned} \tag{49}$$

Inserting the mode expansion into the equations of motion, we see

$$\begin{aligned} \partial_\mu \partial^\mu A_\nu &= \partial_\mu \partial^\mu \sum_{n=0}^{\infty} A_\nu^{(n)}(x) a_{(n)}(y) = 0, \\ \Rightarrow &= \sum_n \left\{ a_{(n)}(y) \Box A_\nu^{(n)}(x) + A_\nu^{(n)}(x) \partial_5 \partial^5 a_{(n)}(y) \right\} = 0. \end{aligned} \tag{50}$$

Also,

$$\begin{aligned} \partial_\mu \partial^\mu A_5 &= \partial_\mu \partial^\mu \sum_{n=0}^{\infty} A_5^{(n)}(x) b_{(n)}(y) = 0, \\ \Rightarrow &= \sum_n \left\{ b_{(n)}(y) \Box A_5^{(n)}(x) + A_5^{(n)}(x) \partial_5 \partial^5 b_{(n)}(y) \right\} = 0. \end{aligned} \tag{51}$$

This gives us the familiar solutions for $a_{(n)}(y)$ and $a_{(n)}(y)$ as before:

$$\begin{aligned} a_{(n)}(y) &= \cos\left(\frac{ny}{R}\right), \\ b_{(n+1)}(y) &= \sin\left(\frac{(n+1)y}{R}\right). \end{aligned} \tag{52}$$

Finally, as we learned in the lecture, we can *break* the gauge symmetry by introducing a parity operator. In this case, we are explicitly breaking the gauge symmetry in the action by endowing the field $A_\mu(x,y)$ with specific transformation properties. This can be done by writing down an operator which gives $A_\mu(x,y)$ boundary conditions *other than* $(++)$. If this is the case, then we will clearly need a new mode expansion, and one can repeat the calculation in a similar manner and find that

$$a_{(n)}(y) = \left\{ \begin{array}{ll} \cos\left(\frac{ny}{R}\right) & (++) \\ \cos\left(\frac{(n+\frac{1}{2})y}{R}\right) & (+-) \\ \sin\left(\frac{(n+\frac{1}{2})y}{R}\right) & (-+) \\ \sin\left(\frac{(n+1)y}{R}\right) & (--) \end{array} \right\}. \tag{53}$$

2.2 Something Like the Higgs Mechanism.

We now examine Equation (47) term by term. First, in terms of the five dimensional fields...

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu), \\ &= 2(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu). \end{aligned} \tag{54}$$

Substituting into the first term,

$$\partial_\mu A_\nu(x,y)\partial^\mu A^\nu(x,y) = \sum_{n,m} \cos\left(\frac{ny}{R}\right)\cos\left(\frac{my}{R}\right)\partial_\mu A_\nu^{(n)}(x)\partial^\mu A^{\nu(m)}(x). \tag{55}$$

Now we can integrate over the fifth dimension, and see

$$\int_0^{2\pi R} dy \partial_\mu A_\nu(x,y)\partial^\mu A^\nu(x,y) = \sum_{n=0}\left(2^{\delta_{n,0}}\pi R\right)\partial_\mu A_\nu^{(n)}(x)\partial^\mu A^{\nu(n)}(x). \tag{56}$$

Renaming the indices gives us the other term in Equation (54). Finally, we find

$$\int_0^{2\pi R} dy F_{\mu\nu}F^{\mu\nu} = 2\pi R\Big\{F_{\mu\nu}^{(0)}F^{\mu\nu(0)} + \frac{1}{2}\sum_{n=1}F_{\mu\nu}^{(n)}F^{\mu\nu(n)}\Big\}. \tag{57}$$

Next, we can look at the $F_{\mu 5}F^{\mu 5}$ term. One must be careful in computing the terms here and take care that the metric is inserted properly. For instance

$$\partial_5 = -\partial^5 \equiv \frac{\partial}{\partial y}, \tag{58}$$

$$A_5 = -A^5. \tag{59}$$

We want to write things in terms of A_5 (by choice, see Lecture 2 Notes, Eq. (6)), so

$$\begin{aligned} F_{\mu 5}F^{\mu 5} &= \partial_\mu A_5 \partial^\mu A^5 - \partial_\mu A_5 \partial^5 A^\mu - \partial_5 A_\mu \partial^\mu A^5 + \partial_5 A_\mu \partial^5 A^\mu, \\ \Rightarrow F_{\mu 5}F^{\mu 5} &= -\partial_\mu A_5 \partial^\mu A_5 + \partial_\mu A_5 \partial_5 A^\mu + \partial_5 A_\mu \partial^\mu A_5 - \partial_5 A_\mu \partial_5 A^\mu. \end{aligned} \tag{60}$$

Substituting in the Kaluza-Klein mode expansion, we find

$$\begin{aligned} F_{\mu 5}F^{\mu 5} = &- \sum_{n,m=1}\sin\left(\frac{ny}{R}\right)\sin\left(\frac{my}{R}\right)\partial_\mu A_5^{(n)}(x)\partial^\mu A_5^{(m)}(x) \\ &- \sum_{n,m=1}\frac{m}{R}\sin\left(\frac{ny}{R}\right)\sin\left(\frac{my}{R}\right)\partial_\mu A_5^{(n)}(x)A^{\mu(m)}(x) \\ &- \sum_{n,m=1}\frac{n}{R}\sin\left(\frac{ny}{R}\right)\sin\left(\frac{my}{R}\right)A_\mu^{(n)}(x)\partial^\mu A_5^{(m)}(x) \\ &- \sum_{n,m=1}\frac{nm}{R^2}\sin\left(\frac{ny}{R}\right)\sin\left(\frac{my}{R}\right)A_\mu^{(n)}(x)A^\mu_{(m)}(x) \end{aligned} \tag{61}$$

Now, we can integrate out the fifth dimension:

$$\begin{aligned}\int_0^{2\pi R} dy F_{\mu 5}F^{\mu 5} &= -\sum_{n=1}\pi R\Big\{\partial_\mu A_5^{(n)}(x)\partial^\mu A_5^{(n)}(x) + \frac{n^2}{R^2}A_\mu^{(n)}(x)A^{\mu(n)}(x) \\ &\quad +\frac{n}{R}\left(A_\mu^{(n)}(x)\partial^\mu A_5^{(n)}(x) + \partial_\mu A_5^{(n)}(x)A^{\mu(n)}(x)\right)\Big\}. \end{aligned} \tag{62}$$

Putting everything together, we see that

$$\begin{aligned}\mathcal{S}_{eff} &= \frac{-2\pi R}{4g^2}\int d^4x\Big\{F_{\mu\nu}^{(0)}(x)F^{\mu\nu(0)}(x) + \frac{1}{2}\sum_{n=1}F_{\mu\nu}^{(n)}(x)F^{\mu\nu(n)}(x) \\ &\quad -\sum_{n=1}\Big[\partial_\mu A_5^{(n)}(x)\partial^\mu A_5^{(n)}(x) + \frac{n^2}{R^2}A_\mu^{(n)}(x)A^{\mu(n)}(x) \\ &\quad +\frac{n}{R}\left(A_\mu^{(n)}(x)\partial^\mu A_5^{(n)}(x) + \partial_\mu A_5^{(n)}(x)A^{\mu(n)}(x)\right)\Big]\Big\}. \end{aligned} \tag{63}$$

where the effective four dimensional coupling constant is identified:

$$g_{eff}^2 \equiv \frac{g^2}{2\pi R}. \tag{64}$$

An interesting point is that the effective coupling constant for the Kaluza-Klein modes is now *larger* by a factor of $\sqrt{2}$. This means that the KK modes couple *more strongly* to the other states in the theory. Other than that, however, Equation (63) is *exactly* what we hoped to get— the action of a tower four dimensional vector fields with mass $m_n^2 = \frac{n^2}{R^2}$:

$$\mathcal{S}_{KK} = \frac{-1}{4g_{eff}^2}\int d^4x\frac{1}{2}\sum_{n=1}\left\{F_{\mu\nu}^{(n)}F^{\mu\nu(n)} - \frac{2n^2}{R^2}A_\mu^{(n)}A^{\mu(n)}\right\}, \tag{65}$$

where we have taken a factor of two from the last term. Redefining the fields, we see that

$$\mathcal{S}_{KK} = \sum_{n=1}\int d^4x\left\{\frac{-1}{4}F_{\mu\nu}^{(n)}F^{\mu\nu(n)} + \frac{1}{2}\frac{n^2}{R^2}A_\mu^{(n)}A^{\mu(n)}\right\}, \tag{66}$$

which is indeed the action for a massive gauge boson.

Finally, note that the third polarization of the massive gauge boson is exactly the component along the fifth direction, A_5—the higgs mechanism happens for each Kaluza-Klein mode. This can be made more explicit by showing that the scalar field A_5 is a gauge artifact, and that it does not appear in the (gauge fixed) theory. First, we will choose to add a gauge-fixing term (for each KK mode) to the lagrangian in Equation (63)

$$\mathcal{S}_{g.f.} = -\frac{1}{2}\int d^4x\xi^{-1}\left[\partial_\mu A^\mu + 2\xi\frac{2\pi R}{4g^2}\frac{n}{R}A_5\right]^2. \tag{67}$$

Note that if we take $n = 0$, note that we are left with the typical R_ξ gauge. Multiplying the terms in Equation (67), we find

$$\mathcal{S}_{g.f.} = -\frac{1}{2}\int d^4x\xi^{-1}\left[\partial_\mu A^\mu \partial_\nu A^\nu + 4\xi^2 \frac{(2\pi R)^2}{16g^4}\frac{n^2}{R^2}(A_5)^2 + 4\xi\frac{n}{R}\frac{2\pi R}{4g^2}\partial_\mu A^\mu A_5\right]. \tag{68}$$

The last term can be integrated by parts to give

$$\mathcal{S}_{g.f.} = -\frac{1}{2}\int d^4x\left[\xi^{-1}\partial_\mu A^\mu \partial_\nu A^\nu + \xi \frac{(2\pi R)^2}{16g^4}\frac{n^2}{R^2}(A_5)^2 - 4\frac{n}{R}\frac{2\pi R}{4g^2}A^\mu\partial_\mu A_5\right]. \tag{69}$$

If we add this gauge fixing action to Equation (63), we see that the cross term relating the derivative of $A_\mu^{(n)}$ and A_5 cancels. Furthermore, one can choose $\xi \to \infty$, the so-called "unitary gauge", and decouple the scalar degree of freedom from the theory—the second term in Equation (69) tells us that the scalar degree of freedom gets a mass term proportional to $\sqrt{\xi}$. The analogy with the higgs mechanism is now complete.

2.3 A New Contribution to the Beta Functions.

First consider the vacuum polarization diagram, with a massless photon coupled to an infinite number of fermions with masses m_n. In the usual manner, one may write

$$\Pi_{\mu\nu}(p^2) = \sum_n -e^2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\,\left\{\frac{\gamma_\mu\left(k\cdot\gamma + m_n\right)\gamma_\nu\left((k+p)\cdot\gamma + m_n\right)}{\left(k^2 - m_n^2\right)\left((k+p)^2 - m_n^2\right)}\right\}. \tag{70}$$

One then expects, by the Ward Identities, to write

$$\Pi_{\mu\nu}(p^2) = \Pi(p^2)\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right) \Rightarrow \Pi(p^2) = \frac{1}{3p^2}g^{\mu\nu}\Pi_{\mu\nu}(p^2). \tag{71}$$

In the normal manner (by introducing a Feynman x and changing the integration variables, one finds

$$\Pi(p^2) = \frac{-8e^2}{3p^2}\sum_n\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{-\ell^2 + x(1-x)p^2 + 2m_n^2}{\left[\ell^2 + p^2x(1-x) - m_n^2\right]^2}. \tag{72}$$

Working with Euclidean momenta, this becomes

$$\Pi(p^2) = \frac{-8e^2}{3p^2}\sum_n\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\frac{\ell^2 - x(1-x)p^2 + 2m_n^2}{\left[\ell^2 + p^2x(1-x) + m_n^2\right]^2}. \tag{73}$$

If we were working with QED (or, only the zero mode fermion running around the loop), one would now proceed with dimensional regularization as usual. One could proceed in the same manner here, by shifting to $4-\epsilon$ dimensions and evaluating the Euclidean integral. If one does that, then we end up with a nasty expression, involving an infinite series of logs. Instead, we can introduce a Schwinger parameter t:

$$\frac{1}{x^2} = \int_0^\infty dt\, te^{-xt}. \tag{74}$$

Then Equation (73) can be written as

$$\begin{aligned}\Pi(p^2) &= \frac{-8e^2}{3p^2}\sum_n \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\int_0^\infty dt\, t e^{-t\left\{\ell^2+p^2x(1-x)+m_n^2\right\}} \\ &\quad \times \left\{\ell^2 - x(1-x)p^2 + 2m_n^2\right\}, \end{aligned} \tag{75}$$

which we will write (suggestively) as

$$\Pi(p^2) = \frac{-8e^2}{3p^2}\sum_n \int_0^1 dx \int_0^\infty dt\, t e^{-t\left\{p^2x(1-x)+m_n^2\right\}} \tag{76}$$

$$\times \int \frac{d^4\ell}{(2\pi)^4} e^{-t\ell^2}\left\{\ell^2 - x(1-x)p^2 + 2m_n^2\right\}, \tag{77}$$

The momentum integrals are now straightforward to evaluate, for example:

$$\int \frac{d^4\ell}{(2\pi)^4} e^{-t\ell^2} = \frac{1}{(2\pi)^4}\int_0^\infty d\ell \ell^3 e^{-t\ell^2}\int d\Omega_4 = \frac{2\pi^2}{(2\pi)^4}\frac{\Gamma(2)}{2t^2} = \frac{1}{16\pi^2 t^2}, \tag{78}$$

and one finds

$$\Pi(p^2) = \frac{-e^2}{6\pi^2p^2}\sum_n \int_0^1 dx \int_0^\infty \frac{dt}{t} e^{-t\left\{p^2x(1-x)+m_n^2\right\}}\left\{\frac{2}{t} + 2m_n^2 - x(1-x)p^2\right\}. \tag{79}$$

The first term can be integrated to give

$$\int_0^\infty \frac{dt}{t^2} e^{-t\left\{p^2x(1-x)+m_n^2\right\}} = (-1)^3 \int_0^\infty \frac{dt}{t}\left\{p^2x(1-x)+m_n^2\right\} e^{-t\left\{p^2x(1-x)+m_n^2\right\}}. \tag{80}$$

This gives

$$\begin{aligned}\Pi(p^2) &= \frac{e^2}{2\pi^2}\sum_n \int_0^1 dx\, x(1-x)\int_0^\infty \frac{dt}{t} e^{-t\left\{p^2x(1-x)+m_n^2\right\}}, \\ \Rightarrow \Pi(0) &= \frac{e^2}{12\pi^2}\sum_n \int_0^\infty \frac{dt}{t} e^{-tm_n^2} \end{aligned} \tag{81}$$

Luckily for us, we can go a bit further. The Jacobi θ function is defined as

$$\theta_3(t) \equiv \sum_n e^{i\pi n^2 t}. \tag{82}$$

Equation (81) can be written in terms of the Jacobi θ functions:

$$\Pi(0) = \frac{e^2}{12\pi^2}\int_0^\infty \frac{dt}{t}\theta_3\left(\frac{it}{\pi R^2}\right) \tag{83}$$

In general, this integral gives both UV and IR divergences. In practice, however, this expression is only applicable between the compactification scale, M_{C}, and the cutoff, M_{S}—motivated by this, then, we introduce hard UV and IR cutoffs of t, which has units of energy squared:

$$\Pi(0) = \frac{e^2}{12\pi^2}\int_{M_{\mathrm{S}}^{-2}}^{M_{\mathrm{C}}{}^{-2}} \frac{dt}{t}\theta_3\left(\frac{it}{\pi R^2}\right). \tag{84}$$

Notice that this expression reduces to the familiar result from QED (coupling renormalization) when we set $n = 0$ in Equation (81). This makes the θ_3 function equal to unity, and the integral in Equation (84) evaluates to a logarithm of the ratio of UV to IR scales.

Using the approximation

$$\theta_3\left(\frac{it}{\pi R^2}\right) \cong \sqrt{\frac{\pi}{t}}R, \tag{85}$$

we find after integrating

$$\Pi(0) \sim Rt^{-1/2}\Big|_{M_{\mathrm{S}}^{-2}}^{M_{\mathrm{C}}{}^{-2}}, \tag{86}$$

where we identify $R = M_{\mathrm{C}}{}^{-1}$. Finally,

$$\Rightarrow \Pi(0) \sim \left(\frac{M_{\mathrm{S}}}{M_{\mathrm{C}}} - 1\right). \tag{87}$$

This implies that the KK tower gives a new contribution to the running of the coupling constant, for $\mu < M_{\mathrm{C}}$:

$$\alpha^{-1}(\mu) \sim \left(\frac{M_{\mathrm{S}}}{M_{\mathrm{C}}} - 1\right) + \log\frac{M_{\mathrm{S}}}{\mu}, \tag{88}$$

where the log term is due to the states that contribute to the gauge coupling evolution below the compactification scale.

3 Lecture 3: Stringy Orbifold GUTs

3.1 A Terrible Prediction for Newton's Constant.

First, we'll start with the ten dimensional effective field theory:

$$\mathcal{S} = -\int d^{10}x e^{-2\phi}\left\{\frac{4}{\alpha'^4}\mathcal{R} + \frac{1}{\alpha'^3}\ \mathrm{Tr}\ F^2 + \ldots\right\}. \tag{89}$$

Integrating out six dimensions is straightforward:

$$\mathcal{S} = -\int d^4x V_6 e^{-2\phi}\left\{\frac{4}{\alpha'^4}\mathcal{R} + \frac{1}{\alpha'^3}\ \mathrm{Tr}\ F^2 + \ldots\right\}. \tag{90}$$

Of course, there will be other stuff that will depend on the compactification, as we found in the previous exercises. This "other stuff" is of great interest to most people, but we're not worried about it right now—we're only interested in comparing our results to the familiar 4-d results. Specifically,

$$\frac{1}{2g_{\rm GUT}} = \frac{e^{-2\phi}V_6}{\alpha'^3} \Rightarrow \alpha_{\rm GUT} = \frac{e^{2\phi}\alpha'^3}{8\pi V_6}, \tag{91}$$

and

$$\frac{1}{16\pi G_N} = \frac{4e^{-2\phi}V_6}{\alpha'^4} \Rightarrow G_N = \frac{e^{2\phi}\alpha'^4}{64\pi V_6}. \tag{92}$$

Now we can eliminate the dilaton and volume dependence to find

$$G_N = \frac{1}{8}\alpha_{\rm GUT}\alpha'. \tag{93}$$

If we assume that all of the symmetry breaking happens at the string scale, we should make the identification that $M_{\rm S} \sim \sqrt{\alpha'^{-1}} \approx M_{\rm GUT} \approx 3 \times 10^{16}$ GeV . Barring large threshold corrections, it seems reasonable to take $\alpha_{\rm GUT}^{-1} \approx 24$. Then we can calculate the Planck mass:

$$M_{\rm PL} = \sqrt{\frac{8{M_{\rm GUT}}^2}{\alpha_{\rm GUT}}} \approx \sqrt{8 \times 24 \times (3 \times 10^{16}\text{ GeV })^2} \approx 4.2 \times 10^{17}\text{ GeV }, \tag{94}$$

which is wrong by about a factor of 30—the prediction for Newton's constant is wrong by nearly three orders of magnitude. So it's clear that if we want to get unification in the heterotic string, something has to happen around the GUT scale.[2]

3.2 A 6d Orbifold GUT.

We will do this exercise in each of its parts.

3.2.1 Transformation of SO(8) Vectors/Spinors

The vectors of SO(8) are given by

$$\mathbf{r} = (\pm1,0,0,0),(0,\pm1,0,0),(0,0,\pm1,0),\text{ and }(0,0,0,\pm1). \tag{95}$$

Because we are working to get a six dimensional effective field theory ($T^4/\mathbb{Z}_N$), we take the twist as

$$2\cdot\mathbf{v_6} \equiv \mathbf{v_3} = (0,\frac{1}{3},\frac{2}{3},-1). \tag{96}$$

Because the twist acts trivially on the first and last entries (mod 1) of the SO(8) vector, we understand that those are the two (complex) directions which are non-compact. Recall,

[2]See K. Dienes, Physics Reports **287**, 447-525 (1997). FIND EPRINT HEP-TH/9602045.

also, that we are working in the light cone coordinates, and have gauged away two of the (transverse) directions. Now, we calculate:

$$
\begin{aligned}
(\pm 1,0,0,0)\cdot \mathbf{v_3} &= (0,0,0,\pm 1)\cdot \mathbf{v_3} = 0 \text{ mod } 1, && (97)\\
(0,1,0,0)\cdot \mathbf{v_3} &= (0,0,-1,0)\cdot \mathbf{v_3} = \frac{1}{3} \text{ mod } 1, && (98)\\
(0,0,1,0)\cdot \mathbf{v_3} &= (0,-1,0,0)\cdot \mathbf{v_3} = \frac{2}{3} \text{ mod } 1. && (99)
\end{aligned}
$$

The degrees of freedom associated with the four polarizations of the (six dimensional) gauge bosons are given in Equation (97), while the scalars in the six dimensional hypermultiplets are given in Equations (98) and (99). When we compactify to four dimensions, the states with $(0,0,0,\pm 1)$ will become scalar degrees of freedom.

The spinors of SO(8) are given by

$$(\pm,\pm,\pm,\pm),(\pm,\pm,\mp,\mp),(\pm,\mp,\pm,\mp), \text{ and } (\pm,\mp,\mp,\pm), \tag{100}$$

where $\pm \equiv \pm\frac{1}{2}$. They transform under $\mathbf{v_3}$ as

$$
\begin{aligned}
(\pm,\pm,\pm,\pm)\cdot \mathbf{v_3} &= (\pm,\mp,\mp,\pm) = 0 \text{ mod } 1, && (101)\\
(+,+,-,-)\cdot \mathbf{v_3} &= (-,+,-,+) = \frac{1}{3} \text{ mod } 1, && (102)\\
(+,-,+,-)\cdot \mathbf{v_3} &= (-,-,+,+) = \frac{2}{3} \text{ mod } 1. && (103)
\end{aligned}
$$

It is easy to see from Equations (97) - (103) exactly how we'll make gauge multiplets and hypermultiplets. We haven't actually constructed states (remember the SO(8) vectors/spinors are only the right movers), but the $d=6$, $\mathcal{N}=1$ gauge multiplet will contain

$$V_{d=6}^{\mathcal{N}=1} \supset \begin{pmatrix} (\pm 1,0,0,0) & (\pm,\pm,\pm,\pm) \\ (0,0,0,\pm 1) & (\pm,\mp,\mp,\pm) \end{pmatrix}. \tag{104}$$

When we do the dimensional reduction to $d=4$, the vector multiplet in Equation (104) can be written in terms of a $d=4$, $\mathcal{N}=1$ vector multiplet and a chiral multiplet:

$$
\begin{aligned}
V_{d=6}^{\mathcal{N}=1} &\to V_{d=4}^{\mathcal{N}=1} \oplus \Sigma_{d=4}^{\mathcal{N}=1} && (105)\\
V_{d=4}^{\mathcal{N}=1} &\supset \begin{pmatrix} (\pm 1,0,0,0) \\ (\pm,\pm,\pm,\pm) \end{pmatrix}, && (106)\\
\Sigma_{d=4}^{\mathcal{N}=1} &\supset \begin{pmatrix} (0,0,0,\pm 1) \\ (\pm,\mp,\mp,\pm) \end{pmatrix}. && (107)
\end{aligned}
$$

Likewise, the spinors sit in a $d=6$, $\mathcal{N}=1$ hypermultiplet:

$$\Phi_{d=6,\mathcal{N}=1} \supset \left(\begin{array}{cc|cc} (0,1,0,0) & (+,+,-,-) & (0,0,1,0) & (+,-,+,-) \\ (0,0,-1,0) & (-,+,-,+) & (0,-1,0,0) & (-,-,+,+) \end{array} \right). \tag{108}$$

In the dimensional reduction, the hypermultiplet can be written as two ($d = 4$, $\mathcal{N} = 1$) chiral multiplets:

$$\Phi_{d=6,\mathcal{N}=1} \quad \to \quad \phi_{d=4,\mathcal{N}=1} \oplus \phi^c_{d=4,\mathcal{N}=1}, \tag{109}$$

$$\phi_{d=4,\mathcal{N}=1} \quad \supset \quad \begin{pmatrix} (0,1,0,0) & (+,+,-,-) \\ (0,0,-1,0) & (-,+,-,+) \end{pmatrix}, \tag{110}$$

$$\phi^c_{d=4,\mathcal{N}=1} \quad \supset \quad \begin{pmatrix} (0,-1,0,0) & (-,-,+,+) \\ (0,0,1,0) & (+,-,+,-) \end{pmatrix}. \tag{111}$$

As an aside, it is now clear why $\mathcal{N} = 2$ SUSY can never give anything like the MSSM—the matter fields come in $d = 4, \mathcal{N} = 2$ hypermultiplets (or equivalently, a $d = 6, \mathcal{N} = 1$ hypermultiplet), which (in the $\mathcal{N} = 1$ language) means that a chiral field is always accompanied by it's conjugate partner. This means that we can never get a chiral spectrum.

3.2.2 The 6d Gauge Group

In order to calculate the gauge group in six dimensions, it is helpful to know the simple roots of E_8. The simple roots correspond to the uncharged gauge bosons. By applying the projection conditions to the simple roots of E_8, and then computing the Cartan matrix, we can determine the gauge group.

First, the simple roots of E_8 are

$$\begin{array}{lcrrrrrrrr} \alpha_1 & = & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \alpha_2 & = & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ \alpha_3 & = & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ \alpha_4 & = & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ \alpha_5 & = & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ \alpha_6 & = & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ \alpha_7 & = & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_8 & = & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array} \tag{112}$$

The conventions used here match Slansky's conventions, and NOT the conventions in the Big Book of CFT by DiFrancesco, et al.

The projection condition for an untwisted sector state is (for $\mathbf{P}$ living in the E_8 root lattice)

$$\mathbf{P} \cdot \mathbf{V}_3 - \mathbf{r} \cdot \mathbf{v}_3 = 0 \bmod 1 \tag{113}$$

where the shift vector for the six dimensional model is given by (see Equation (9a) of Lecture 3 notes) is defined as[3]:

$$\mathbf{V}_3 \equiv 2 \cdot \mathbf{V}_6 = \left(\frac{2}{3}, -1, -1, 0, 0, 0, 0, 0\right). \tag{114}$$

[3]We're only listing the components of $\mathbf{V}$ which act on the first E_8 here.

Earlier, we argued that $\mathbf{r} \cdot \mathbf{v}_3 = 0 \bmod 1$ for the gauge bosons, which means that we're looking for the simple roots of E_8 which satisfy

$$\alpha_i \cdot \mathbf{V}_3 = 0 \bmod 1. \tag{115}$$

Finally, the gauge bosons must be invariant under the action of the Wilson Line, as well:

$$\alpha_i \cdot \mathbf{W}_3 = 0 \bmod 1, \tag{116}$$

where $\mathbf{W}_3$ is defined in Equation (9c):

$$\mathbf{W}_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right). \tag{117}$$

There is an additional projection condition in the *four dimensional* spectrum, namely $\alpha_i \cdot \mathbf{W}_2 = 0 \bmod 1$, but because we have not yet compactified on the SO(4) torus yet, this condition does not apply.

Now we can ask, which of the α_is satisfy these conditions? Clearly (115) only projects out α_1, however, the requirement that the α_i be invariant under the Wilson line projects out α_7 and α_8. The simple roots which survive are

$$\begin{array}{lcrrrrrrrr}
\alpha_2 & = & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
\alpha_3 & = & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
\alpha_4 & = & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
\alpha_5 & = & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
\alpha_6 & = & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0.
\end{array} \tag{118}$$

The gauge group which we are working with is rank 5, which means probably either SO(10) or SU(6), unless the gauge group is semi-simple.

In order to figure out *which* gauge group we're working with, we can proceed in one of two ways. One way is to look for the other E_8 weights which satisfy the projection conditions (115) and (116). If you find them all, you know that they should transform in the adjoint representation of the gauge group—in this case either the **35** for SU(6) or the **45** for SO(10).

Another way to do this is to calculate the Cartan Matrix, which uniquely determines the gauge group when given the simple roots. The Cartan Matrix is defined as

$$[\mathbb{A}]_{ij} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_j^2}. \tag{119}$$

We find

$$\mathbb{A} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \tag{120}$$

which is exactly the Cartan Matrix for SU(6).

3.2.3 Matter Reps in the Untwisted Sector

In order to find the charged matter in the untwisted sector, the projection conditions in the kth twisted sector of a $\mathbb{Z}_N$ orbifold become

$$\begin{aligned} \mathbf{P} \cdot \mathbf{V}_3 &= \mathbf{r} \cdot \mathbf{v}_3 = k/N \bmod 1, && (121) \\ \mathbf{P} \cdot \mathbf{W}_3 &= 0 \bmod 1. && (122) \end{aligned}$$

In this case, $N = 3$ because formally we're working on $T^6/\mathbb{Z}_3$. Further, we already know that we're working with SU(6), we know that the smallest representations are the $\mathbf{6}, \mathbf{15}, \mathbf{20}$ and $\mathbf{21}$.

Since we've already worked out the transformation laws of the SO(8) vectors and spinors in Equations (97) - (103), we can start there. Without loss of generality, we'll consider the SO(8) vector $(0,1,0,0)$, which transforms as $\mathbf{v}_3 \cdot (0,1,0,0) = \frac{1}{3}$. In order to build a good state, we need to find all left movers which transforms the same way. We find the following E_8 lattice vectors obey $\mathbf{P} \cdot \mathbf{V}_3 = \frac{1}{3} \bmod 1$:

$$\begin{aligned} 16 &\times \left(+\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, [\text{even \# of } +]\right), \\ 16 &\times \left(+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, [\text{even \# of } +]\right), \\ 16 &\times \left(+\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, [\text{odd \# of } +]\right), \\ 16 &\times \left(+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, [\text{odd \# of } +]\right), \\ 14 &\times \left(-1, \underline{\pm 1, 0, 0, 0, 0, 0, 0}\right), \end{aligned} \qquad (123)$$

where the underline means that we take all combinations of ± 1 and six zeros. The conjugate states (which transform as $\mathbf{P} \cdot \mathbf{V}_3 = \frac{2}{3} \bmod 1$) come by exchanging the $+$s and $-$s above. Finally, we need to check which of these weights obey $\mathbf{P} \cdot \mathbf{W}_3 = 0 \bmod 1$. We find

$$\begin{aligned} 10 &\times \left(+\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, \underline{+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}}\right), \\ 10 &\times \left(+\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \underline{+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}}\right), \\ 1 &\times (-1, 1, 0, 0, 0, 0, 0, 0), \end{aligned}$$

where the underline means that we take all permutations of $+\frac{1}{2}$ and $-\frac{1}{2}$. Given the representations of SU(6), we should suspect that we have either a $\mathbf{21}$, a $\mathbf{20} + 1$, or some combination of $\mathbf{6} + \overline{\mathbf{6}}$ and singlets.

In order to actually calculate how the states actually transform, we'll have to figure out the Dynkin labels of each of these E_8 weights. For some group with simple roots $\alpha_1, \alpha_2, \ldots$, we have:

$$\mathrm{DL}(\mathbf{P}) = (\alpha_1 \cdot \mathbf{P}, \alpha_2 \cdot \mathbf{P} \ldots). \qquad (124)$$

Then the "highest weight" of a representation is the weight which corresponds to a Dynkin label with all positive entries, and a table of highest weights of highest weights can be found in Slansky's Physics Reports article referenced earlier. So, for example:

$$\begin{aligned} \mathbf{P} &= (-1,-1,0,0,0,0,0,0), \\ \Rightarrow \mathrm{DL}(\mathbf{P}) &= (0,0,0,0,0). \end{aligned} \tag{125}$$

Thus the state is an SU(6) singlet as it is orthogonal to all of the simple roots of SU(6). A less trivial example comes from looking at the other 20 states. So, for example,

$$\begin{aligned} \mathbf{P} &= \left(+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right), \\ \Rightarrow \mathrm{DL}(\mathbf{P}) &= (0,0,1,0,0), \end{aligned} \tag{126}$$

which is exactly the Dynkin label for the **20**. In general, one must compute Dynkin labels until the highest weight is found. For example, had we chosen another of the 20 weights above, we would find

$$\mathbf{P} = \left(+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},+\frac{1}{2},+\frac{1}{2}\right), \tag{127}$$

$$\Rightarrow \mathrm{DL}(\mathbf{P}) = (0,-1,1,0,0), \tag{128}$$

which means that **P** listed in Equation (127) is not the highest weight of the representation.

3.2.4 Bonus!

It should be clear, after completing this calculation, why one has trouble finding larger than adjoint representations. Because we're always starting with the adjoint of E_8, it is difficult to see how one could find larger (than adjoint) representations in any low energy effective field theory, by simply applying the above projection conditions. This is not a general statement for *all* string models (again, see the review by Kieth Dienes, mentioned earlier), but it is a general statement for this class of string models.

3.3 Local GUTs in Orbifold Compactifications

Finding states in the twisted sector is a bit of a challenge because the masslessness condition is modified, see Equations (119) and (120) of the Notes—don't worry about oscillator modes, as they never come in very large representations. Again we'll start with the simple roots of E_8, and only keep those that survive the projection conditions. In general, the uncharged gauge bosons which comprise the local GUT satisfy

$$\mathbf{P} \cdot \mathbf{X} = 0 \text{ mod } 1, \tag{129}$$

where $\mathbf{X}$ is defined *locally* on the orbifold, in terms of the fixed point (see p.6, for example, in Lecture 3 Notes, or the Introduction of[4]). For the fixed point we're interested in, we have simply

$$\mathbf{X} = \mathbf{V}_6 + 0 \times \mathbf{W}_3 + 0 \times \mathbf{W}_2 = \mathbf{V}_6. \tag{130}$$

Clearly, the E_8 roots which survive the projection are

$$(0,0,0,\underline{\pm 1, \pm 1, 0, 0, 0}), \quad (0, \underline{\pm 1, \pm 1}, 0, 0, 0, 0, 0). \tag{131}$$

In order to find the gauge group, we should find a suitable basis and calculate the Cartan Matrix. We choose

$$\begin{array}{rcllllllll} \alpha_1 &=& 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ \alpha_2 &=& 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ \alpha_3 &=& 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ \alpha_4 &=& 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ \alpha_5 &=& 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \alpha_6 &=& 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ \alpha_7 &=& 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{132}$$

We can again compute the Cartan Matrix of the surviving 6 roots and find:

$$\mathbb{A} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \tag{133}$$

The Cartan Matrix for SO(10) is

$$\mathbb{A}_{SO(10)} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}, \tag{134}$$

so it seems that our local GUT is precisely SO(10) × SU(2) × SU(2).

In order to show that there's a spinor (**16**) of SO(10), we must find the right mover and the left mover which satisfy the masslessness condition in the notes, Lecture 3 Equations (1) and (2). If we take the twist to be $\mathbf{v} = (0 \frac{1}{6} \frac{1}{3} - \frac{1}{2})$, then the vacuum energy can be found from Equation (121) in the notes. We find

$$a_R^1 = -\frac{1}{2} + \frac{1}{2}\left\{\frac{5}{36} + \frac{2}{9} + \frac{1}{4}\right\} = -\frac{7}{36}. \tag{135}$$

[4]W. Buchmüller, K. Hamaguchi, O. Lebedev, and M. Ratz. Nuclear Physics **B785**, 149-209 (2007). FIND EPRINT HEP-TH/0606187

This gives us the mass equation

$$|\mathbf{r}+\mathbf{v}|^2 = \frac{7}{18} \tag{136}$$

Now, consider the SO(8) spinor $\left|\frac{1}{2} - \frac{1}{2} - \frac{1}{2}\frac{1}{2}\right\rangle$. It satisfies the mass relationship:

$$|\mathbf{r}+\mathbf{v}|^2 = \frac{1}{4} + \frac{1}{9} + \frac{1}{36} = \frac{7}{18}. \tag{137}$$

Next, we calculate the vacuum energy in the right moving sector:

$$a_L^1 = -1 + \frac{1}{2}\left(\frac{5}{36} + \frac{2}{9} + \frac{1}{4}\right) = -\frac{25}{36} \tag{138}$$

Showing that there is an SO(10) spinor is as easy as finding some E_8 lattice vector that is all $\pm\frac{1}{2}$s and which obeys the masslessness condition

$$|\mathbf{P}+\mathbf{V}|^2 = \frac{50}{36} \tag{139}$$

We find that

$$\mathbf{P} = \left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, [\text{even \# of } +]\right)\left(-\frac{1}{2}, \left(\frac{1}{2}\right)^6, -\frac{1}{2}\right) \tag{140}$$

works. The $\left(\frac{1}{2}\right)^6$ means 6 $\frac{1}{2}$s. How many different ways can we get an even number of $+\frac{1}{2}$s? There are 5 ways to have four $+\frac{1}{2}$s, there are $\frac{5\times 4}{2} = 10$ ways to get two $+\frac{1}{2}$s, and there is one way to get no $+\frac{1}{2}$s, so we do indeed have a sixteen dimensional representation. As before, we can actually *calculate* how this weight transforms by computing the Dynkin label of the state $\mathbf{P}+\mathbf{V}$, however, in order to get something meaningful, we'd have to be sure that we had the highest weight of the representation.

Finally we can show how the projection conditions work on the left mover:

$$\begin{aligned} 2\mathbf{P}+\mathbf{V} &= \left(-\frac{2}{3}, \frac{1}{2}, \frac{1}{2}, [\text{even \# of } +1]\right)\left(-\frac{1}{2}, \frac{5}{6}, \left(\frac{1}{2}\right)^5, -\frac{1}{2}\right), \\ \Rightarrow (2\mathbf{P}+\mathbf{V})\cdot\mathbf{V} &= -\frac{94}{36} \end{aligned} \tag{141}$$

This is fortunate, because the right mover transforms as

$$\begin{aligned} 2\mathbf{r}+\mathbf{v} &= \left(\frac{1}{2}, -\frac{1}{3}, -\frac{1}{6}, 0\right), \\ \Rightarrow -(2\mathbf{r}+\mathbf{v})\cdot\mathbf{v} &= \frac{22}{36}. \end{aligned} \tag{142}$$

We have no oscillators so $\phi = 1$, and $\gamma = 1$ in the first twisted sector. This means that the GSO projection acts on the states as

$$\Delta = \gamma\phi\ \exp\left\{i\pi\left[(2\mathbf{P}+\mathbf{V})\cdot\mathbf{V} - (2\mathbf{r}+\mathbf{v})\right]\right\} = \exp\left\{i\pi\left[-\frac{94}{36} + \frac{22}{36}\right]\right\} = 1. \tag{143}$$