PiTP Summer School 2009

Plan for my lectures

Volker Springel

Lecture 1 ▶ **Basics of collisionless dynamics and the N-body approach**

Lecture 2 ▶ **Gravitational solvers suitable for collisionless dynamics, parallelization**

Lecture 3 ▶ **More parallelization, Introduction to smoothed particle hydrodynamics**

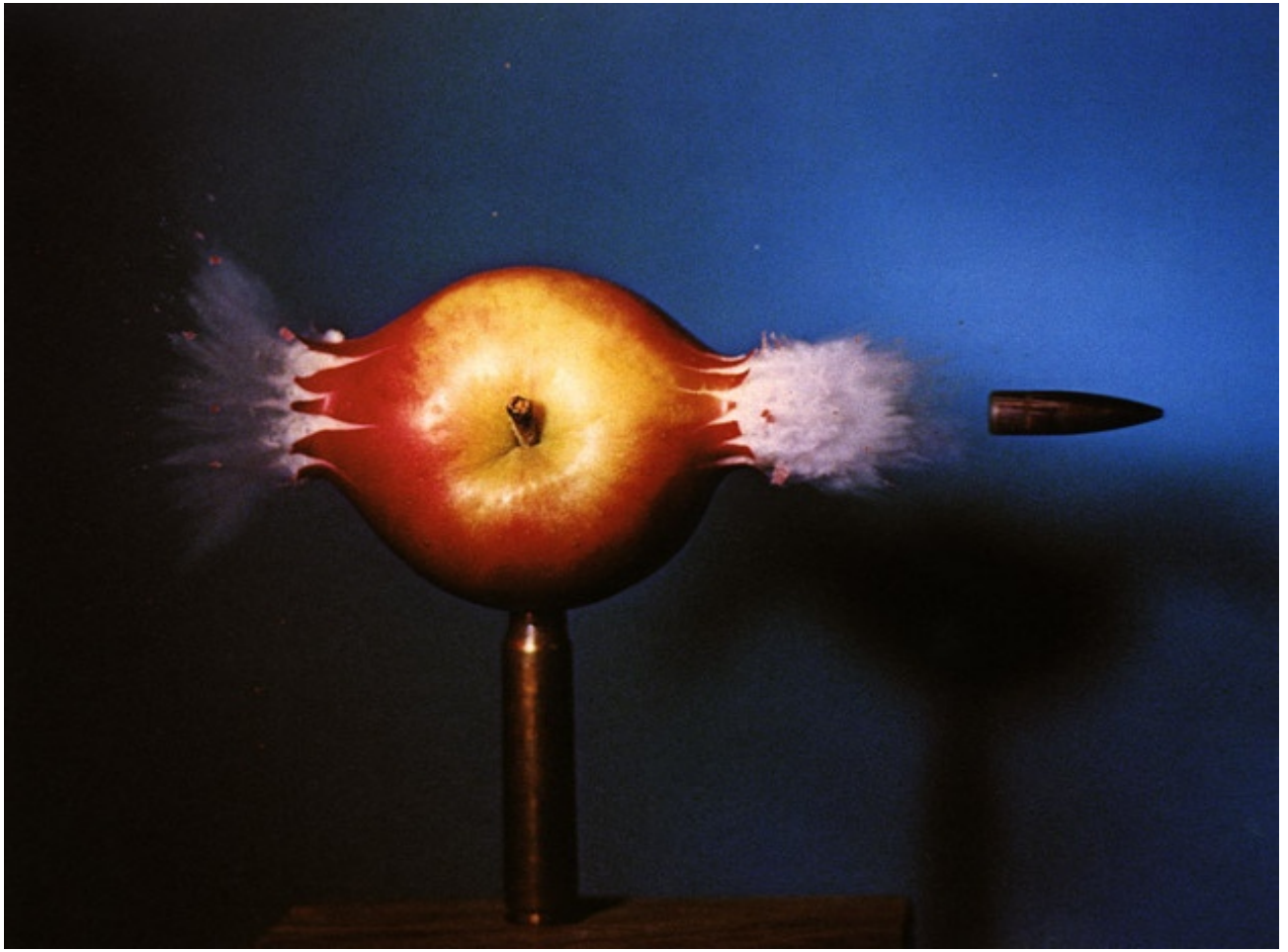
Lecture 4 ▶ **Algorithmic aspects of SPH, caveats, applications**

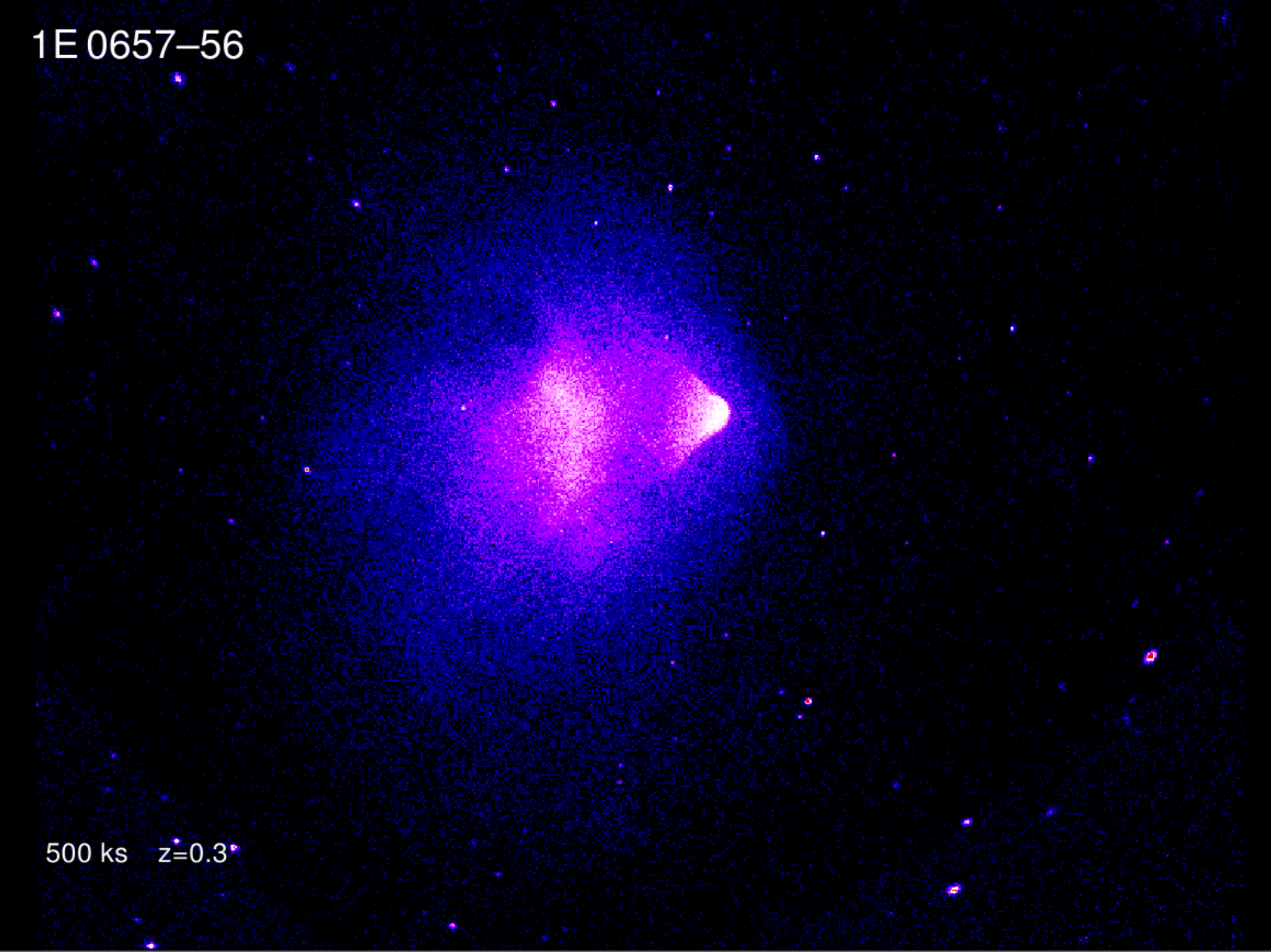
Lecture 5 ▶ **Comparison of SPH to finite volume methods, Moving-mesh hydrodynamics**

Example applications of SPH and collisionless dynamics in cosmology

An example application of SPH and collisionless dynamics in cosmology

The bullet cluster




1E 0657–56
500 ks
z=0.3

NASA Press Release Aug 21, 2006:

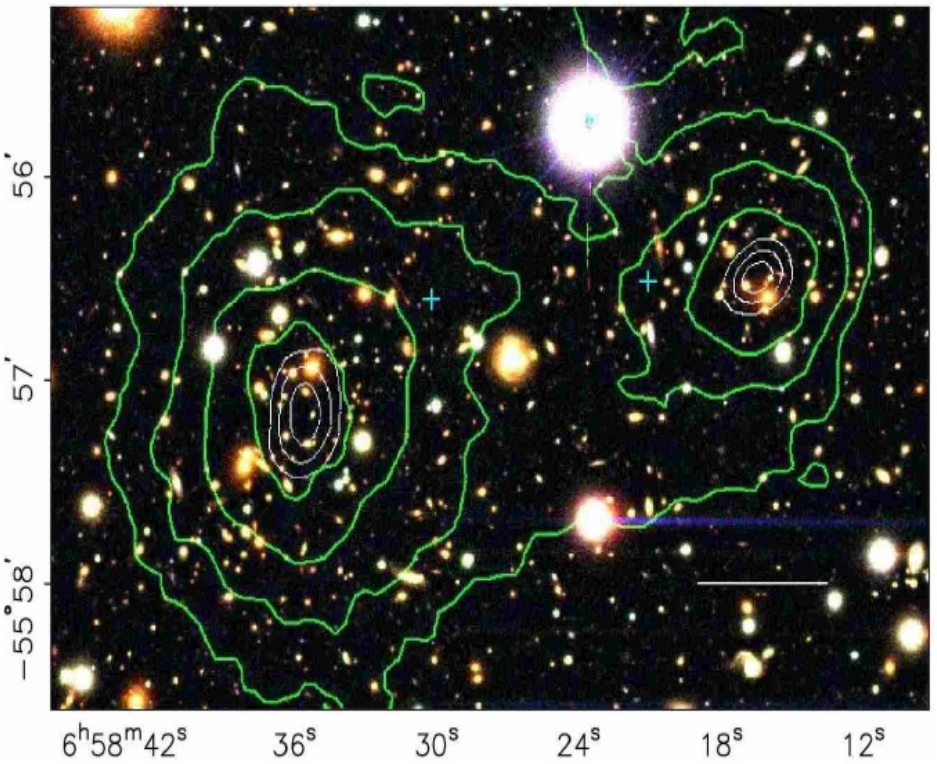
1E 0657-56: NASA Finds Direct Proof of Dark Matter

New weak lensing mass reconstructions have confirmed an offset between mass peaks and X-ray emission

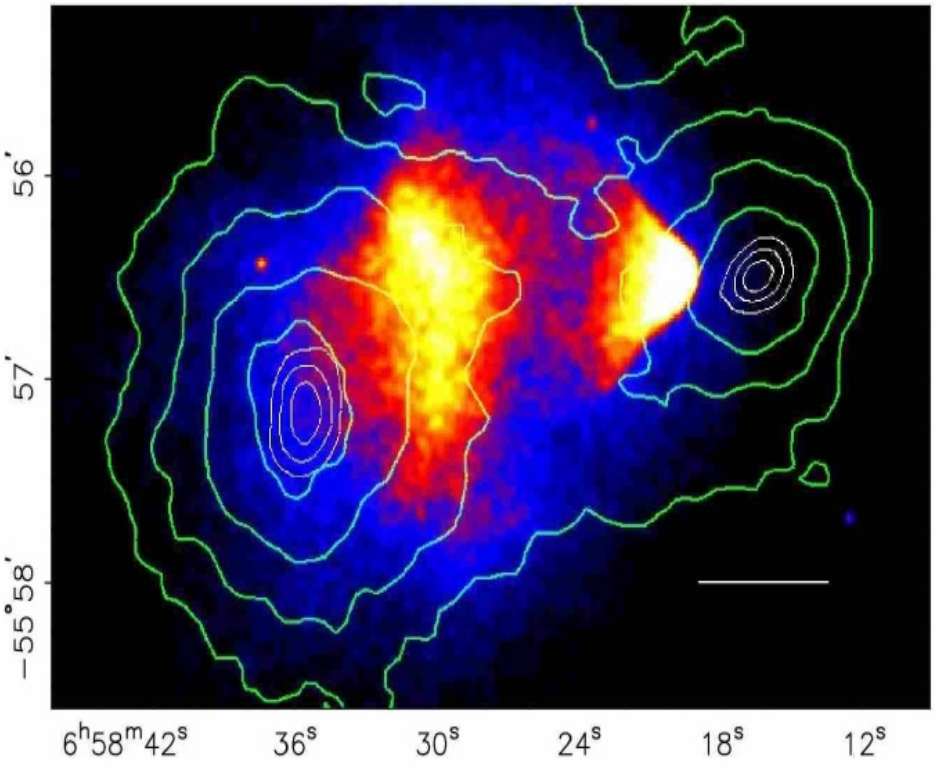
MASS CONTOURS FROM LENSING COMPARD TO X-RAY EMISSION

Clowe et al. (2006)

Magellan Optical Image



500 ksec Chandra exposure

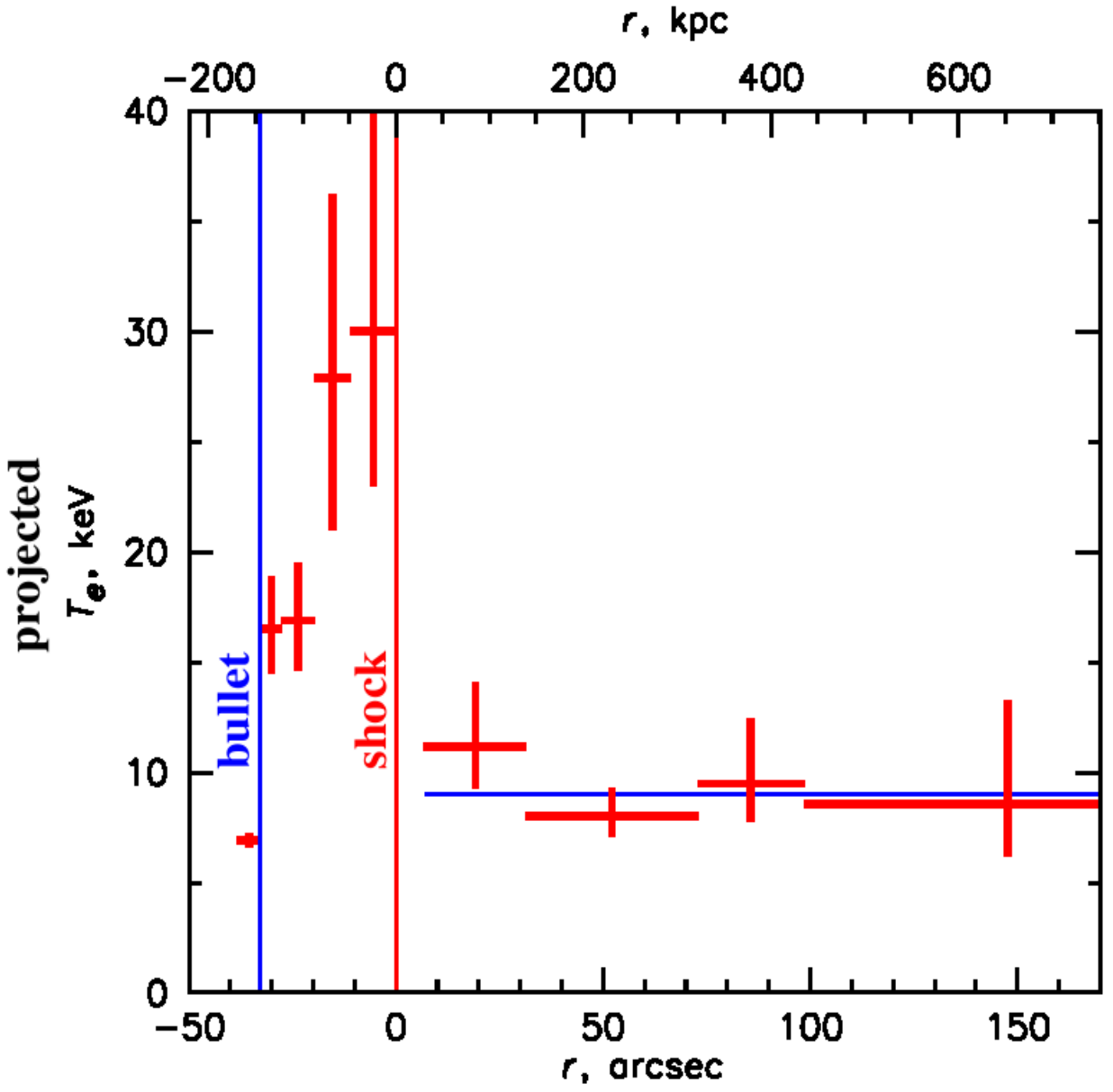


weak lensing mass contours overlaid

The temperature profile through the nose of the shock shows a strong shock and a cold front

X-RAY TEMPERATURE PROFILE FROM CHANDRA OBSERVATIONS

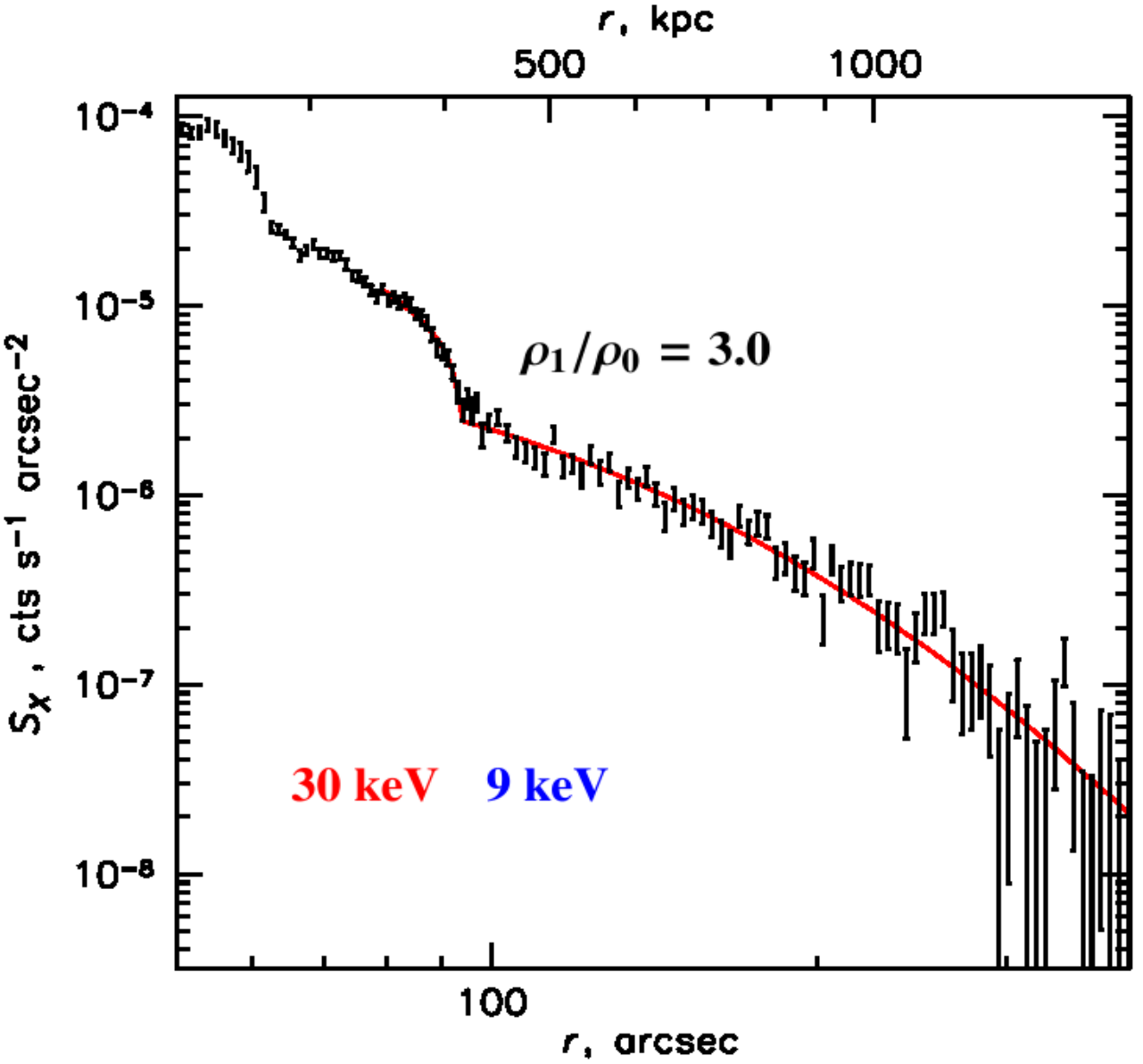
Markevitch et al. (2006)



Fitting the density jump in the X-ray surface brightness profile allows a measurement of the shock's Mach number

X-RAY SURFACE BRIGHTNESS PROFILE

Markevitch et al. (2006)



shock strength:

M = 3.0 ± 0.4

shock velocity:

v_s = 4700 km/s

Usually, shock velocity has been identified with velocity of the bullet.

How rare is the bullet cluster?

DISTRIBUTION OF VELOCITIES OF THE MOST MASSIVE SUBSTRUCTURE IN THE MILLENNIUM RUN

Hayashi & White (2006)

Adopted mass model from Clowe et al. (2004):

NFW-Halo with:

M_{200} = 2.96 x 10^{15} $M_\odot$
R_{200} = 2.25 Mpc

V_{200} = 2380 km/sec
V_{shock} = 4500 km/sec

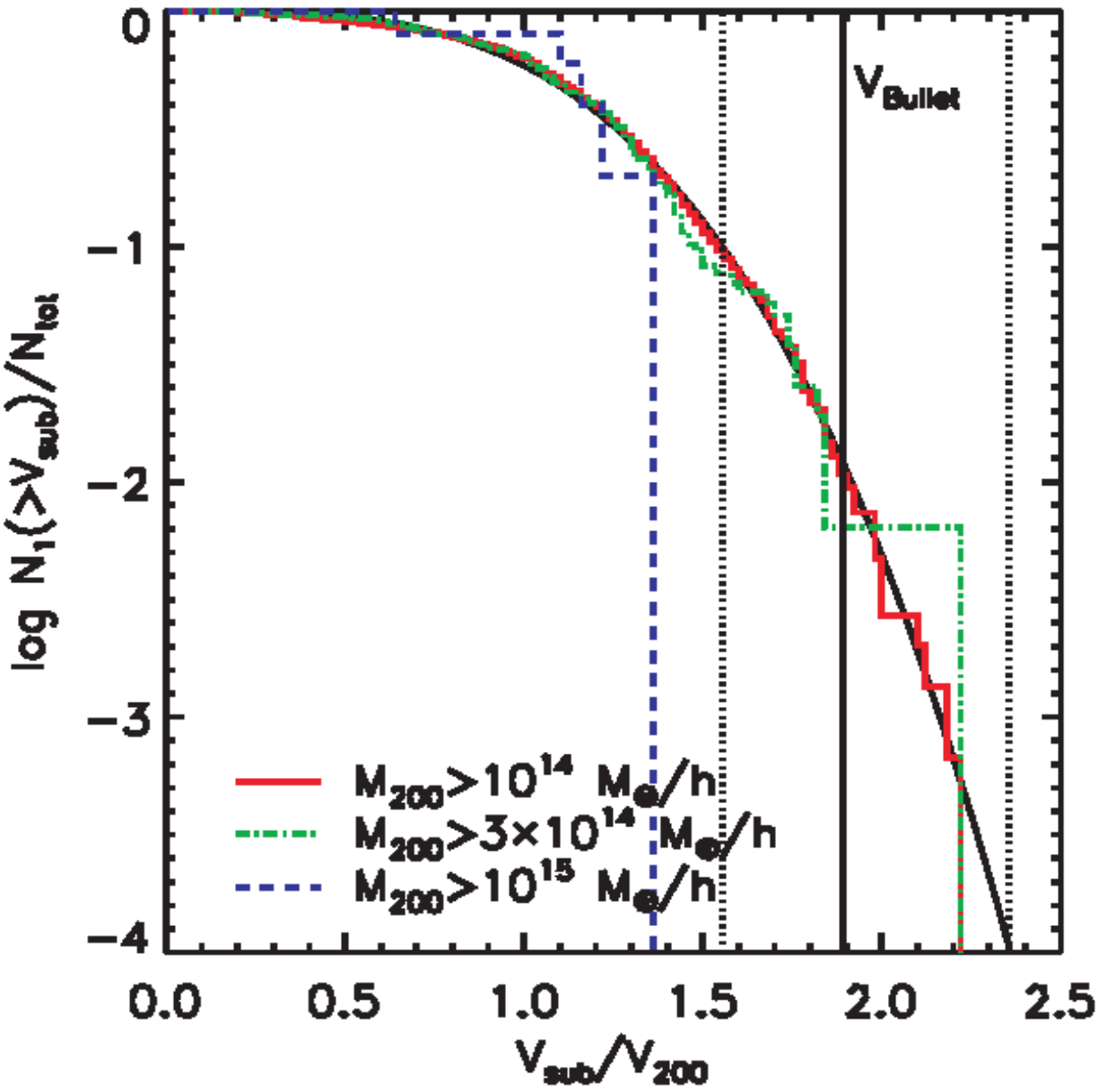
V_{sub}/V_{shock} = 1.9 chance: 10^{-2}

But, revised data from Clowe et al. (2006) and Markevitch el al. (2006):

M_{200} = 1.5 x 10^{15} $M_\odot$

V_{200} = 1680 km/sec
V_{shock} = 4740 km/sec

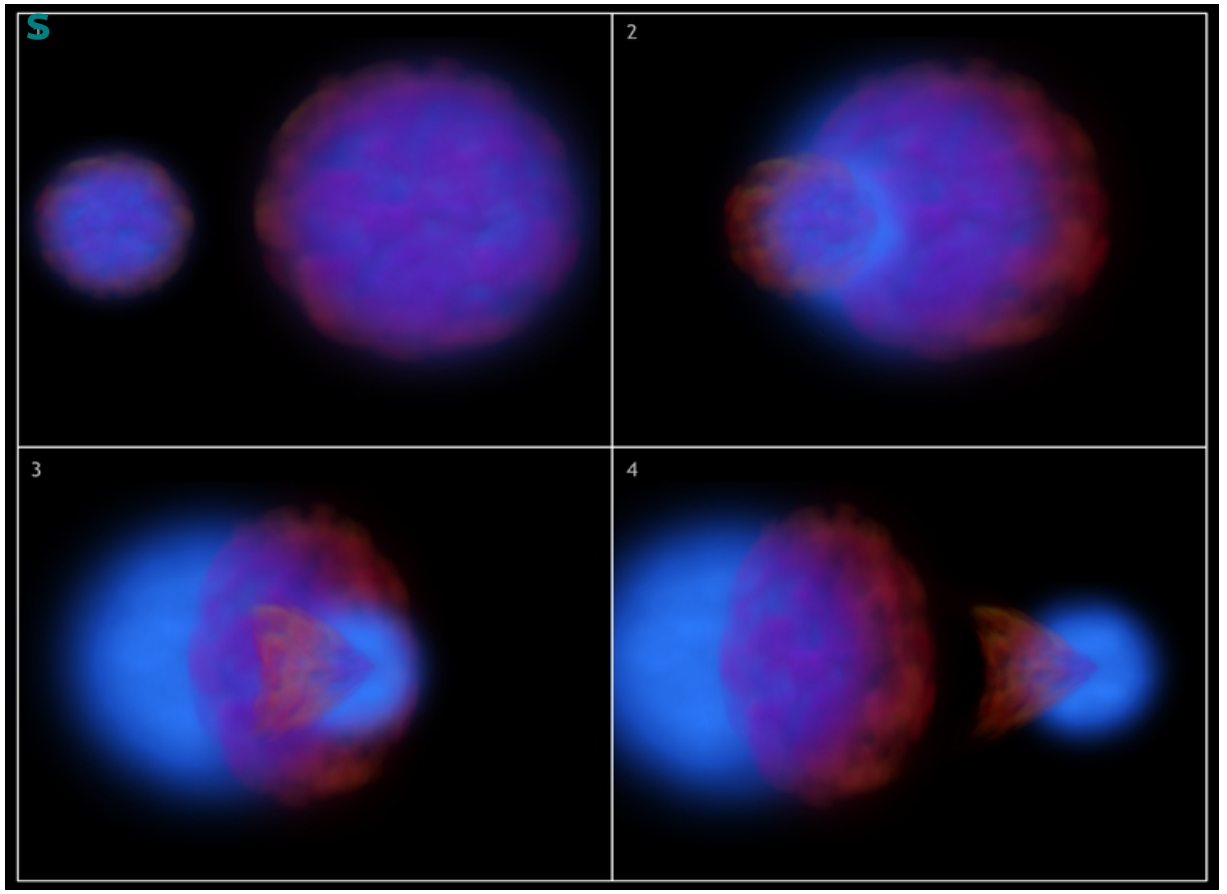
V_{sub}/V_{shock} = 2.8 chance: 10^{-7}



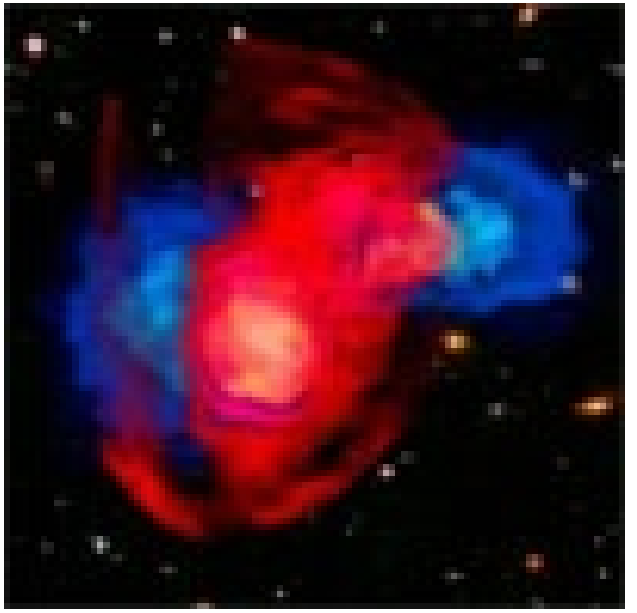
Simulating realistic models of the bullet cluster is key for a proper interpretation of the dynamical state of the system

ANIMATIONS OF THE BULLET CLUSTER MERGER

NASA/CXC/M.Weis



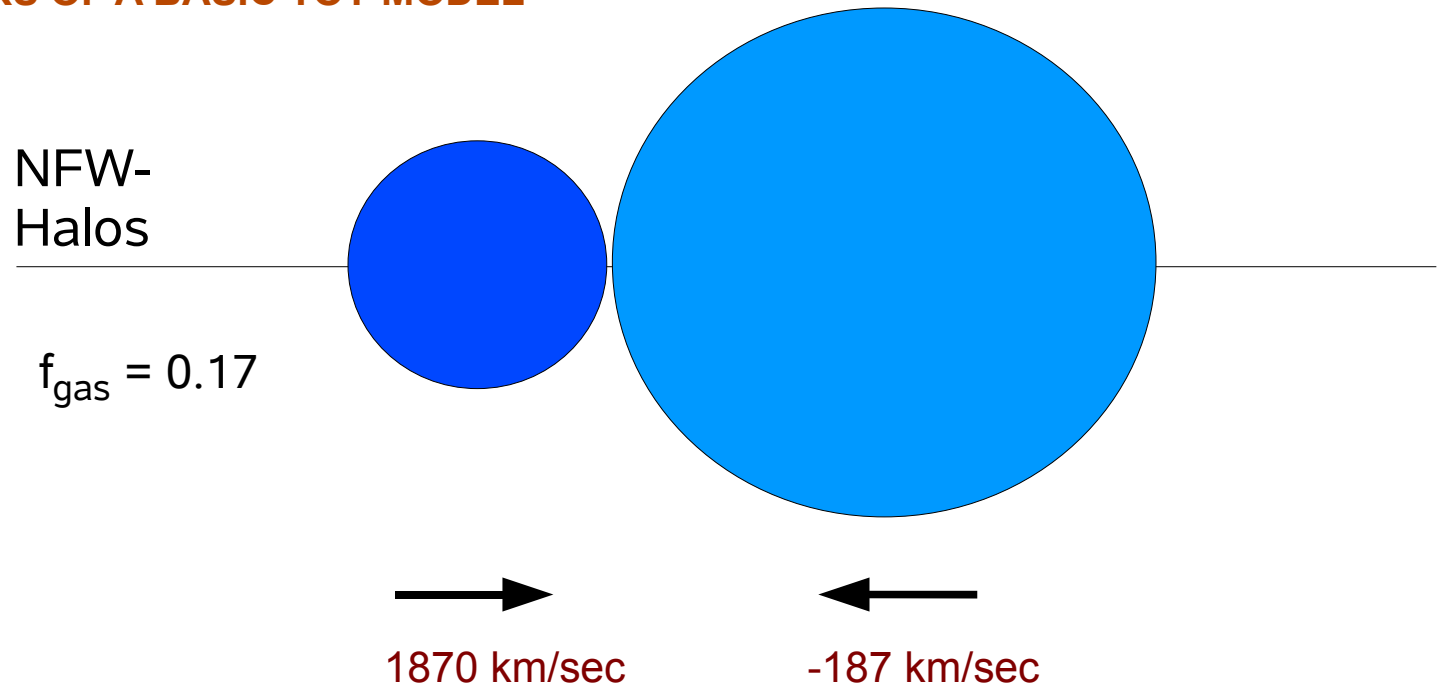
KIPAC/John Wise



(Note: The distance of the mass peaks is 720 kpc on the sky...)

A simple toy merger model of two NFW halos on a zero-energy collision orbit

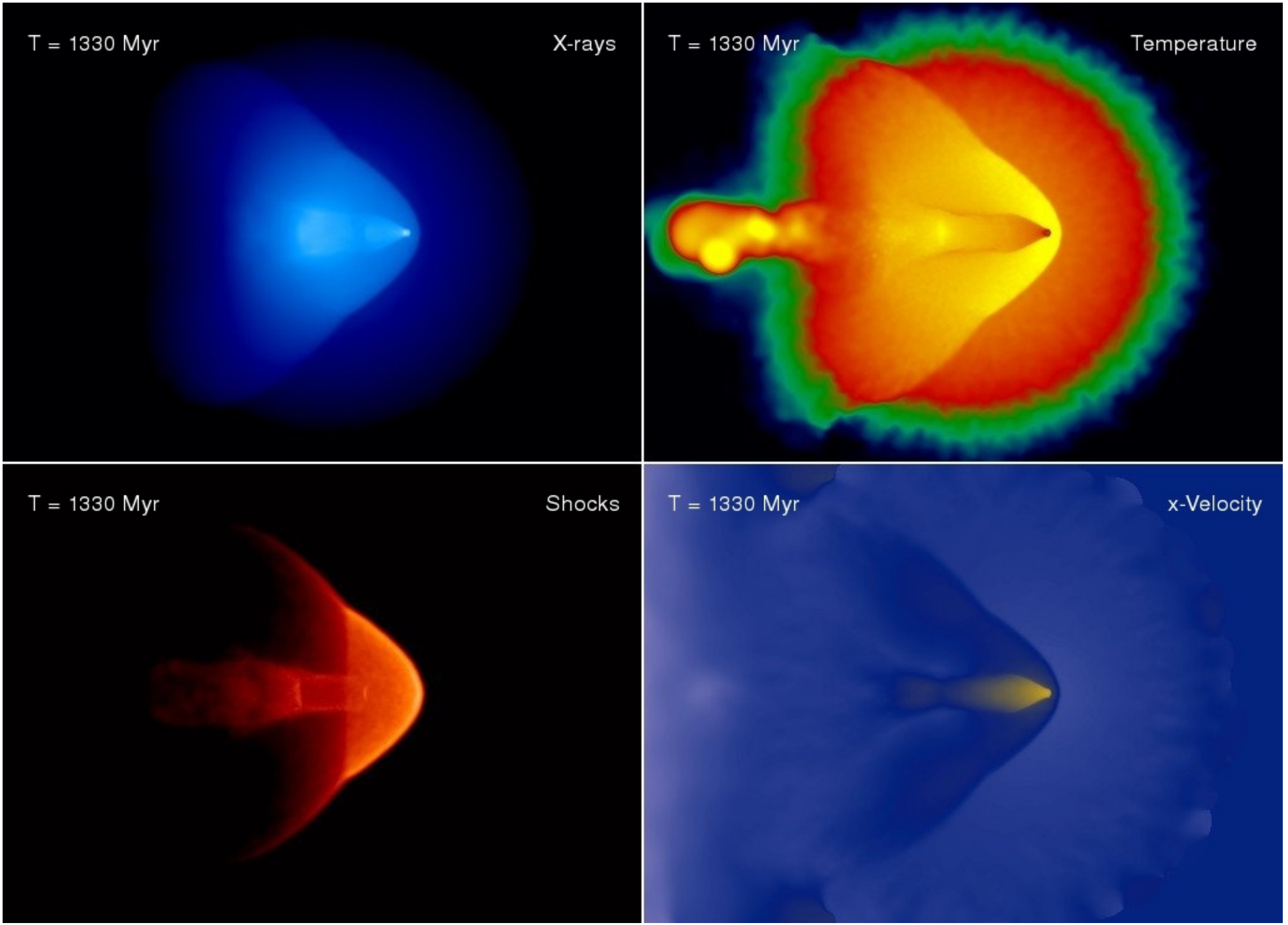
PARAMETERS OF A BASIC TOY MODEL



Mass model from **Clowe et al. (2006):**

M_{200} = 1.5 x 10^{14} $M_\odot$	**M_{200} = 1.5 x 10^{15} $M_\odot$**
R_{200} = 1.1 Mpc	R_{200} = 2.3 Mpc
c = 7.2	c = 2.0
V_{200} = 780 km/sec	V_{200} = 1680 km/sec

VIDEO OF THE TIME EVOLUTION OF A SIMPLE BULLET CLUSTER MODEL

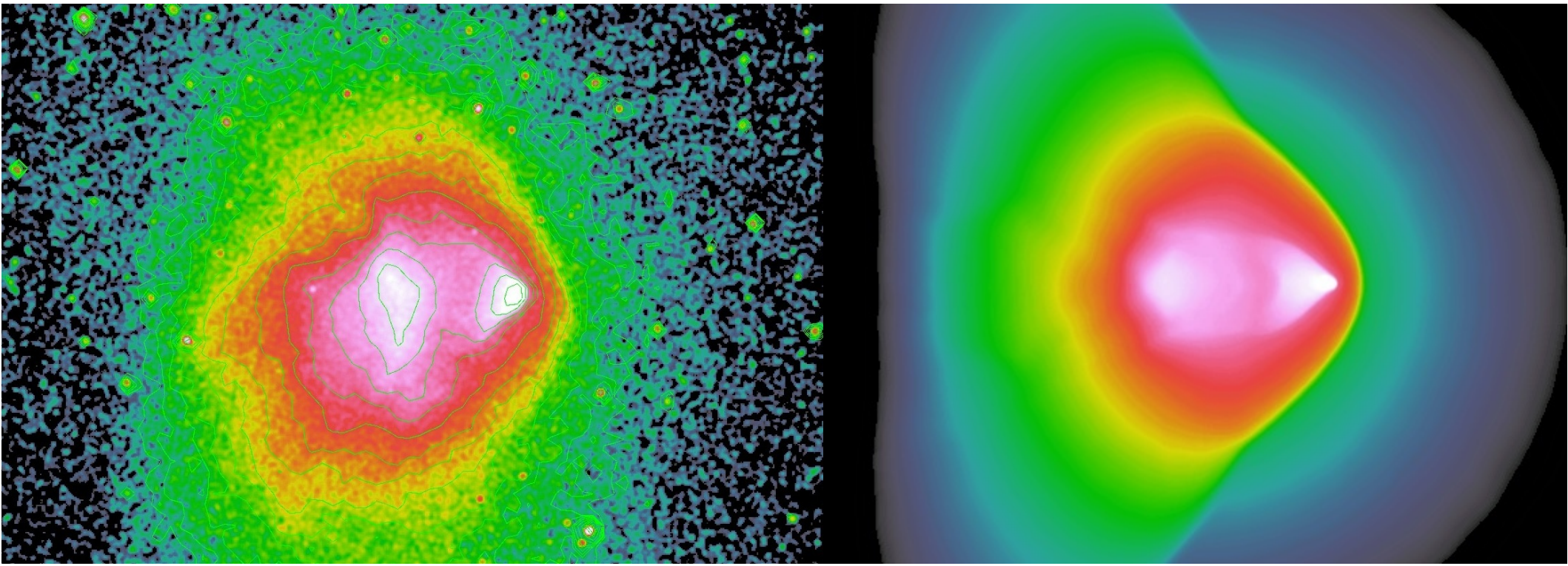


Drawing the observed X-ray map and the simulation images with the same color-scale simplifies the comparison

SIMULATED X-RAY MAP COMPARED TO OBSERVATION

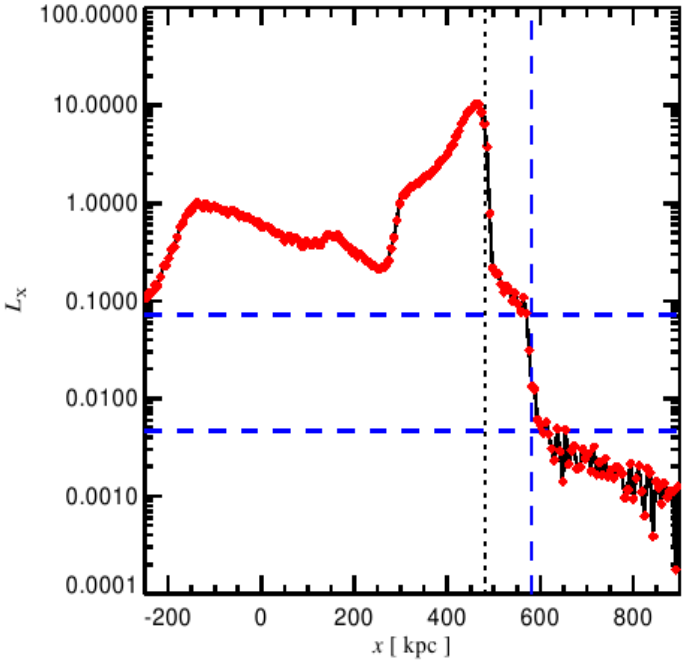
Candra 500 ks image

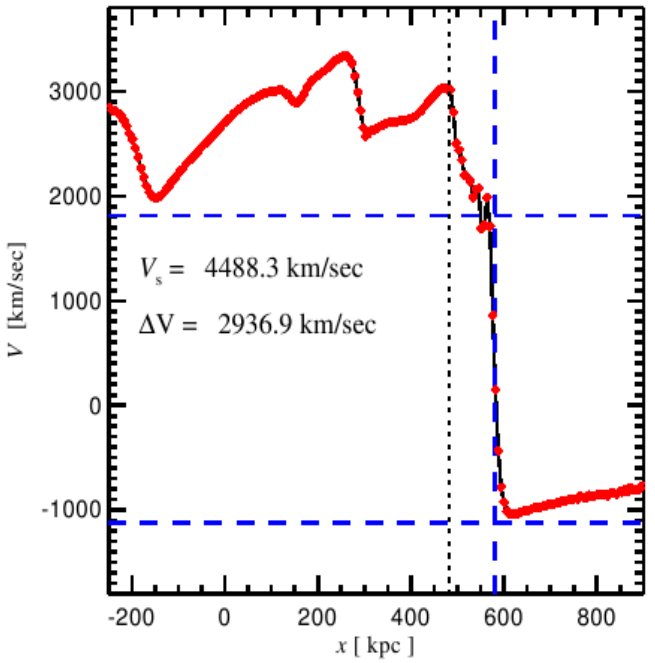
bullet cluster simulation

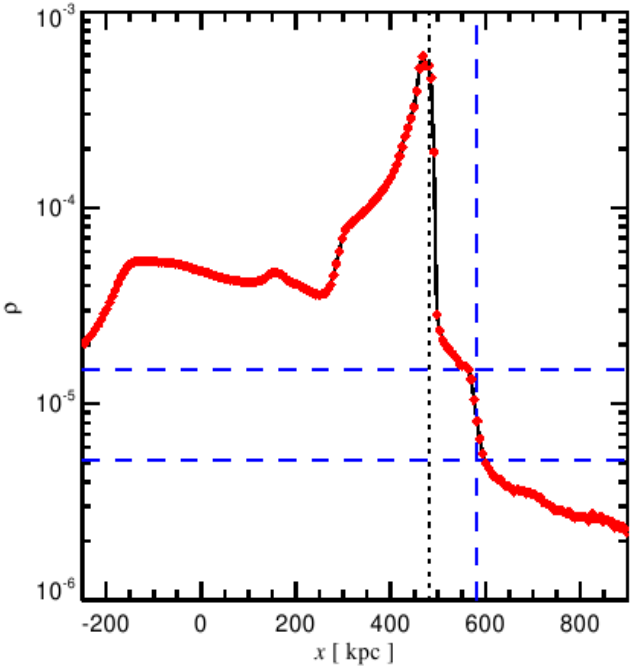


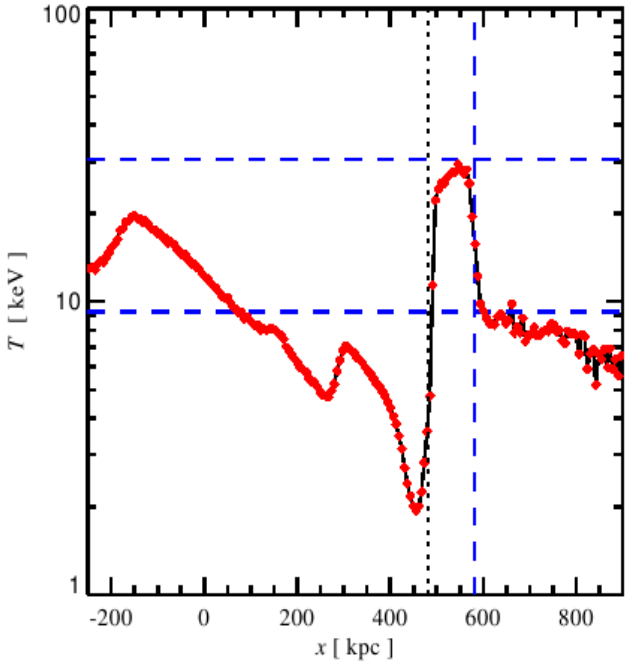
Springel & Farrar (2007)

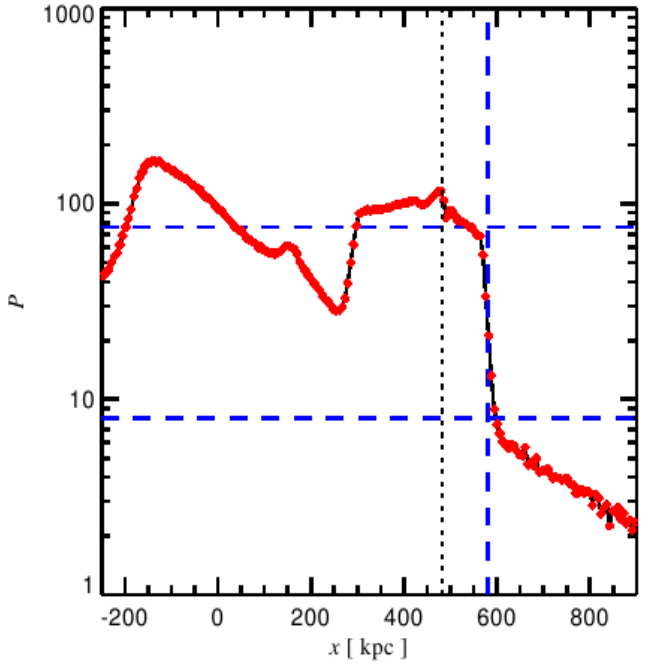
Profiles along the collision axis reveal a strong shock and a cold front

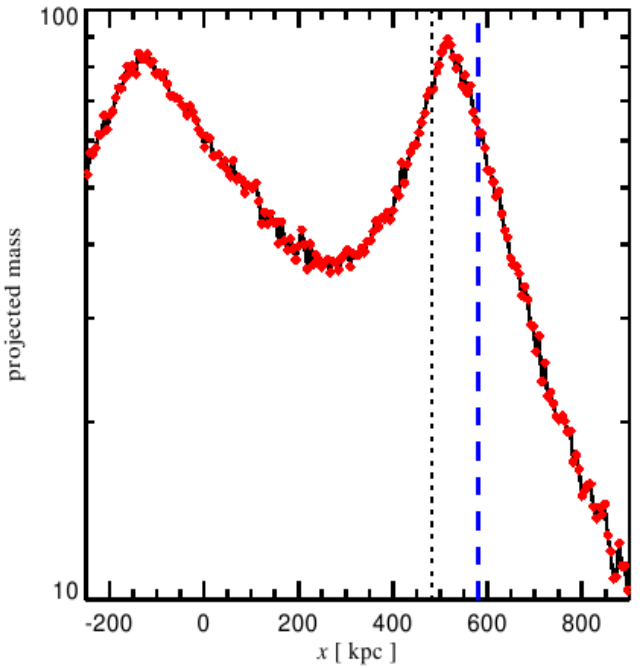






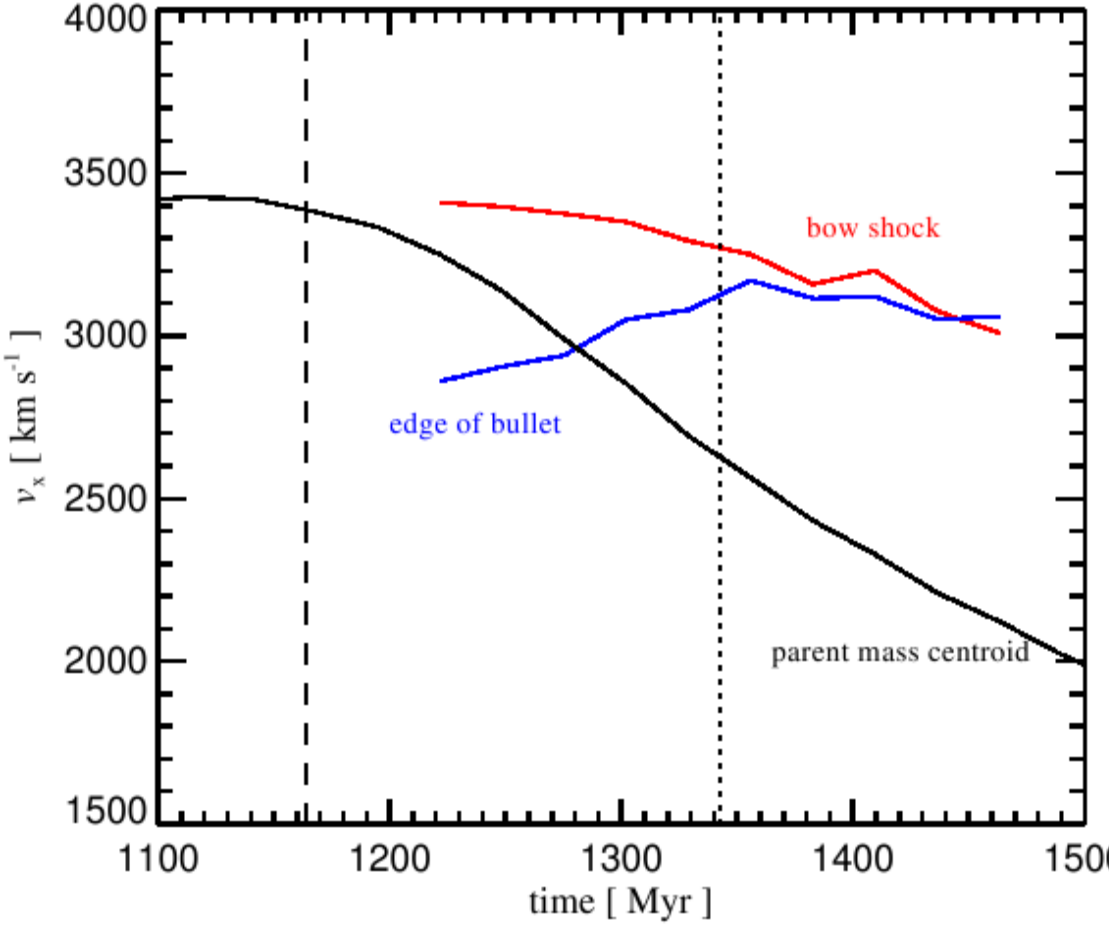


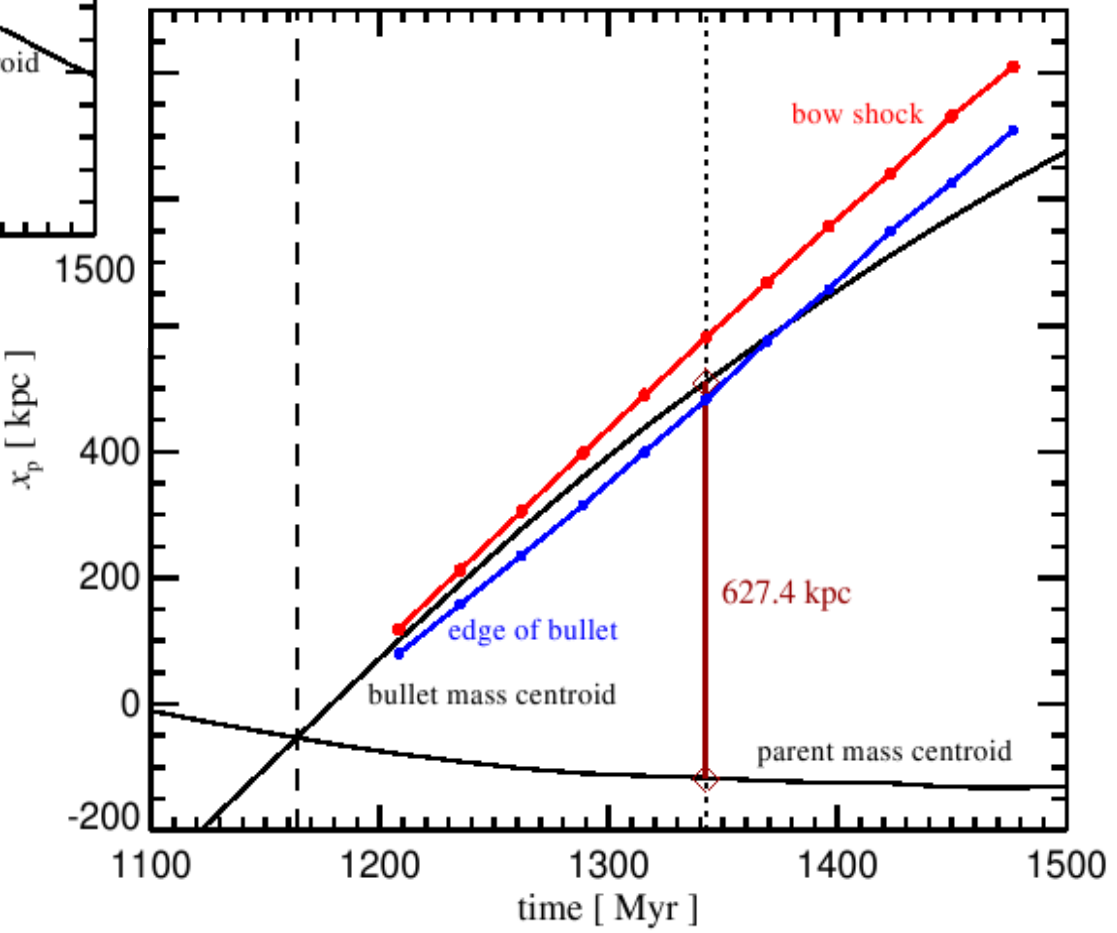




Despite a shock speed of ~4500 km/s, the bullet moves considerably slower

VELOCITIES AND POSITIONS OF MAIN BULLET CLUSTER FEATURES AS A FUNCTION OF TIME





Shock speed: 4500 km/s

Pre-shock infall: -1100 km/s

Shock speed relative to bullet: -800 km/s

Speed of bullet: 2600 km/s

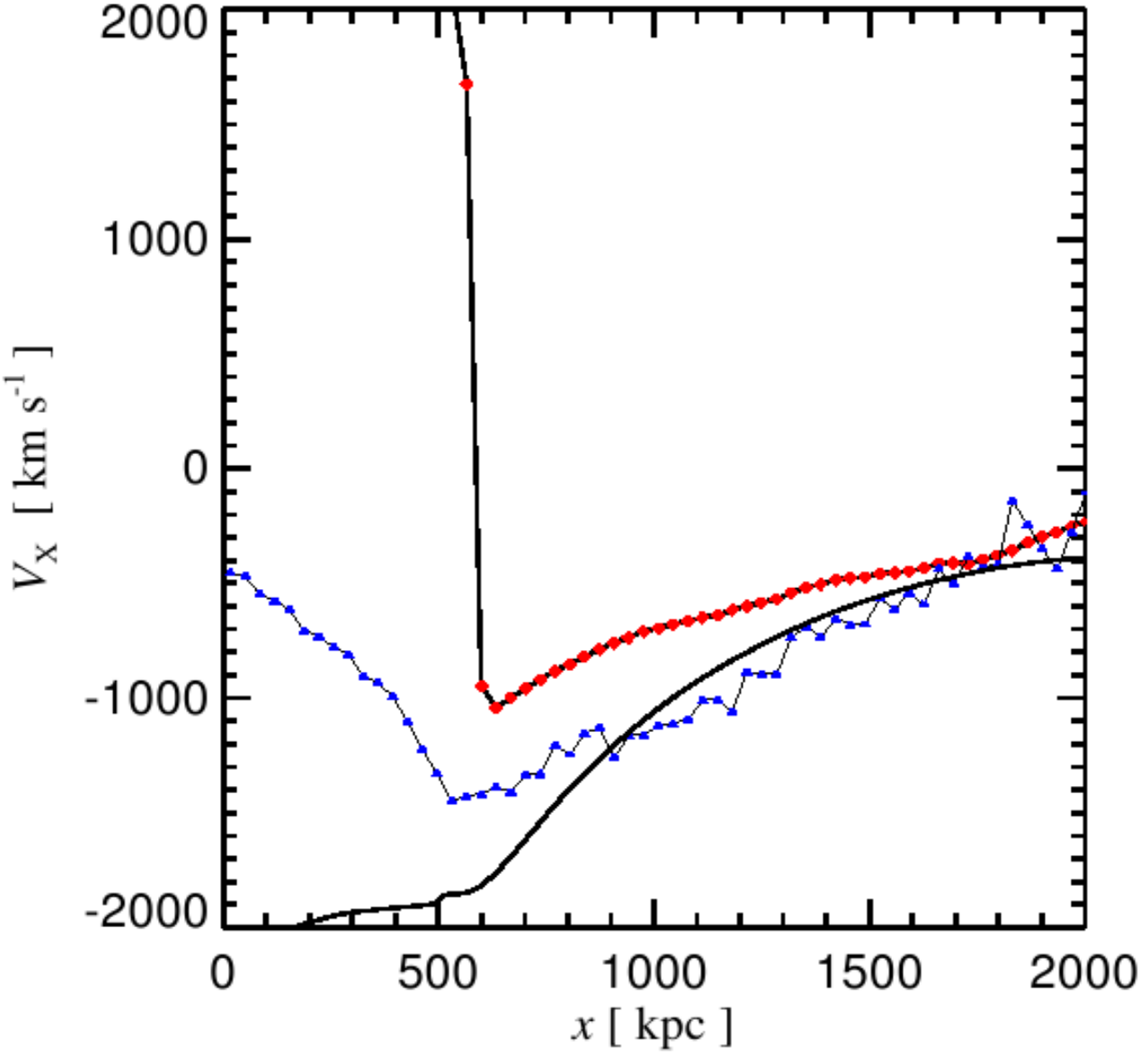
The change of the potential can be used to estimate the energy gained by the dark matter particles in the parent cluster region ahead of the shock

ESTIMATE OF THE EXPECTED DARK MATTER STREAMING VELOCITY COMPARED TO THE SIMULATION

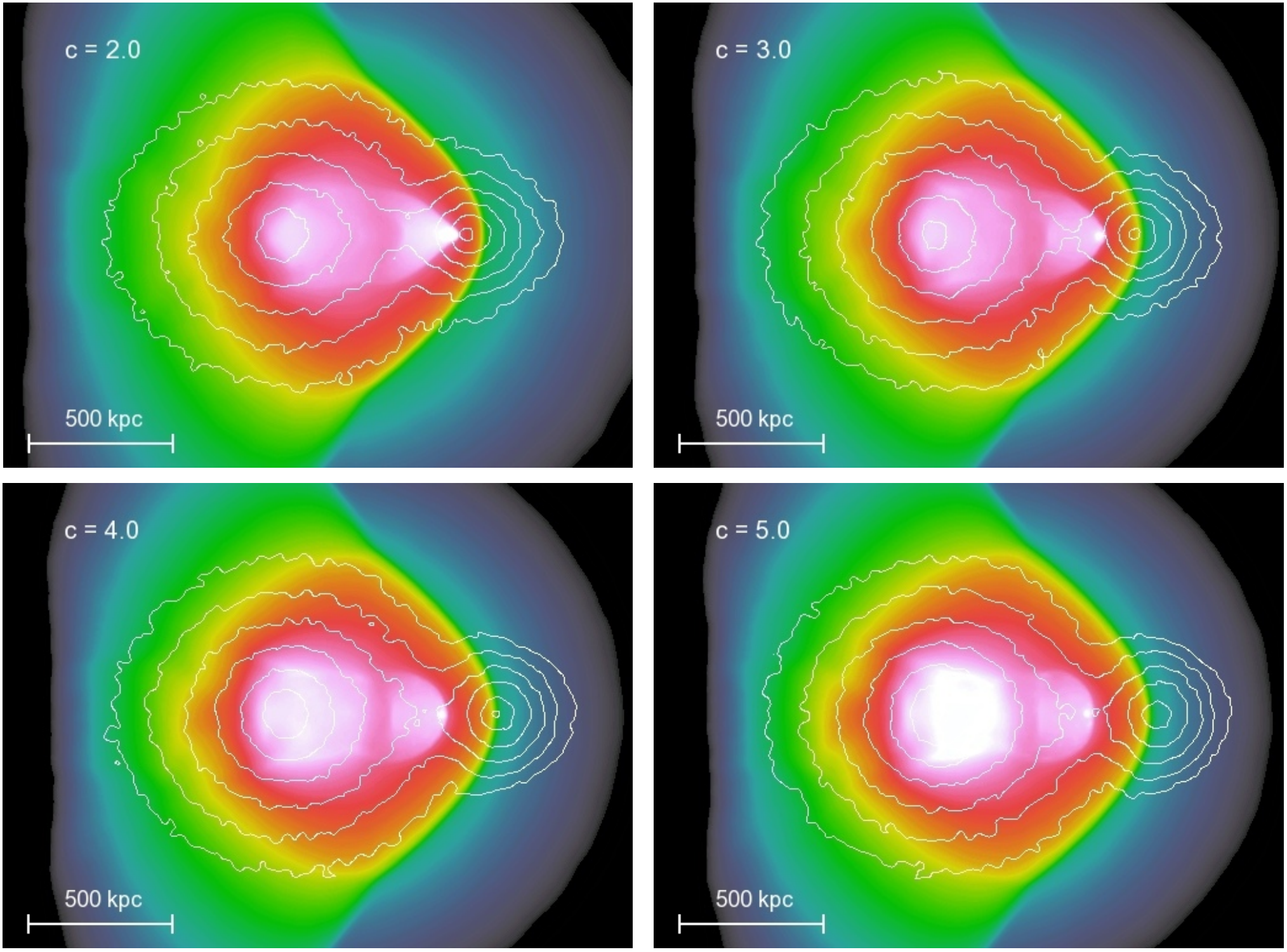
$$e_i = v_i^2/2 + \Phi(\vec{x}_i)$$

$$\mathrm{d}e_i/\mathrm{d}t = \partial\Phi/\partial t$$

$$\tilde{v}_r^2 \simeq v_r^2 - 2\Delta\Phi$$

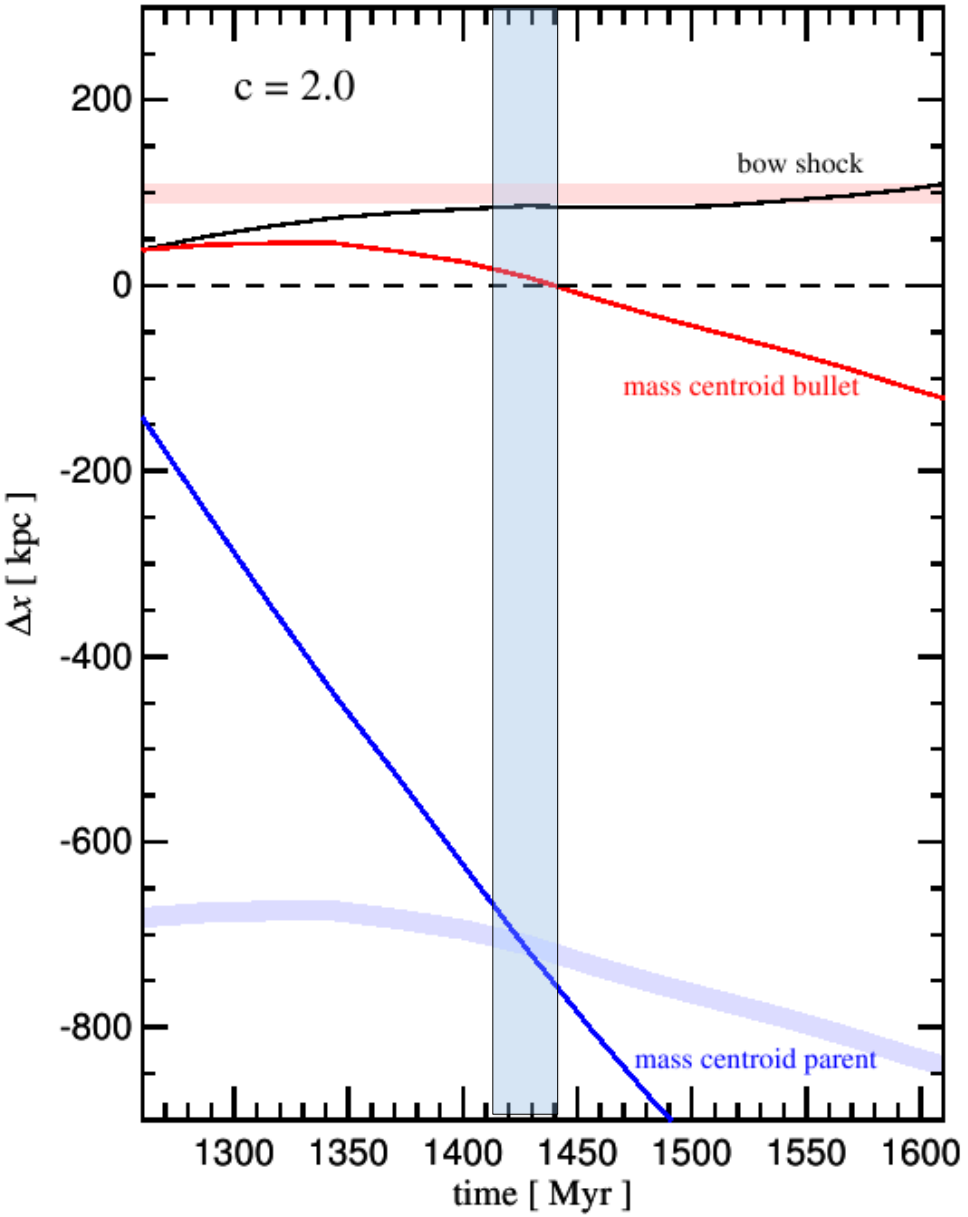


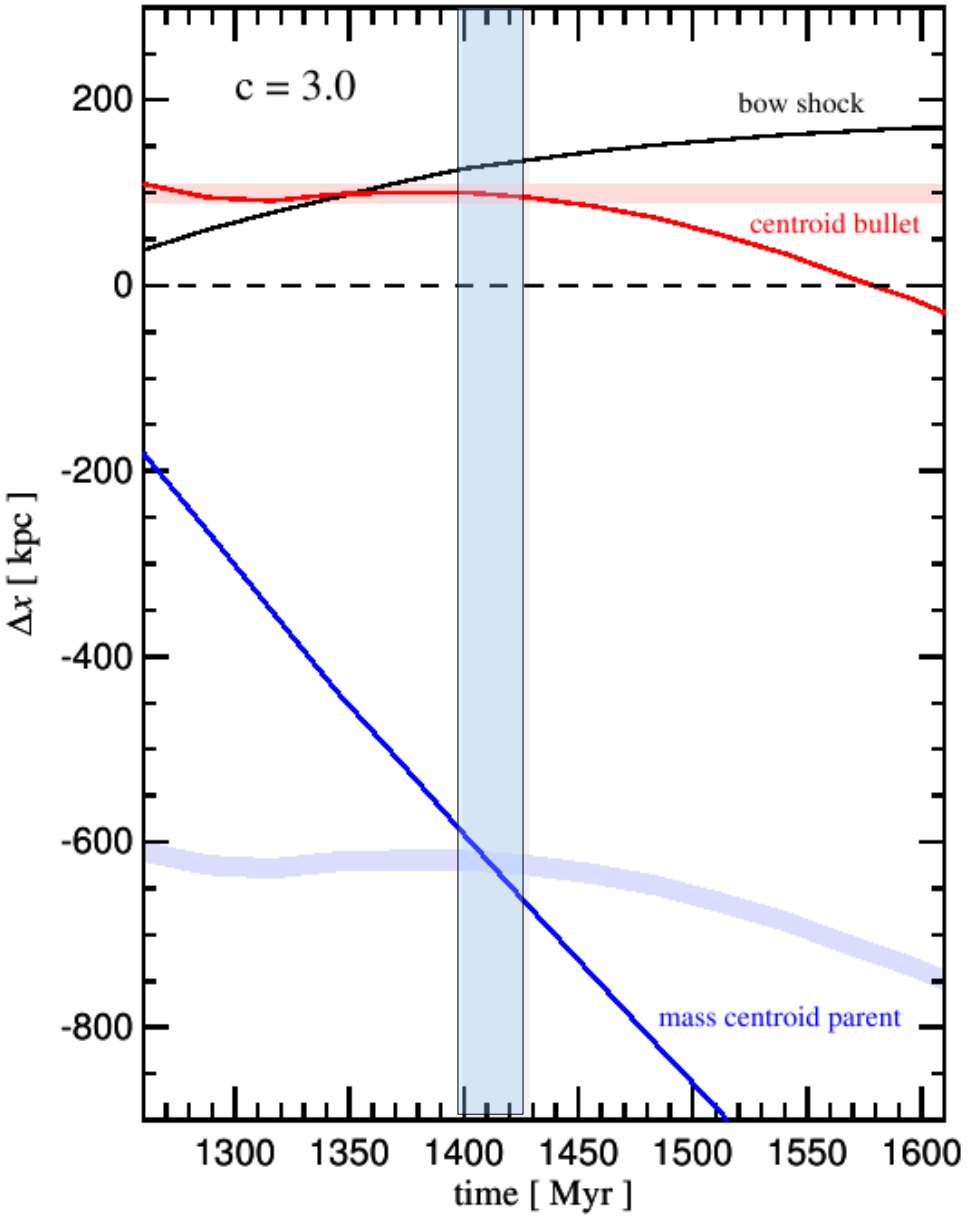
Changing the structural properties of the parent cluster



For c=3.0 for the parent cluster, the model can simultaneously fit the cluster-bullet separation and the offset between the bullet's gas and mass peak

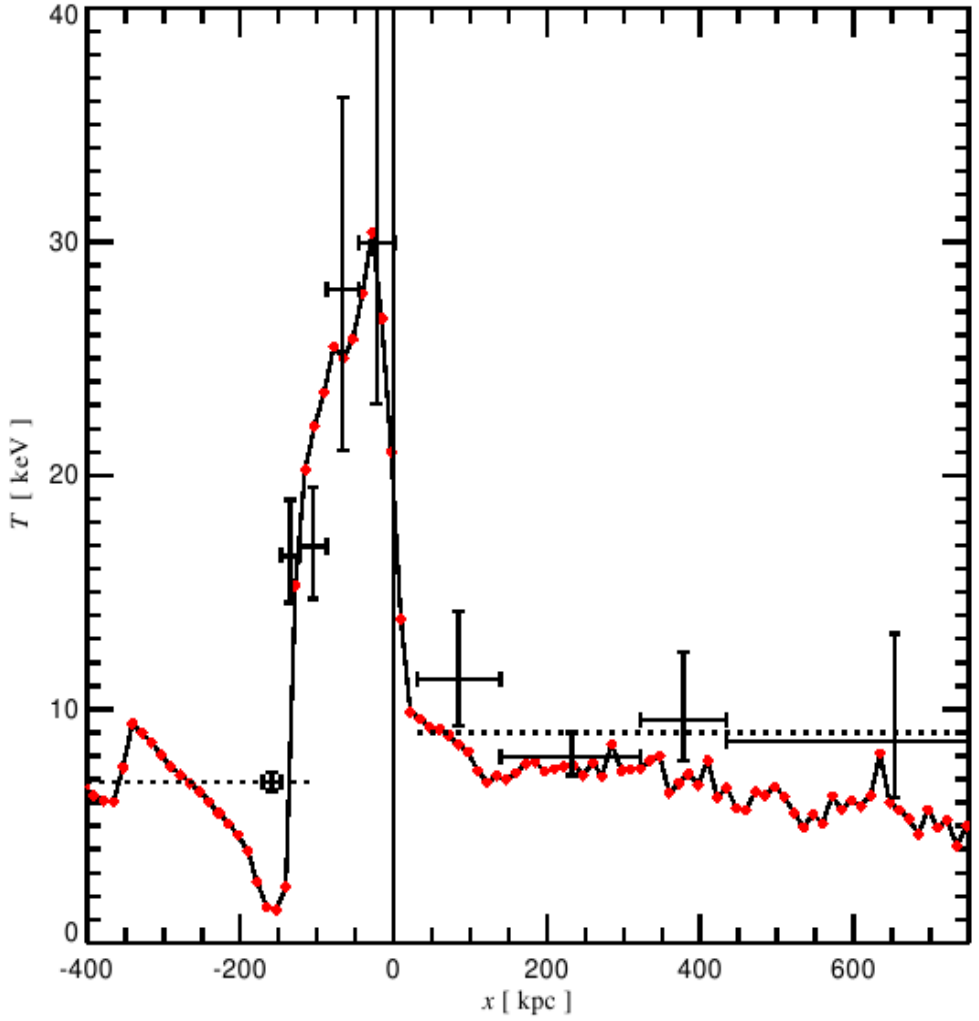
SEPARATION OF FEATURS AS A FUNCTION OF TIME



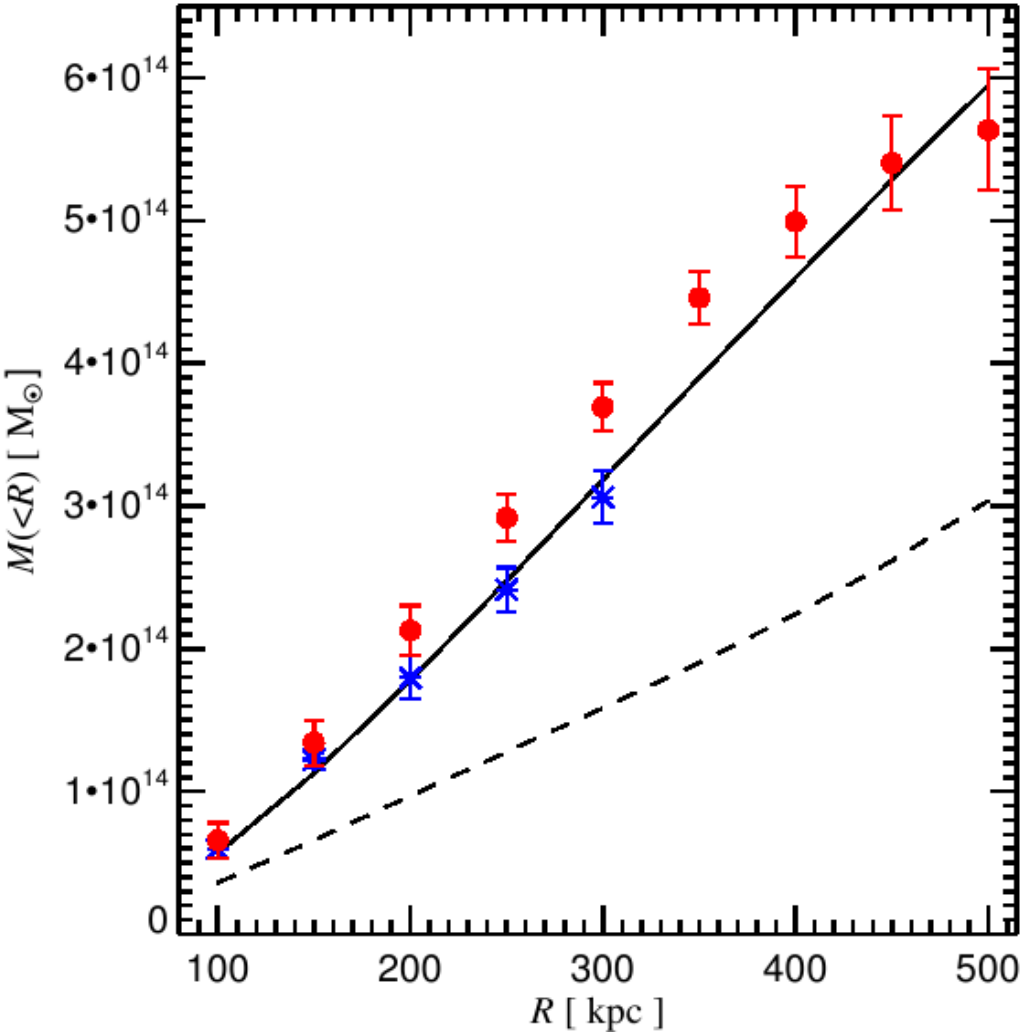


The model also matches the observed temperature and mass profiles

COMPARISON OF SIMULATED TEMPERATURE AND MASS PROFILE WITH OBSERVATIONS



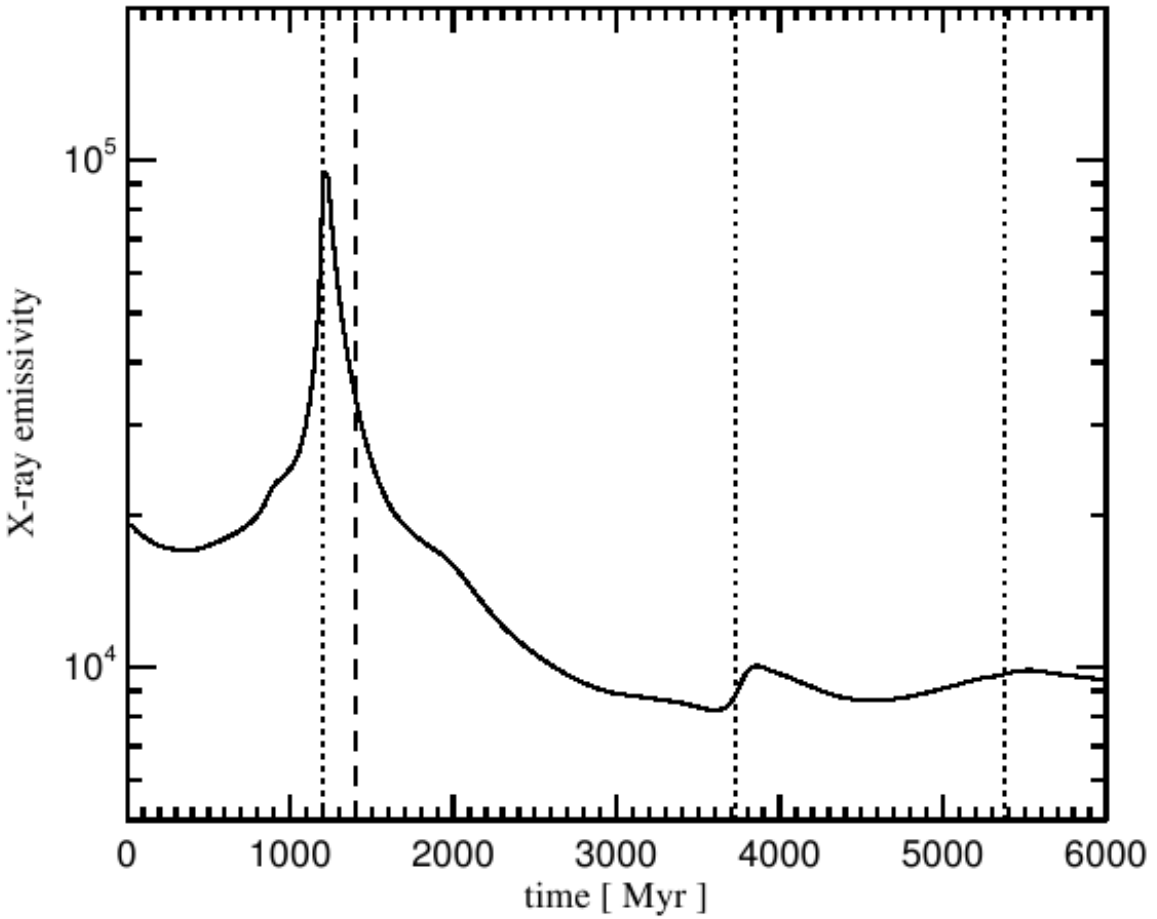
Data from **Markevitch et al. (2006)**

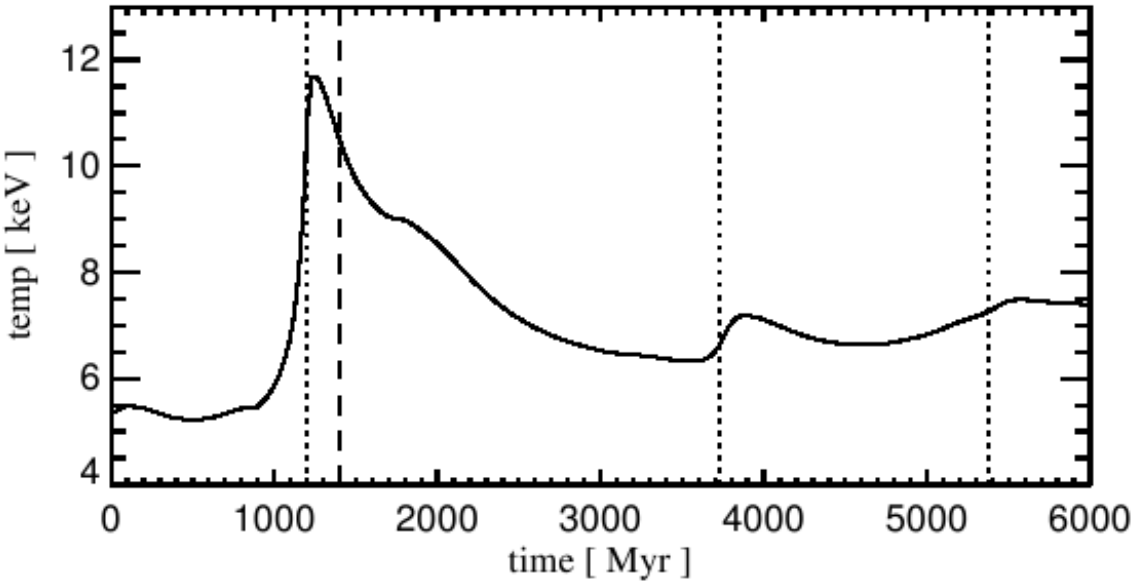


Data from **Bradac et al. (2006)**

The simulation model can be used to predict the future evolution of the bullet cluster

TIME EVOLUTION OF X-RAY EMISSION AND X-RAY TEMPERATURE





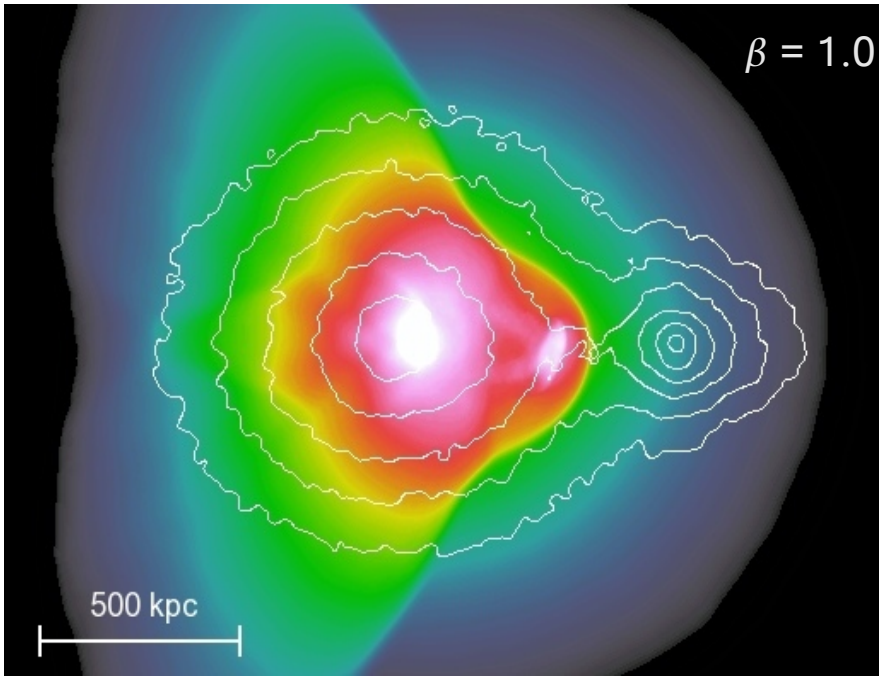
Models with a “fifth force” in the dark sector can speed up the bullet, but seem not required to match the bullet system

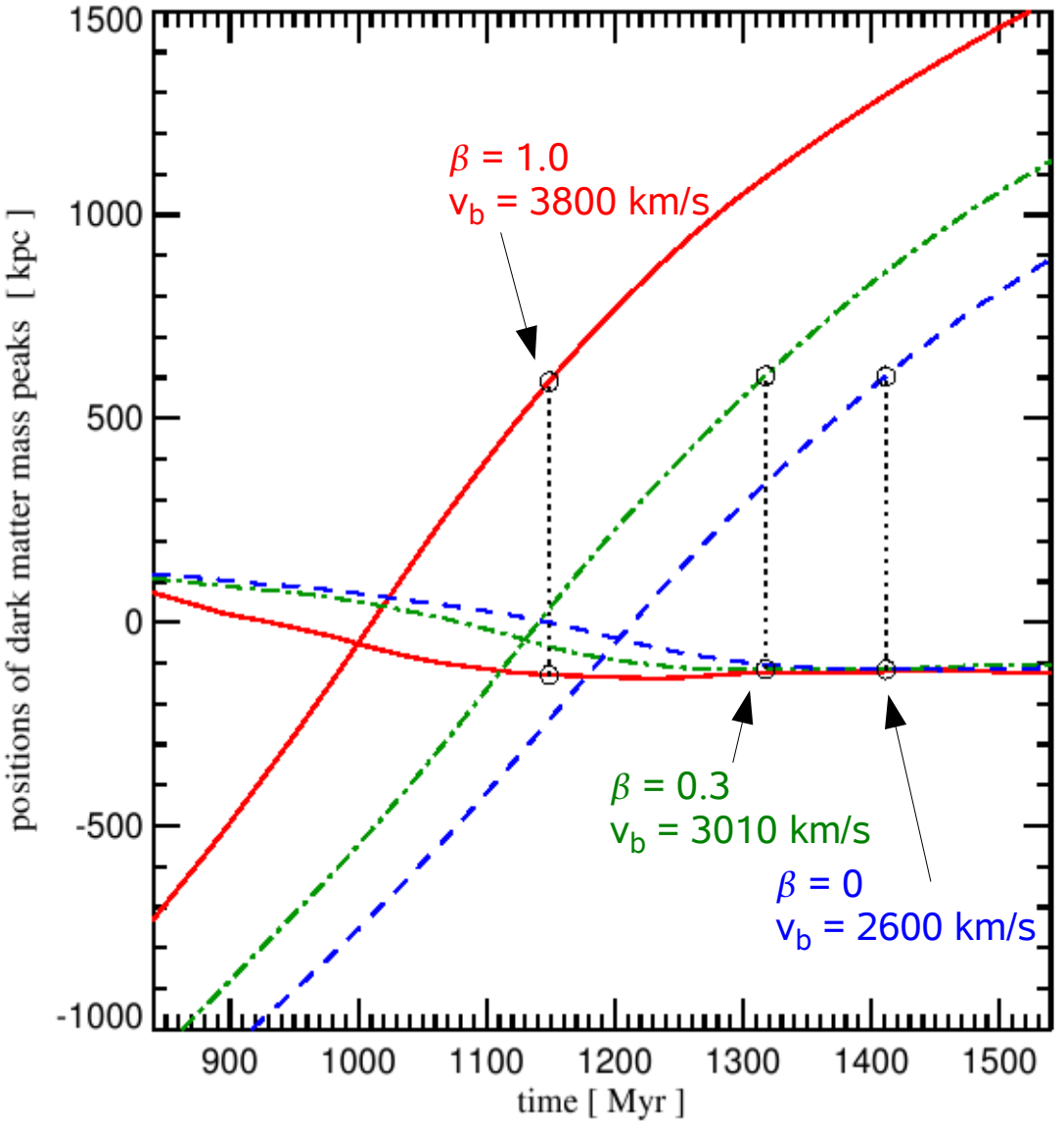
SPEED OF THE BULLET IN FIFTH FORCE MERGERS

(proposed by **Farrar & Rosen 2006)**

$$\phi_s(r) = -\beta \frac{Gm}{r} \exp\left(-\frac{r}{r_s}\right)$$

- $\beta = 1.0$, $r_s = 4$ Mpc
- $\beta = 0.3$, $r_s = 4$ Mpc





Examples for non-standard physics with SPH

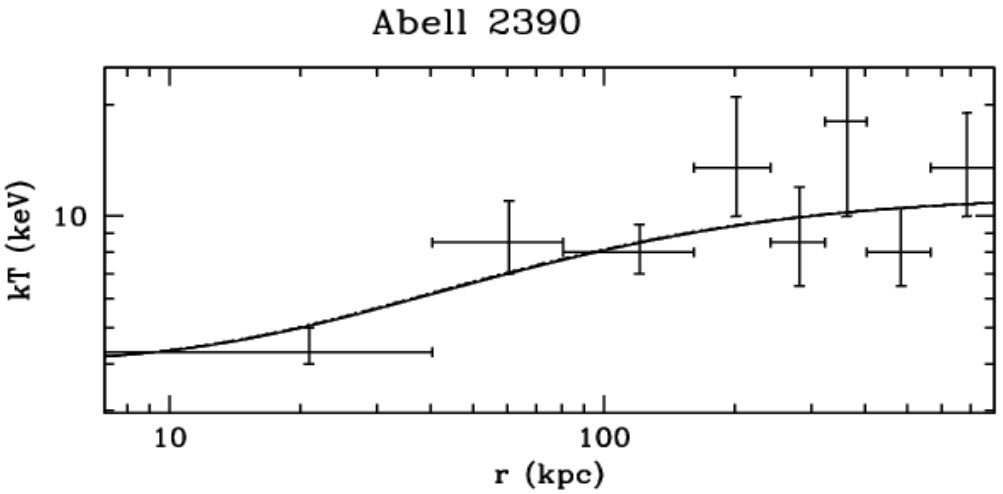
Thermal conduction may partially offset radiative cooling in central cluster regions

THE CONDUCTION IDEA

Inner region of clusters (~10-50 kpc) is cooler than the rest of the cluster → Is thermal conduction from the outer hot regions of the cluster the heat source?

Zakamska & Narayan (2003)

- Assume hydrostatic equilibrium with a balance between cooling and conductive heating
- Temperature profiles of five clusters can be well fit, requiring conductives of the order 30% Spitzer-value



BUT: Magnetic fields are the natural enemy of conduction....

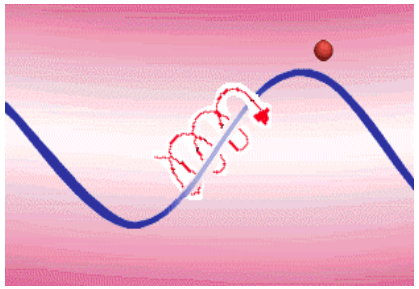
Magnetic fields are the natural enemy of thermal conduction

THE QUEST TO UNDERSTAND THE EFFECTIVE CONDUCTIVITY

Spitzer (1962) Conductivity of unmagnetized plasma:

$$\kappa_{\rm sp} = 1.31\, n_e \lambda_e k \left(\frac{kT_e}{m_e}\right)^{1/2} \qquad \lambda_e n_e = \frac{3^{3/2}(kT_E)^2}{4\sqrt{\pi}e^4 \ln\Lambda}$$

$$\rightarrow \kappa_{\rm sp} \propto T_e^{5/2}$$



If we have an ordered magnetic field:

$$\kappa_{||} \sim \kappa_{\rm sp}/3$$
$$\kappa_{\perp} \sim (r_L/\lambda)^2 \kappa_{\rm sp}$$

In clusters:

$$B \sim 10^{-6}\,{\rm G}$$
$$r_L \sim 10^{-12}\lambda$$

Rechester & Rosenbluth(1978)
Chandran & Cowley (1998)
Malyshkin & Kulsrud (2001)

If the field is **tangled**, the effective conductivity can be heavily suppressed:

$$\kappa_{\rm eff} \sim \kappa_{\rm sp}/100$$

Narayan & Medvedev (2001)

If the field is **chaotic on a range of turbulent scales**, conduction may almost reach the Spitzer value:

$$\kappa_{\rm turb} \sim \kappa_{\rm sp}/3$$

A robust and accurate implementation of thermal conduction in SPH

SPH DISCRETIZATION OF CONDUCTION

Conduction equation:

$$\mathbf{j} = -\kappa \nabla T \qquad \rho \frac{\mathrm{d}u}{\mathrm{d}t} = -\nabla \mathbf{j} \qquad \frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{\rho} \nabla (\kappa \nabla T)$$

Second-order derivative tends to be noisy...

SPH discretization:

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \sum_j \frac{m_j}{\rho_i \rho_j} \frac{(\kappa_j + \kappa_i)\ (T_j - T_i)}{|\mathbf{x}_{ij}|^2} \mathbf{x}_{ij} \nabla_i W_{ij}$$

Brookshaw (1985)

Problems encountered in practice:

- Explicit time integration can easily lead to instabilities
- Individual timestepping may easily lead to errors in energy conservation (conductivity depends strongly on temperature)

→ Best solved with implicit **time integration** schemes, which guarantee robustness

Derivation of the Laplace operator in SPH

Start with a Taylor expansion to third order

$$Y(\boldsymbol{x}_j) - Y(\boldsymbol{x}_i) = \nabla Y\Big|_{\boldsymbol{x}_i} \cdot (\boldsymbol{x}_j - \boldsymbol{x}_i) \; + \frac{1}{2}\frac{\partial^2 Y}{\partial x_s \partial x_k}\Big|_{\boldsymbol{x}_i} (\boldsymbol{x}_j - \boldsymbol{x}_i)_s (\boldsymbol{x}_j - \boldsymbol{x}_i)_k \; + \mathcal{O}(\boldsymbol{x}_j - \boldsymbol{x}_i)^3$$

We multiply through with the factor $\dfrac{(\boldsymbol{x}_j - \boldsymbol{x}_i)\nabla_i W(\boldsymbol{x}_j - \boldsymbol{x}_i)}{|\boldsymbol{x}_j - \boldsymbol{x}_i|^2}$ and integrate over space.

Noting that $\displaystyle\int \boldsymbol{x}_{ij} \frac{\boldsymbol{x}_{ij}\nabla_i W_{ij}}{|\boldsymbol{x}_{ij}|^2} \mathrm{d}^3\boldsymbol{x}_j = 0$ $\displaystyle\int (\boldsymbol{x}_{ij})_s (\boldsymbol{x}_{ij})_k \frac{\boldsymbol{x}_{ij}\nabla_i W_{ij}}{|\boldsymbol{x}_{ij}|^2} \mathrm{d}^3\boldsymbol{x}_j = \delta_{sk}$

we arrive at $$\nabla^2 Y\Big|_{\boldsymbol{x}_i} = -2\int \frac{Y(\boldsymbol{x}_j) - Y(\boldsymbol{x}_i)}{|\boldsymbol{x}_{ij}|^2} \boldsymbol{x}_{ij}\nabla_i W_{ij} \, \mathrm{d}^3\boldsymbol{x}_j$$

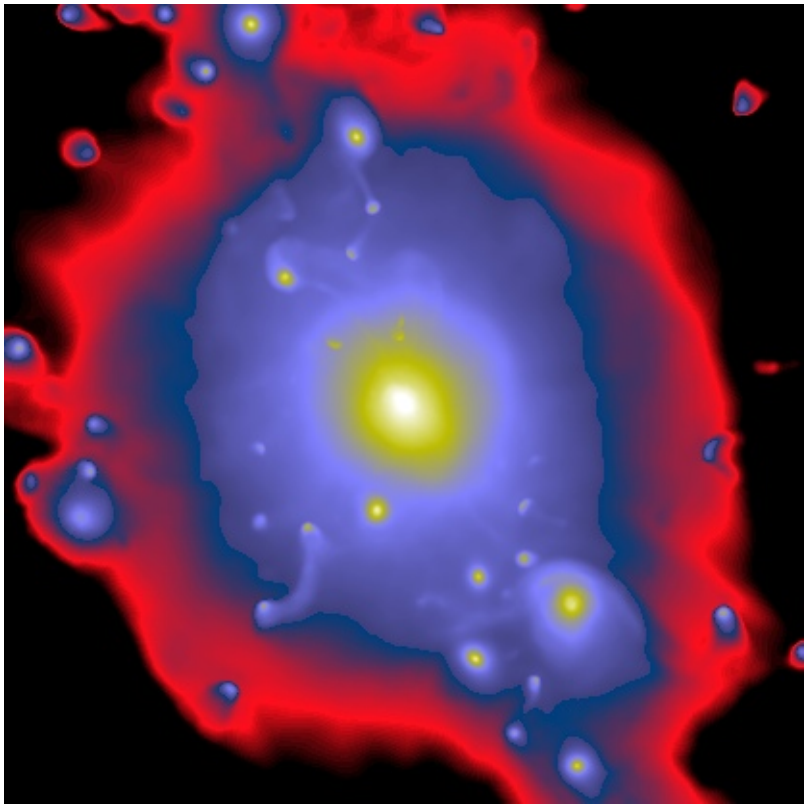
Finally, change integration to SPH sum: $$\nabla^2 Y\Big|_i = -2\sum_j \frac{m_j}{\rho_j} \frac{Y_j - Y_i}{|\boldsymbol{x}_{ij}|^2} \boldsymbol{x}_{ij}\nabla_i W_{ij}$$

Self-consistent cosmologicals simulations of cluster formation can be used to study the impact of conduction on the ICM

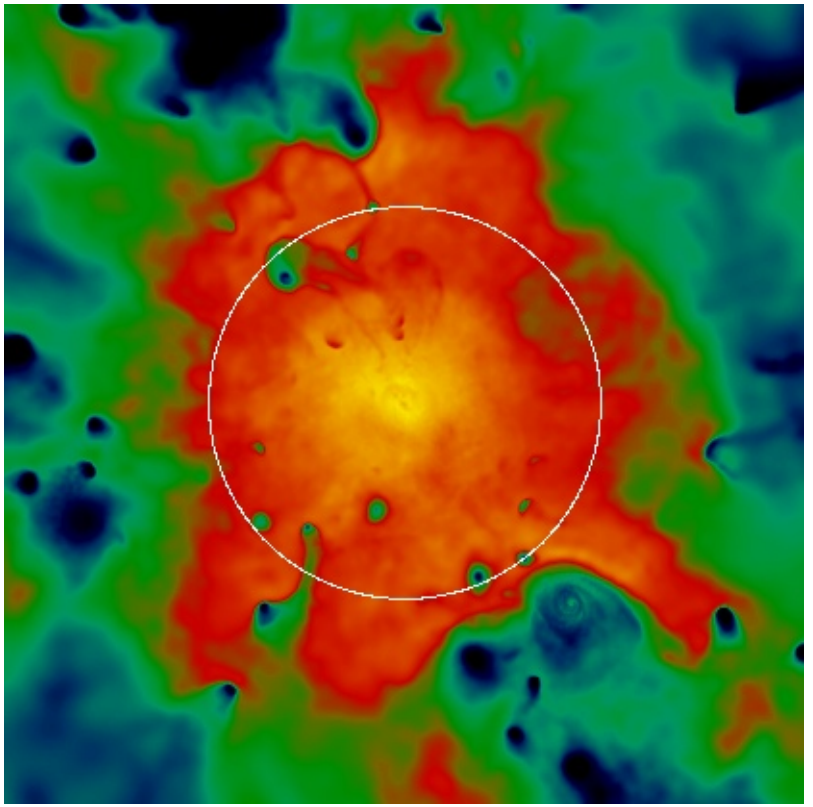
X-RAY AND TEMPERATURE MAPS

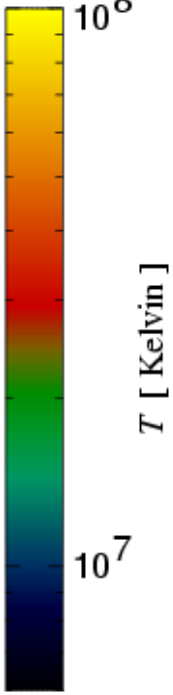
Coma-sized cluster, $M_{vir} \sim 10^{15}\ M_{\odot}$, adiabatic hydrodynamics

Gas density (X-rays)



Mass-weighted temperature



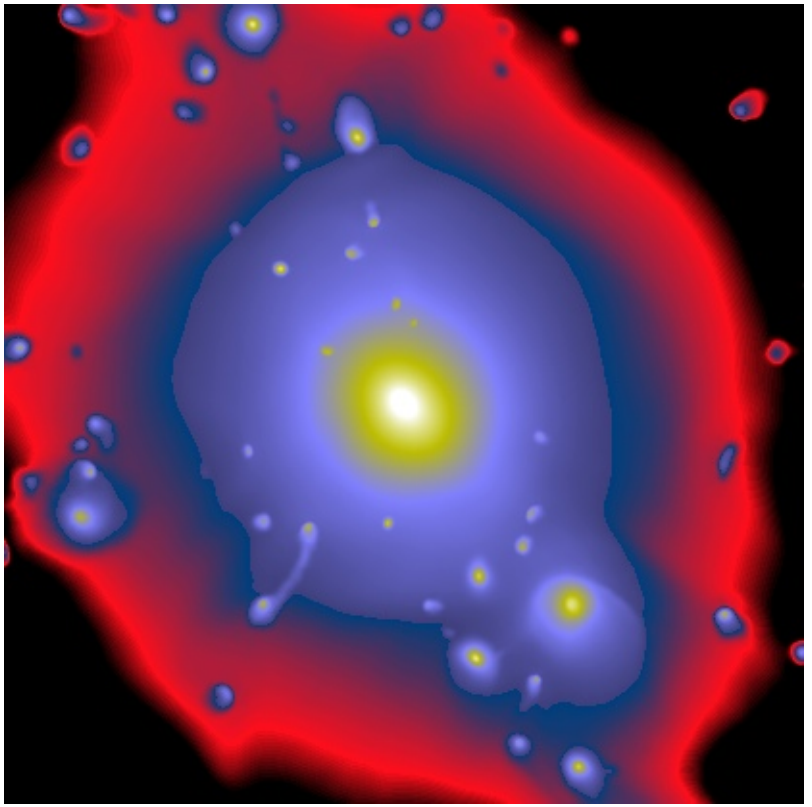


Thermal conduction near the Spitzer value strongly affects rich clusters of galaxies

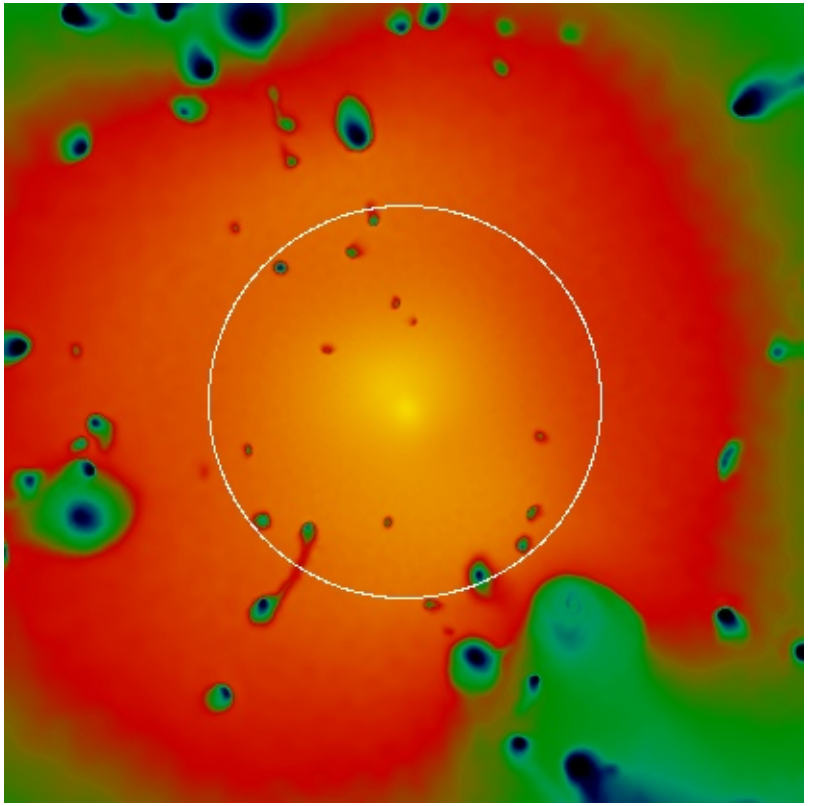
X-RAY AND TEMPERATURE MAPS

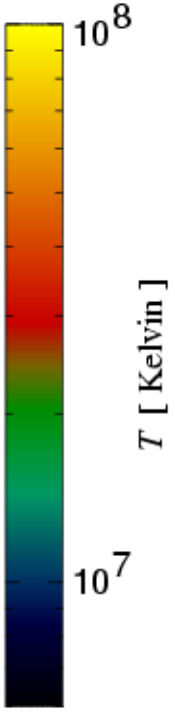
Coma-sized cluster, $M_{vir} \sim 10^{15}\ M_{\odot}$,
adiabatic hydrodynamics, **thermal condution with $\kappa = \kappa_{sp}$**

Gas density (X-rays)



Mass-weighted temperature





Physical viscosity in SPH

One can also derive an SPH discretization of the Navier-Stokes equations

SPH WITH PHYSICAL VISCOUS STRESSES

Viscous stresses modify the momentum flux density tensor: $\Pi_{ik} = p\delta_{ik} + \rho v_i v_k - \sigma_{ik}$

The stress tensor can be written as:

$$\sigma_{ik} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right) + \zeta\delta_{ik}\frac{\partial v_l}{\partial x_l}$$

Shear viscosity coefficient

Bulk viscosity coefficient

The Euler equation of ideal gas dynamics is then replaced by the **Navier Stokes equations**:

$$\rho\left(\frac{\partial v_i}{\partial t} + v_k\frac{\partial v_i}{\partial x_k}\right) = -\frac{\partial p}{\partial x_i} - \rho\frac{\partial \Phi}{\partial x_i} + \frac{\partial}{\partial x_k}\left[\eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)\right] + \frac{\partial}{\partial x_i}\left(\zeta\frac{\partial v_l}{\partial x_l}\right)$$

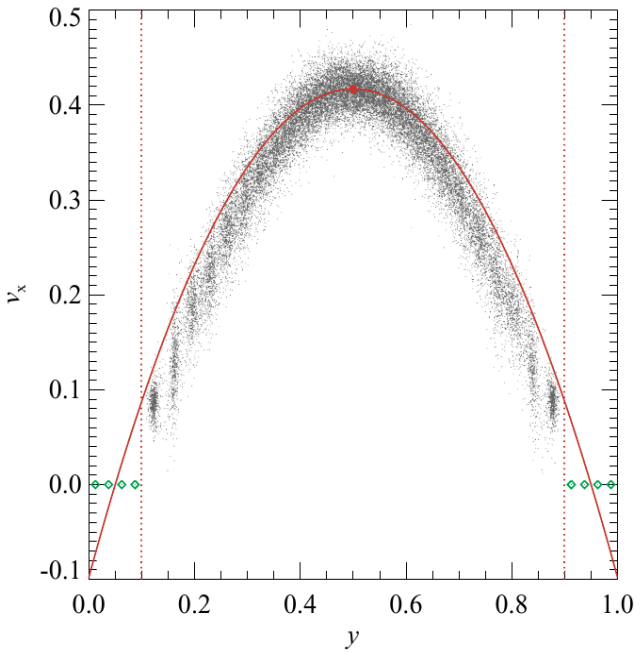
If conduction is also included, the thermal energy equation becomes the **generalized heat transfer equation:**

$$\rho T\frac{\mathrm{d}S}{\mathrm{d}t} = \nabla(\kappa\,\nabla T) + \frac{1}{2}\eta\,\sigma_{\alpha\beta}\sigma_{\alpha\beta} + \zeta(\nabla\, v)^2$$

SPH discretization of the Navier-Stokes equations

SPH WITH PHYSICAL VISCOUS STRESSES

Viscous flow between two plates



$$\left.\frac{\partial v_\alpha}{\partial x_\beta}\right|_i = \frac{1}{\rho_i}\sum_{j=1}^{N} m_j (\boldsymbol{v_j} - \boldsymbol{v_i})|_\alpha \, [\nabla_i W_{ij}(h_i)]|_\beta$$

$$\left.\sigma_{\alpha\beta}\right|_i = \eta\left(\left.\frac{\partial v_\alpha}{\partial x_\beta}\right|_i + \left.\frac{\partial v_\beta}{\partial x_\alpha}\right|_i - \frac{2}{3}\delta_{\alpha\beta}\left.\frac{\partial v_\gamma}{\partial x_\gamma}\right|_i\right) + \zeta\,\delta_{\alpha\beta}\left.\frac{\partial v_\gamma}{\partial x_\gamma}\right|_i$$

$$\left.\frac{\mathrm{d}v_\alpha}{\mathrm{d}t}\right|_{i,\mathrm{shear}} = \sum_{j=1}^{N} m_j \left[\frac{\eta_i \sigma_{\alpha\beta}|_i}{\rho_i^2}\,[\nabla_i W_{ij}(h_i)]|_\beta + \frac{\eta_j \sigma_{\alpha\beta}|_j}{\rho_j^2}\,[\nabla_i W_{ij}(h_j)]|_\beta\right]$$

$$\left.\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right|_{i,\mathrm{bulk}} = \sum_{j=1}^{N} m_j \left[\frac{\zeta_i\,\nabla\cdot\boldsymbol{v}_i}{\rho_i^2}\,\nabla_i W_{ij}(h_i) + \frac{\zeta_j\,\nabla\cdot\boldsymbol{v}_j}{\rho_j^2}\,\nabla_i W_{ij}(h_j)\right]$$

$$\left.\frac{\mathrm{d}A_i}{\mathrm{d}t}\right|_{\mathrm{shear}} = \frac{1}{2}\,\frac{\gamma-1}{\rho_i^{\gamma-1}}\,\frac{\eta_i}{\rho_i}\,\sigma_i^2$$

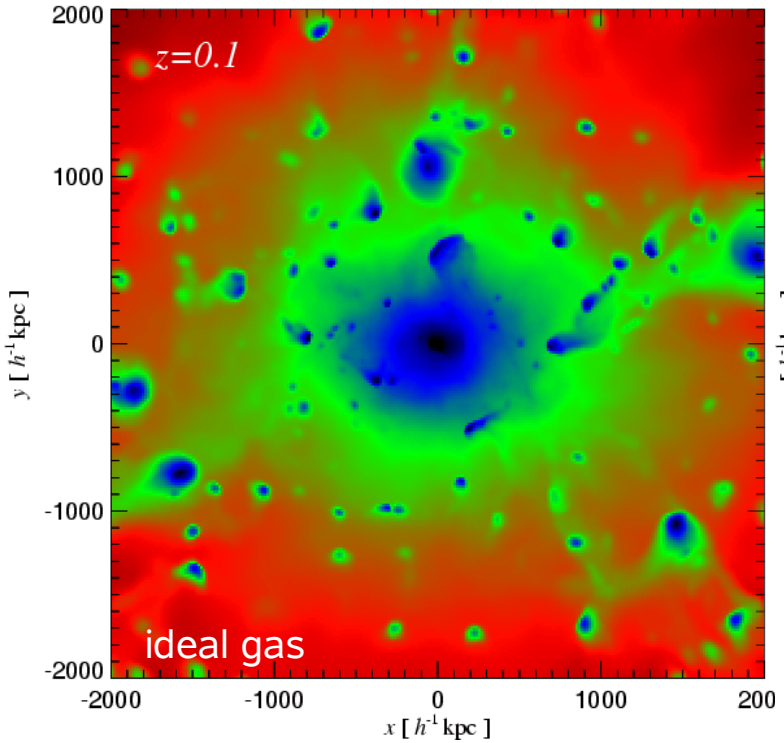
$$\left.\frac{\mathrm{d}A_i}{\mathrm{d}t}\right|_{\mathrm{bulk}} = \frac{\gamma-1}{\rho_i^{\gamma-1}}\,\frac{\zeta_i}{\rho_i}\,(\nabla\cdot\boldsymbol{v}_i)^2$$

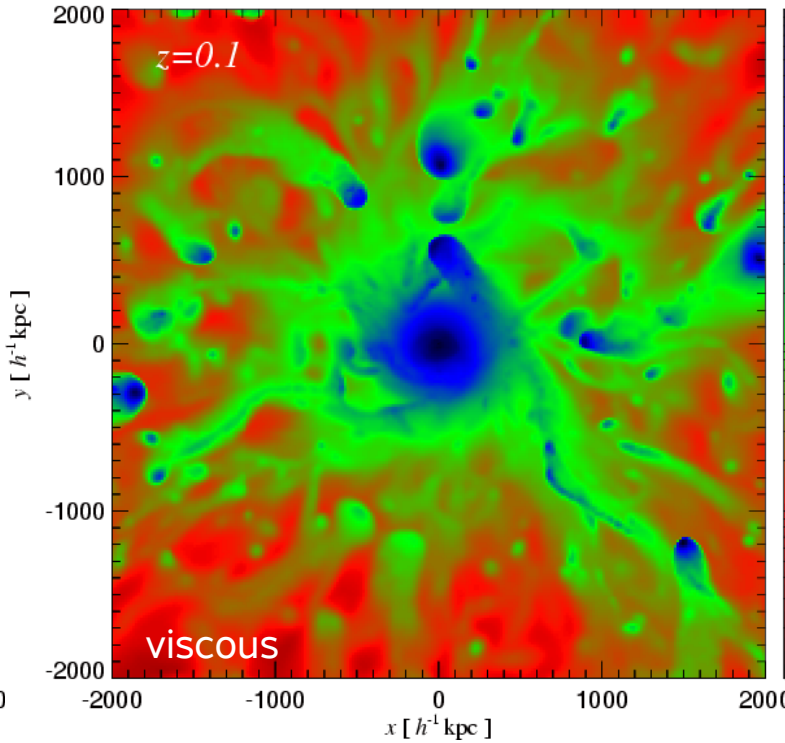
Sijacki & Springel (2006)

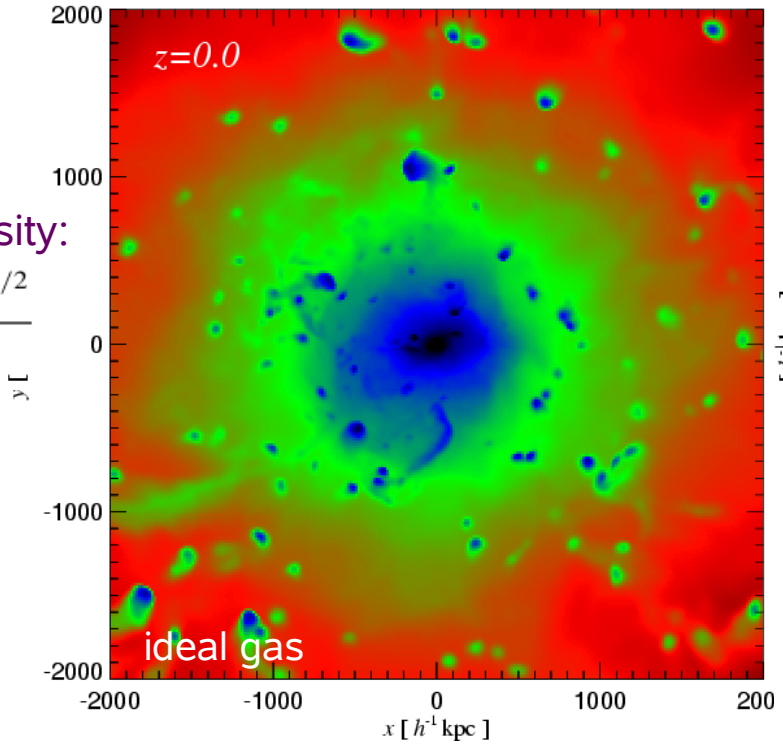
Viscous shear changes gas stripping during cluster assembly

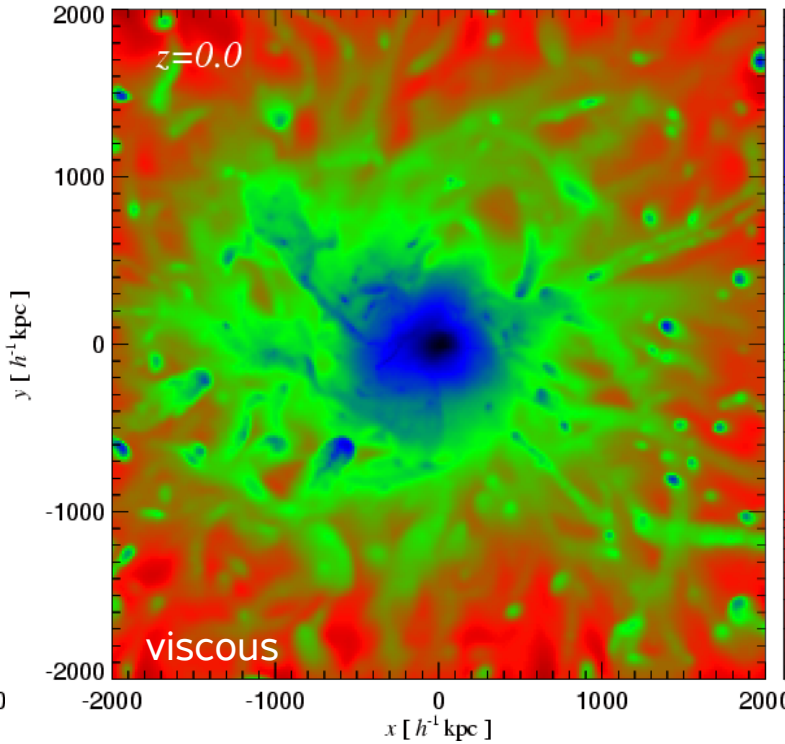
COMPARISON OF PROJECTED GAS DENSITY MAPS

Sijacki & Springel (2006)









Braginskii shear viscosity:

$$\eta = 0.406 \frac{m_{\rm i}^{1/2} (k_{\rm B} T_{\rm i})^{5/2}}{(Ze)^4 \ln \Lambda}$$

Radiative transfer in SPH

Different radiative transfer schemes have been proposed for SPH

TRACING RAYS

Radiative transfer equation in comoving coordinates

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \frac{\hat{\boldsymbol{n}}\cdot\nabla I_\nu}{\bar{a}} - \frac{H}{c}\left(\nu\frac{\partial I_\nu}{\partial \nu} - 3I_\nu\right) = \epsilon_\nu - \kappa_\nu I_\nu$$

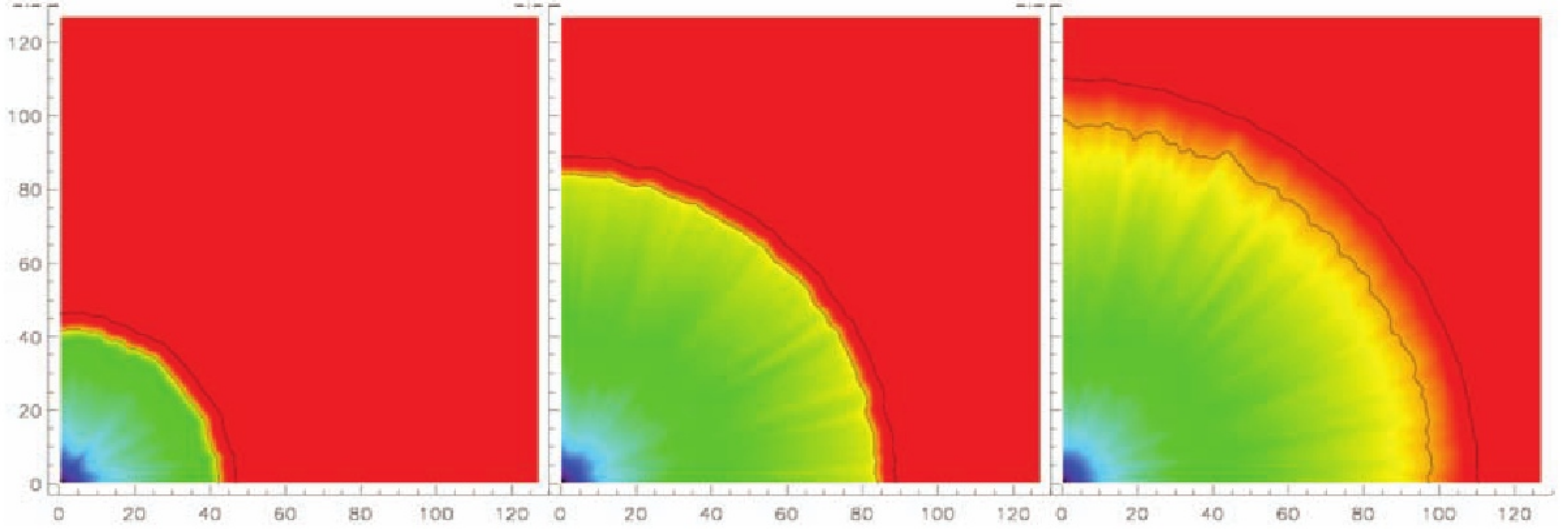
Altay, Croft & Pelupessy (2008) **SPHRAY** code (publicly available)

Individual rays are integrated in a Monte Carlo fashion through the SPH density field. The optical depth along a ray is calculated from the column density.

$$N_{\rm cd} = \int_0^L \rho(\boldsymbol{r})\,{\rm d}l \approx \int_0^L \sum_{j=1}^{N} m_j W(r_{lj}, h_j)\,{\rm d}l$$

Uses intersection tests motivated by computer graphics algorithms for efficiently evaluating SPH sum.

Strömgren sphere problem



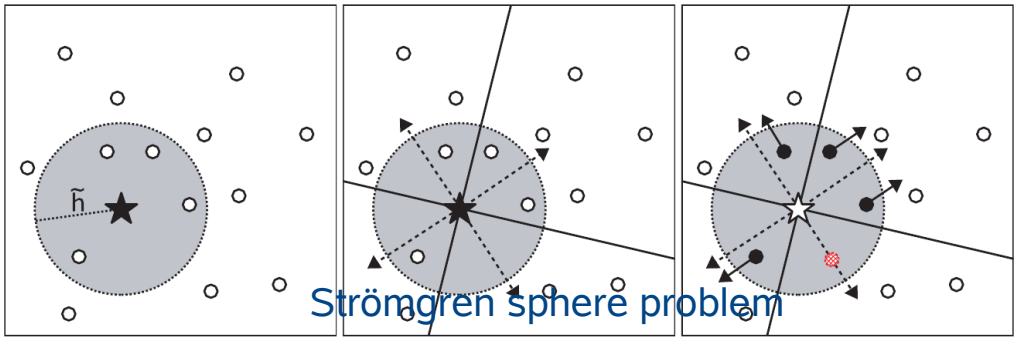
Different radiative transfer schemes have been proposed for SPH

FOLLOWING THE RADIATION FIELD IN CONES

Pawlik & Schaye (2008)

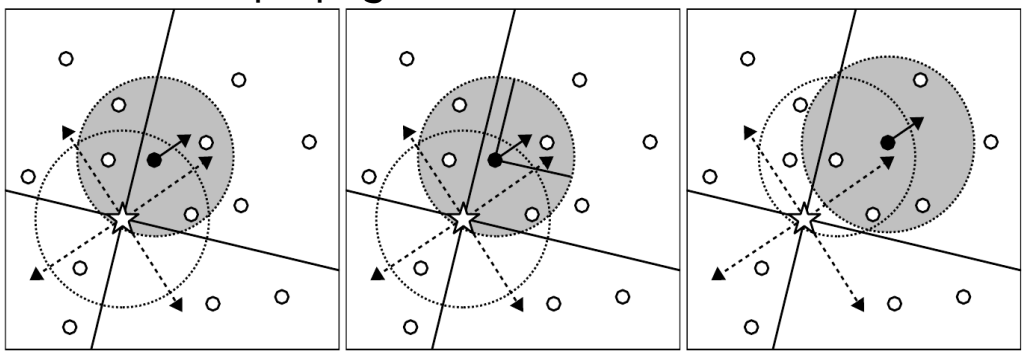
TRAPHIC code

Source radiation is injected into emission cones

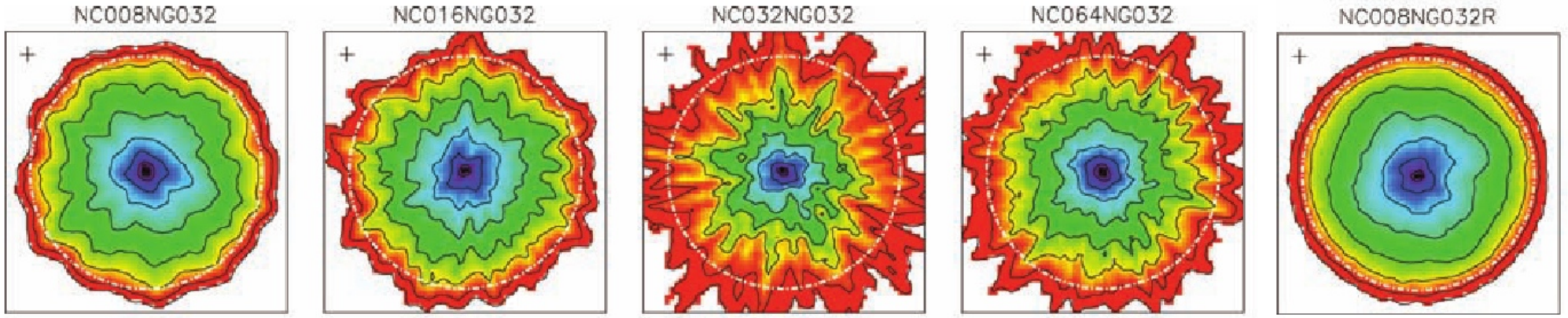


Strömgren sphere problem

In transfer steps, radiation is received from cones and propagated into emission cones



Strömgren sphere problem



Different radiative transfer schemes have been proposed for SPH

MOMENT EQUATIONS

Ansatz: Take moments of the radiative transfer equation and achieve closure with an estimated Eddington tensor.

Mean intensity $$J_\nu = \frac{1}{4\pi}\int \mathrm{d}\Omega\, I_\nu$$

Radiation flux $$F_\nu^i = \frac{1}{4\pi}\int \mathrm{d}\Omega\, n^i I_\nu$$

Radiation pressure $$P_\nu^{ij} = \frac{1}{4\pi}\int \mathrm{d}\Omega\, n^i n^j I_\nu$$

$$\frac{\partial J_\nu}{\partial t} = \frac{c}{a^2}\frac{\partial}{\partial x_j}\left(\frac{1}{\hat{\kappa}_\nu}\frac{\partial J_\nu h^{ij}}{\partial x_i}\right) - c\hat{\kappa}_\nu J_\nu + c j_\nu$$

Eddington tensor $$h^{ij} = \frac{P^{ij}}{\mathrm{Tr}(P)}$$

Radiative transfer becomes an anisotropic diffusion problem.

Optically Thin Variable Eddington Tensor (OTVET) approximation (Gnedin & Abel 2001)

$$P^{ij} \propto \int \mathrm{d}^3x'\,\rho_*(\boldsymbol{x}')\frac{(\boldsymbol{x}-\boldsymbol{x}')_i(\boldsymbol{x}-\boldsymbol{x}')_j}{(\boldsymbol{x}-\boldsymbol{x}')^4}$$

→ Can be calculated efficiently with a tree algorithm alongside gravitational force

Petkova & Springel (2009)

Different radiative transfer schemes have been proposed for SPH

OTVET SCHEME IN GADGET-3

Petkova & Springel (2009)

The anisotropic radiative diffusion problem can be discretized in SPH:

$$\frac{\partial N_i}{\partial t} = \sum_j w_{ij}(N_j - N_i) - c\hat{\kappa}_i N_i$$

$$w_{ij} \equiv \frac{2\,c\,m_{ij}}{\kappa_{ij}\rho_{ij}} \frac{\boldsymbol{x}_{ij}^{\mathrm{T}} \tilde{\boldsymbol{h}}_{ij} \nabla_i W_{ij}}{\boldsymbol{x}_{ij}^2}$$

photon number

Numerical stability requires an **implicit integration** (backwards Euler):

$$N_i^{n+1} = N_i^n + \Delta t \tilde{s}_i m_i + \sum \Delta t\, w_{ij}(N_j^{n+1} - N_i^{n+1}) - \Delta t c \sigma_0 n_{\mathrm{H\,I}} N_i^{n+1}$$

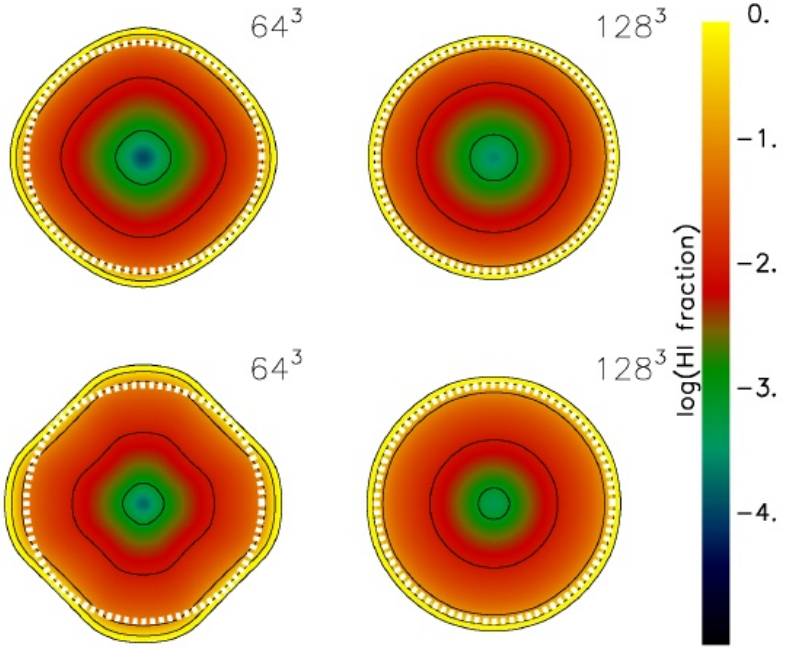
Finding the new raditation field is equivalent to solving a large linear system:

$$\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$$

$$A_{ij} = \delta_{ij}\left(1 + \sum_k \Delta t\, w_{ik} + \Delta t\, c\, \sigma_0 n_{\mathrm{H\,I}}\right) - \Delta t\, w_{ij}$$

As the matrix is symmetric and positive definite, the system can be solved iteratively with the **conjugate gradient method**. Jacobi **preconditioning** can be used to speed up convergence.

Strömgren sphere problem



Magnetic fields in SPH

It is possible to treat MHD in SPH, but divB errors remain problematic in the formulations proposed thus far

SPH MHD FORMULATIONS

(1) Direct discretization of the MHD equations in terms of *B*

Dolag et al. (1999)

Even when div B = 0 initially, the errors usually blow up when the magnetic forces become comparable to thermal pressure forces.

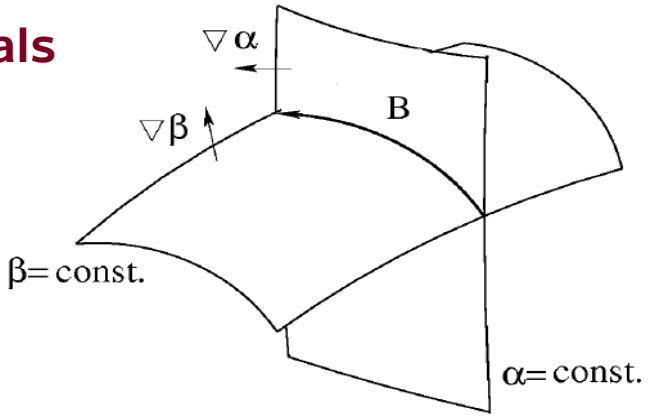
This needs to be controlled by field cleaning and/or smoothing methods, and a judicious choice of the SPH discretization.

(2) Use of the Euler potentials

$$\boldsymbol{B} = \nabla\alpha \times \nabla\beta$$

α and β effectively label field lines, and are simply advected with the flow in ideal MHD.

$$\frac{d\alpha_a}{dt} = 0 \qquad \frac{d\beta_a}{dt} = 0$$



Rosswog & Price (2008)

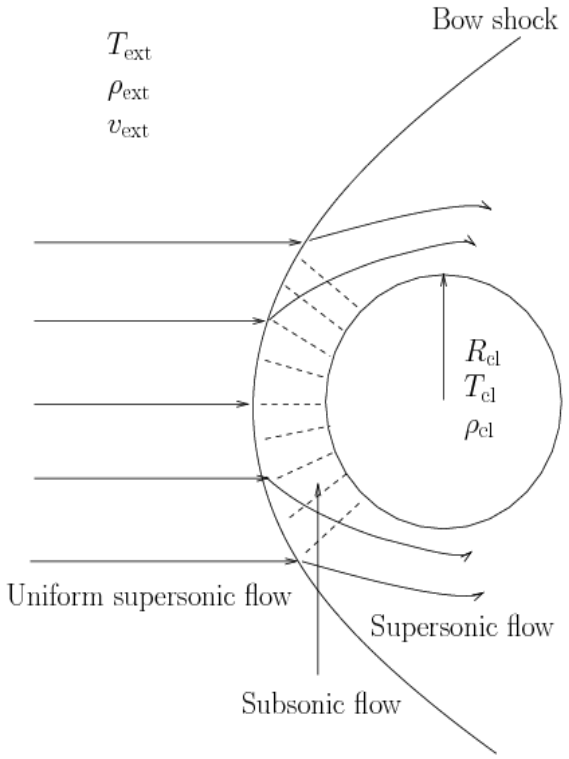
- Euler potentials not unique for a given field, and not all fields can be represented
- Unclear how dissipation should be treated
- Higher-order derivatives give noisy magnetic forces
- Dynamo action and magnetohydrodynamic turbulence may be suppressed (Brandenburg 2009)

(3) Use of the vector potential

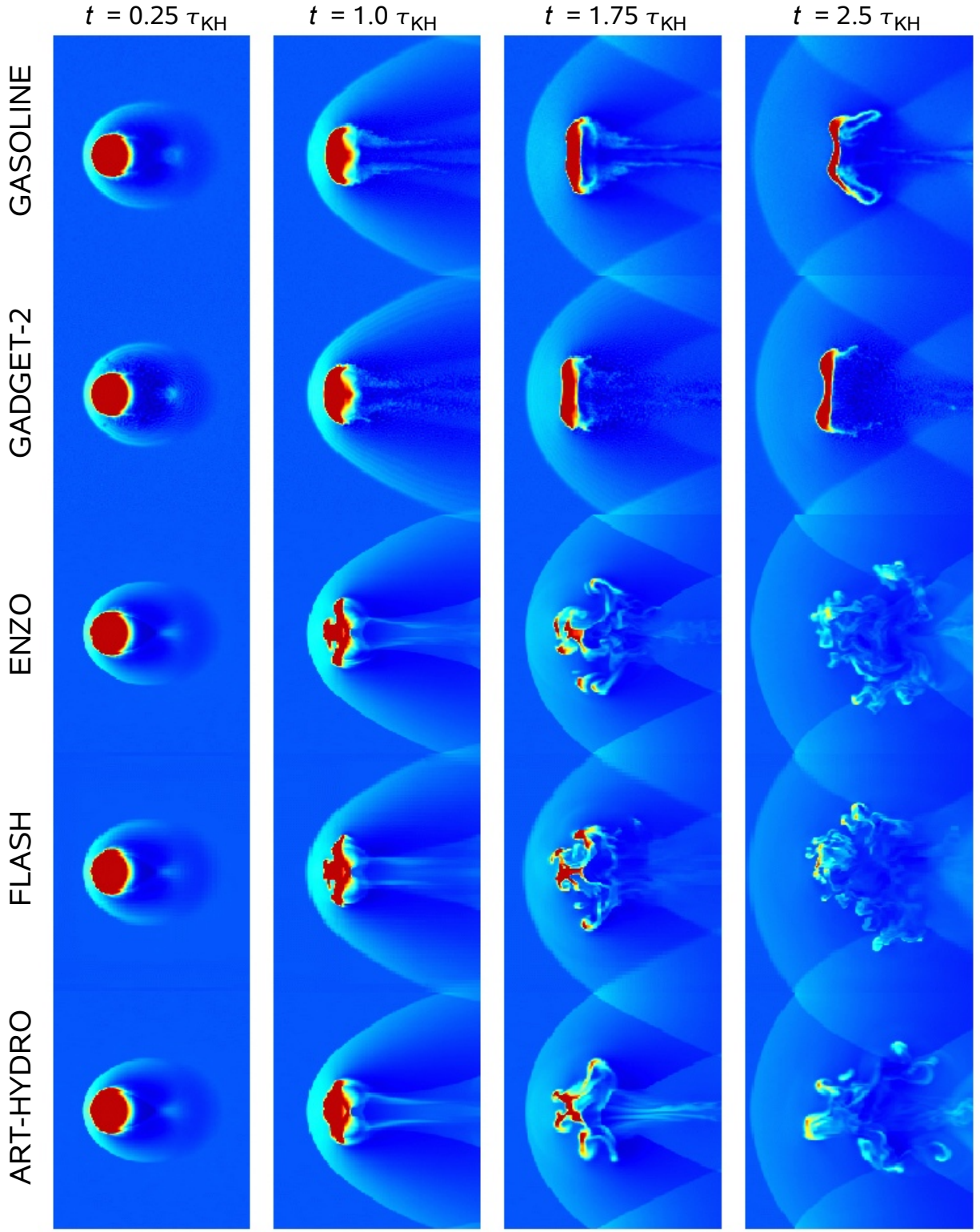
Fluid instabilities and mixing in SPH

A cloud moving through ambient gas shows markedly different long-term behavior in SPH and Eulerian mesh codes

DISRUPTION OF A CLOUD BY KELVIN-HELMHOLTZ INSTABILITIES



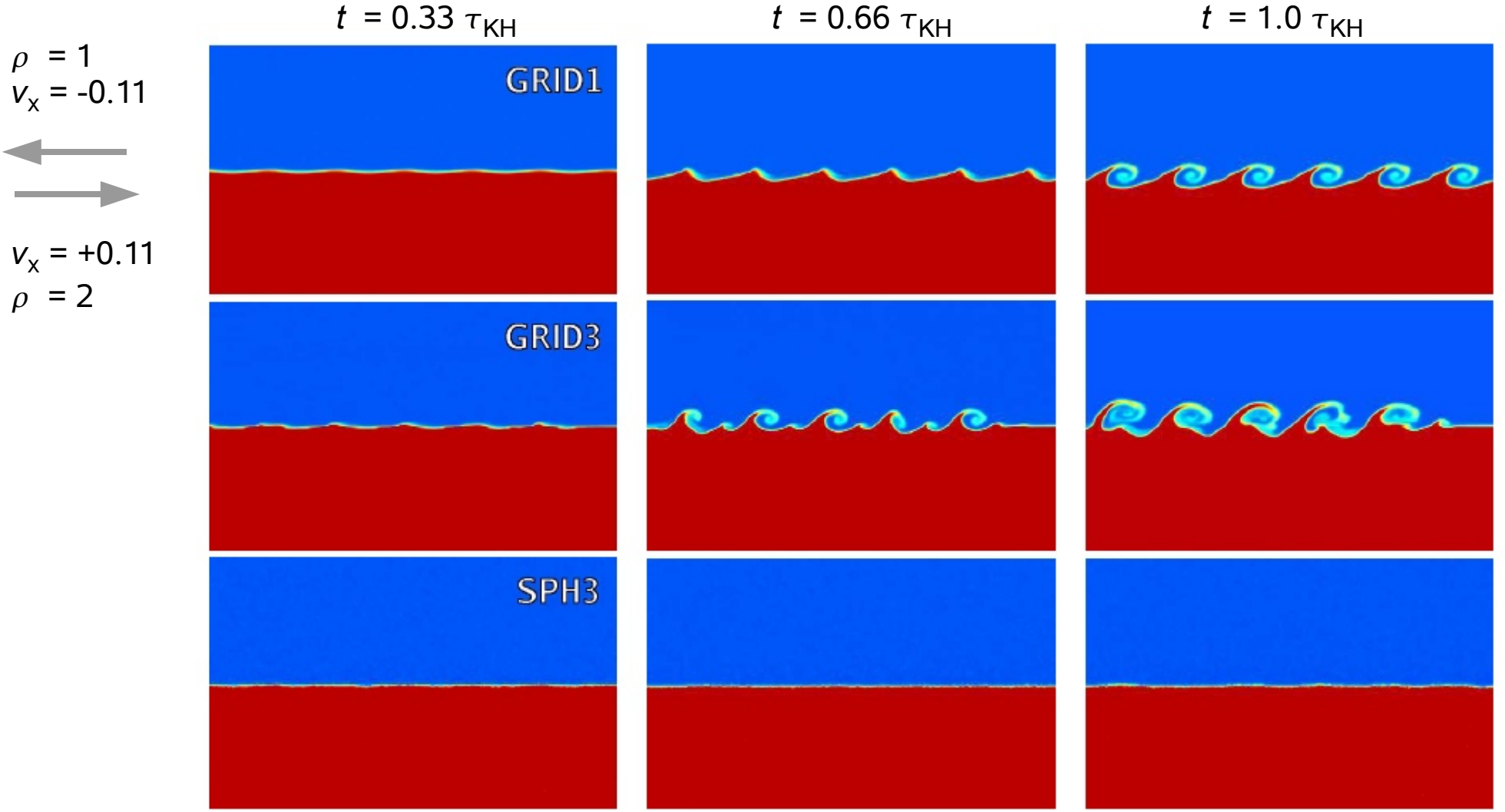
Agertz et al. (2007)



In SPH, fluid instabilities at contact discontinuities with large density jumps tend to be suppressed by a spurious numerical surface tension

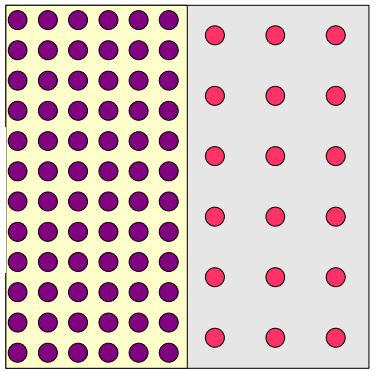
KELVIN-HELMHOLTZ INSTABILITIES IN SPH

Agertz et al. (2007)



Thought experiment on mixing

A simple *Gedankenexperiment* about mixing in SPH



$\rho_1 = 8$ $\quad u_1 = \frac{1}{8}$ $\quad V_1 = \frac{1}{2}$ $\qquad \rho_2 = 1$ $\quad u_2 = 1$ $\quad V_2 = \frac{1}{2}$

The pressure is constant:

$$P_1 = (\gamma - 1)\rho_1 u_1 = \frac{2}{3} \qquad P_2 = P_1$$

The specific entropies are:

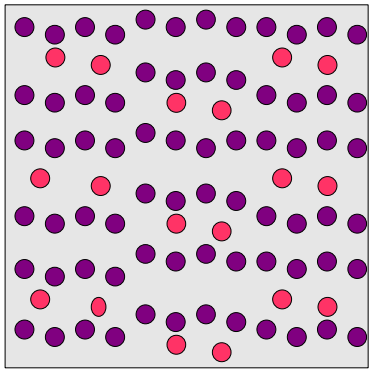
$$A_i = \frac{P_i}{\rho_i^\gamma} \qquad A_1 = \frac{1}{48} \qquad A_2 = \frac{2}{3}$$

Let's calculate the total thermal energy of the system:

$$E_{\rm therm} = \int \frac{A\rho^{\gamma-1}}{\gamma - 1}\,{\rm d}m$$

$$E_{\rm therm} = 1$$

We now mix the particles, keeping their specific entropies fixed:



All particles estimate the same mean density: $M_{\rm tot} = \frac{9}{2}$ $\qquad \overline{\rho} = \frac{9}{2}$

The thermal energy thus becomes:

$$E_{\rm therm} = \frac{M_1 A_1 \overline{\rho}^{2/3}}{2/3} + \frac{M_2 A_2 \overline{\rho}^{2/3}}{2/3}$$

$$E_{\rm therm} = \frac{5}{8}\left(\frac{9}{2}\right)^{2/3} \simeq 1.7$$

⇒ **This mixing process is energetically forbidden!**

What happened to the entropy in our *Gedankenexperiment* ?

In slowly mixing the two phases, we preserve the total thermal energy:

Expect: $\bar{u} = \frac{2}{9}$ $\quad \bar{A} = \frac{2}{3}\frac{\bar{u}}{\bar{\rho}^{2/3}}$ $\quad \bar{A} = \frac{2^{8/3}}{3^{13/3}} \simeq 0.054$

The Sackur-Tetrode equation for the entropy of an ideal gas can be written as:

$$S = \frac{3}{2}\frac{k_B}{\mu} M \left[\ln\left(\frac{P}{\rho^\gamma}\right) + \ln\left(\frac{2\pi\mu^{8/3}}{h^2}\right) + \frac{5}{3} \right]$$

If the mass in a system is conserved, it is sufficient to consider the simplified entropy:

$$\tilde{S} = M \ln A$$

When the system is mixed, the change of the entropy is:

$$\Delta\tilde{S} = M_{\rm tot} \ln \bar{A} - (M_1 \ln A_1 + M_2 \ln A_2)$$

$$\Delta\tilde{S} \simeq 2.55 \quad \geq \quad 0$$

⇒ **Unless this entropy is generated somehow, SPH will have problems to mix different phases of a flow.**

(Aside: Mesh codes can generate entropy outside of shocks – this allows them to treat mixing.)

New developments in SPH that try to address mixing

Artificial heat conduction at contact discontinuities has been proposed as a solution for the suppressed fluid instabilities

ARTIFICIAL HEAT MIXING TERMS

Price (2008) **Wadsley, Veeravalli & Couchman (2008)**

Price argues that in SPH every conservation law requires dissipative terms to capture discontinuities.

The normal artificial viscosity applies to the momentum equation, but discontinuities in the (thermal) energy equation should also be treated with a dissipative term.

For every conserved quantity *A*

$$\sum_j m_j \mathrm{d}A_j/\mathrm{d}t = 0$$

a dissipative term is postulated

$$\left(\frac{\mathrm{d}A_i}{\mathrm{d}t}\right)_{\mathrm{diss}} = \sum_j m_j \frac{\alpha_A v_{\mathrm{sig}}}{\bar{\rho}_{ij}} (A_i - A_j)\hat{\mathbf{r}}_{ij} \cdot \nabla W_{ij}$$

that is designed to capture discontinuities.

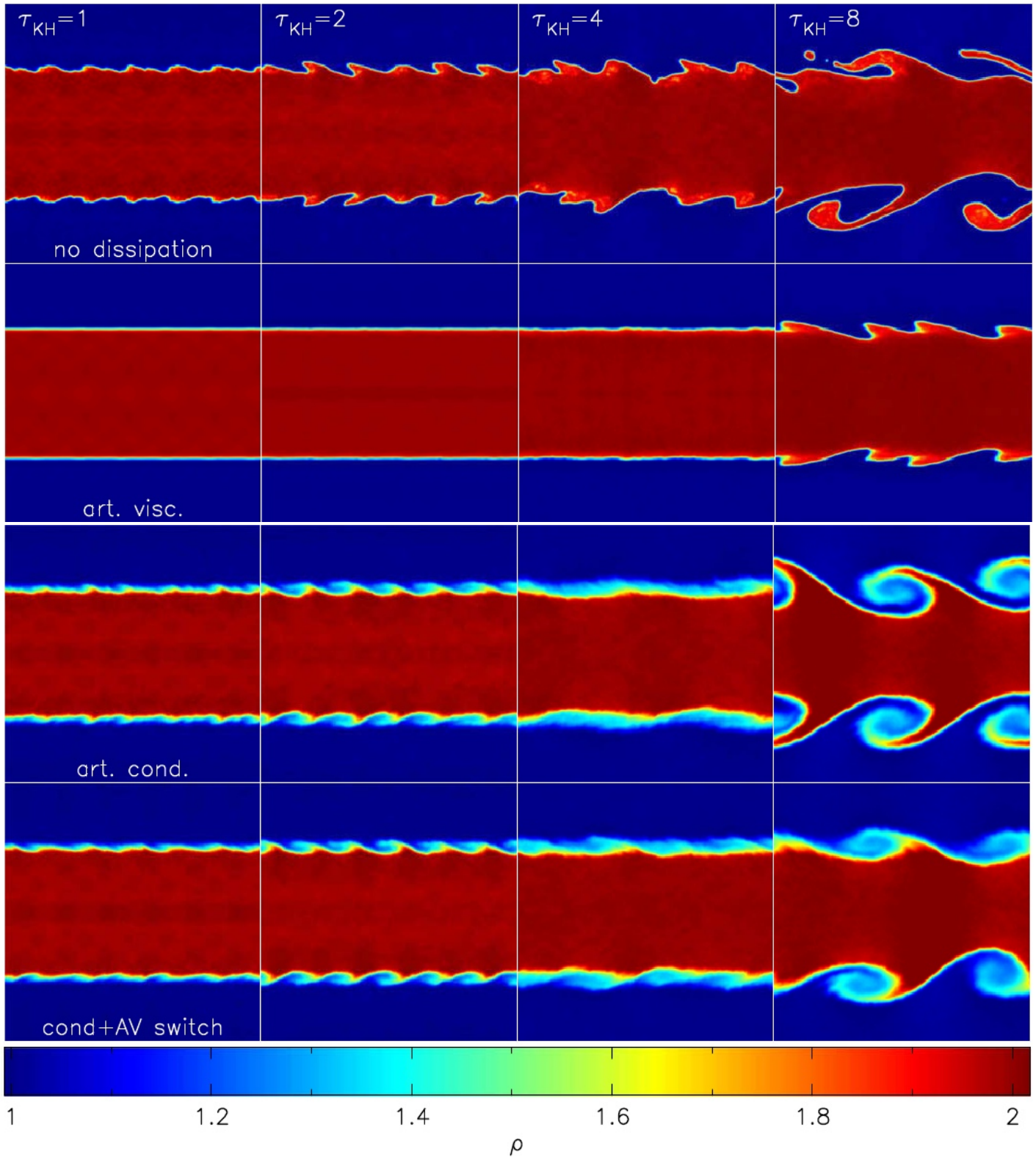
This is the discretized form of a diffusion problem:

$$\left(\frac{\mathrm{d}A}{\mathrm{d}t}\right)_{\mathrm{diss}} \approx \eta \nabla^2 A$$

$$\eta \propto \alpha v_{\mathrm{sig}} |r_{ij}|$$

Artificial heat conduction drastically improves SPH's ability to account for fluid instabilities and mixing

COMPARISON OF KH TESTS FOR DIFFERENT TREATMENTS OF THE DISSIPATIVE TERMS



Price (2008)

Another route to better SPH may lie in different ways to estimate the density

AN ALTERNATIVE SPH FORMULATION

"Optimized SPH" (OSPH) of **Read, Hayfield, Agertz (2009)**

- Density estimate like Ritchie & Thomas (2001):

$$\rho_i = \sum_j^N \left(\frac{A_j}{A_i}\right)^{\frac{1}{\gamma}} m_j \overline{W}_{ij}$$

- Very large number of neighbors (442 !) to beat down noise
- Needs peaked kernel to suppress clumping instability
- This in turn reduces the order of the density estimate, so that a large number of neigbors is required.