PiTP 2009: Computational Astrophysics

Computational Methods for Numerical Relativity

Lecture 3: Adaptive Mesh Refinement

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Outline

- Reminder: why we need AMR, and properties of the solutions that dictate the particular "flavor" of AMR that is adequate
- Berger & Oliger style AMR
  - ideal for hyperbolic wavelike equations, and certain classes of problems in GR
  - extensions for coupled hyperbolic/elliptic systems
    - example: critical phenomena in gravitational collapse
- PAMR/AMRD
  - infrastructure for implementing B&O AMR on clusters (using MPI)

AMR

- Adaptive Mesh Refinement is a technique to make the solution of discrete PDEs more *efficient* for *certain classes* of problem
  - there is a wide range of relevant length scales in the problem, yet the smallest length scales are relatively isolated and not volume filling
  - not known a-priori where the small length scales will develop, or it will be too difficult/cumbersome to construct a non-uniform mesh to efficiently resolve the small length scales
  - computationally too expensive to solve the problem on a single uniform mesh
- AMR allows for solution of such classes of problems by covering the domain with a *mesh hierarchy*, where high resolution meshes are only added where needed to resolve small length scale features
  - NOTE: AMR is *not* a technique to increase the accuracy of a solution; in fact, the AMR solution can never be more accurate than a unigrid solution with resolution corresponding to that of the finest AMR mesh
  - furthermore, AMR generically creates unwanted high-frequency solution components ("noise") at refinement boundaries, and though this can be controlled and made small, it is usually quite challenging to get very high accuracy solutions with AMR
  - Think of AMR as a tool to *get* an answer to a computationally challenging problem in the first place; worry about the n^{th} digit later

Why would AMR be beneficial in GR?

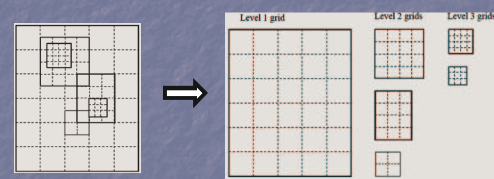
- In most astrophysical scenarios where GR is important and numerical solution is needed, in particular binary compact object mergers and gravitational collapse, there is a clean hierarchy of a modest range of metric length scales that need to be resolved
  - compact object radius → near field zone (10's of gravitational radii) → far field zone (100's gravitational radii)
- *in the strong-field regime* small length-scales are isolated (one or two compact objects) and not volume filling
  - however not always the case in GR, e.g. generic cosmological singularities
- *in the strong-field regime* temporal scales are commensurate with spatial scales; i.e. rapid temporal variation of the metric is typically confined to correspondingly small spatial length scales
- the equations are non-linear, and in many cases we will not *a-priori* know where/when refinement will be needed
- maximum causal speed of propagation (1 !)
- *in the weak-field regime* gravitational wave propagation is the feature of interest
  - this *will* be volume filling, and though the temporal scale for variations is always the same, the spatial scales will reflect the relevant scales of the source at the time of emission

Implications for an AMR algorithm

- The preceding properties suggest
  - it is *not* important to have sophisticated grid structures that can efficiently track features with complicated shapes; rather simple "aligned box-in-box" type strategies will be adequate
  - it *is* however important for the algorithm to provide a mechanism to automatically generated the hierarchy as evolution proceeds (i.e. "adaptive"!)
  - it *is* important to use an algorithm that maintains the same CFL factor everywhere in the domain; i.e. need *time-subcycling*
  - AMR by itself, regardless of how sophisticated the algorithm, will *not* help in tracking gravitational wave emission out to large radii with high accuracy … other technology will be needed to overcome this if it becomes an issue (though in a binary black hole merger the shortest GW wavelength ~ 5 gravitational radii --- not *too* small):
    - changing the spatial coordinates to more efficiently represent the wave structure; e.g. spherical polar coordinates, as the angular structure in the wave will not change by much far from the source, and could efficiently be represented with a relatively small set of multipoles
    - changing the slicing to be asympotically null, to "quickly" propagate the waves to large radii from the source
- In all then, a simplified version of the original Berger and Oliger AMR algorithm (JCP 53, 1984) is ideal for our purposes
  - though some modifications needed if elliptic equations are solved during evolution

Berger and Oliger AMR

- (simplified and extended) Berger and Oliger AMR, as implemented in AMRD [Pretorius & Choptuik, JCP 218,2006]
  - computational domain covered by a hierarchy of *independent uniform rectangular* meshes, where *higher resolution child meshes are aligned with and entirely contained within coarser resolution parent* meshes



mesh hierarchy on computational domain — *memory map of grids in hierarchy*

  - original algorithm allowed for child meshes to be rotated relative to the parent mesh

Berger and Oliger AMR

- (simplified and extended) Berger and Oliger AMR, as implemented in AMRD [Pretorius & Choptuik, JCP 218,2006]
  - recursive time stepping algorithm, so refinements occur in space and time (example in a few slides)
    - a single *unigrid* time step is taken on a parent level *before* ρ_t (temporal refinement ratio) *unigrid* time steps are taken on the child level
      - this ordering is *crucial* to set boundary conditions for interior equations, in particular the elliptics (though alternative strategies are possible for purely hyperbolic systems with explicit time integration, or certain classes of linear elliptic PDEs driven by conserved sources)
      - allows the AMR technology to be implemented independently of the particulars or details of the numerics used to solve them, and conversely shields the user from AMR implementation details
    - after ρ_t steps on the child grid, when the parent and child are in sync again, solution from the child region is *injected* into the overlapping region of the parent level, so that the most accurate solution available at a point is propagated to all levels of the hierarchy containing that point
    - gives near-O(N) (optimal) solution of the PDEs

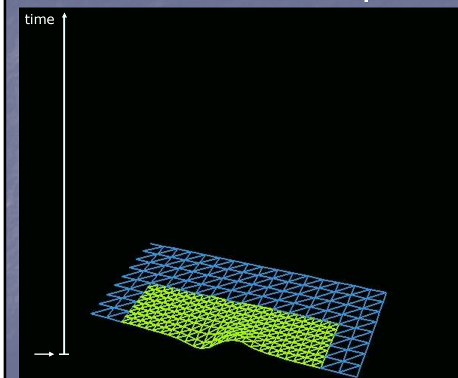
Berger and Oliger AMR

- (simplified and extended) Berger and Oliger AMR, as implemented in AMRD [Pretorius & Choptuik, JCP 218,2006]
  - hierarchy construction driven by truncation error (TE) estimates
    - the B&O proposal to compute this was to periodically make a 2:1 (say) coarsened version of a level in the hierarchy, evolve the two meshes independently for a short time (typically 1 coarse level time step), then *a la* Richardson, subtract the two solutions to give the TE estimate
    - here, use a "self-shadow" hierarchy to obviate the need to duplicate levels
      - due to the recursive nature of the algorithm, just before the fine-to-coarse level injection phase, information to compute TE estimates is naturally available
      - to make this work, simply need to "boot-strap" the procedure by requiring that the coarsest level always be fully refined
      - negligible additional cost … just choose mesh parameters so that the first refined level is the desired "coarsest" level

Berger and Oliger AMR

- (simplified and extended) Berger and Oliger AMR, as implemented in AMRD [Pretorius & Choptuik, JCP 218,2006]
  - algorithm extended to incorporate elliptic PDEs
    - for hyperbolic equations, a poorly resolved interior region of a coarse level will not adversely affect the solution on the parts of the level that are locally of the finest resolution, as the "junk" from the under-resolved region does not have more than 1 time step to propagate to the exterior before it is replaced with finer grid solutions
    - the above does *not* hold for elliptic equations. To deal with elliptics, in a nutshell, modify the algorithm as follows:
      - when *descending* the tree in the recursive time-stepping algorithm, evolve hyperbolics one step, using an extrapolated solution of the variables satisfied by elliptic equations
        - getting stable extrapolation is a bit tricky
      - when *ascending* the tree, post injection, solve the elliptics over the entire sub-hierarchy that is in sync with the given coarse level

B&O AMR Example

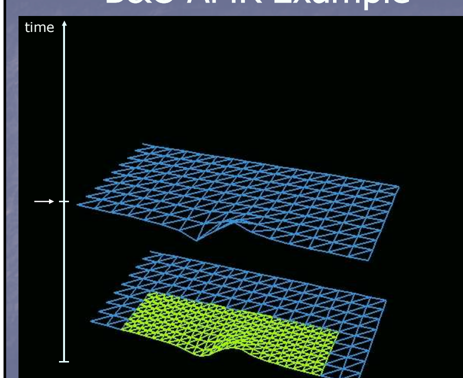


Initial hierarchy

2 Levels

$\rho_{sp} = \rho_t = 2:1$

B&O AMR Example



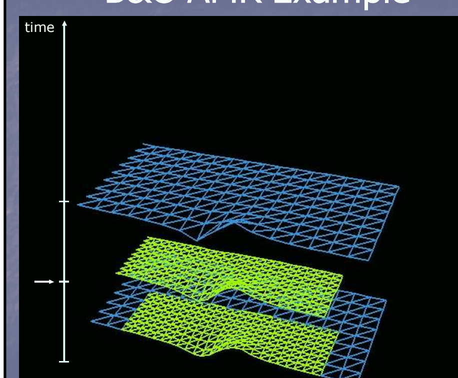
Level 1 time step

- *evolve hyperbolics on level 1*
- *extrapolate elliptics on level 1 from past time levels*

2 Levels

$\rho_{sp} = \rho_t = 2:1$

B&O AMR Example

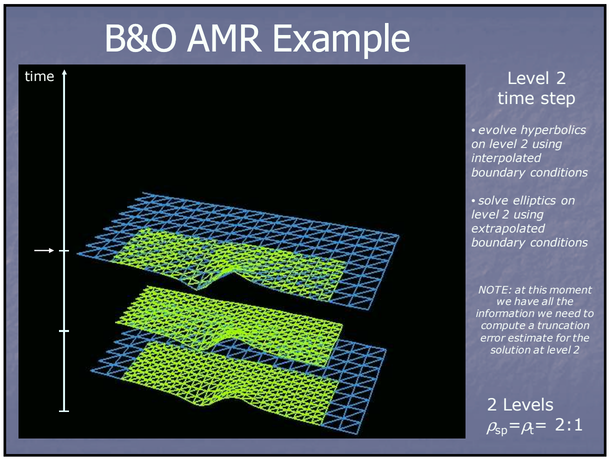


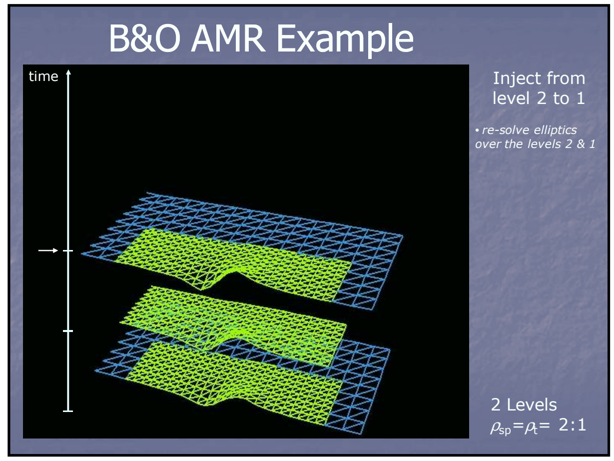
Level 2 time step

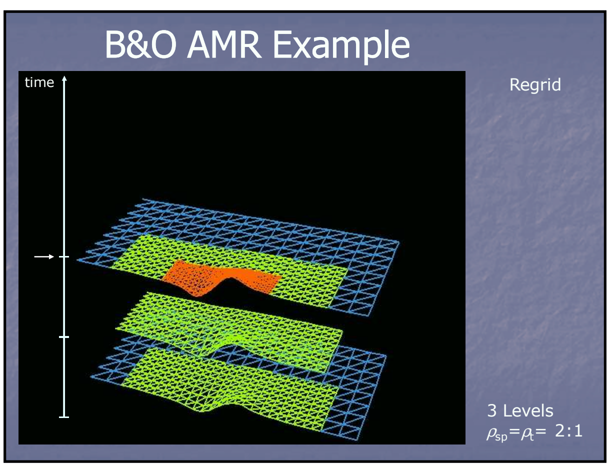
- *evolve hyperbolics on level 2 using interpolated boundary conditions*
- *solve elliptics on level 2 using extrapolated boundary conditions*

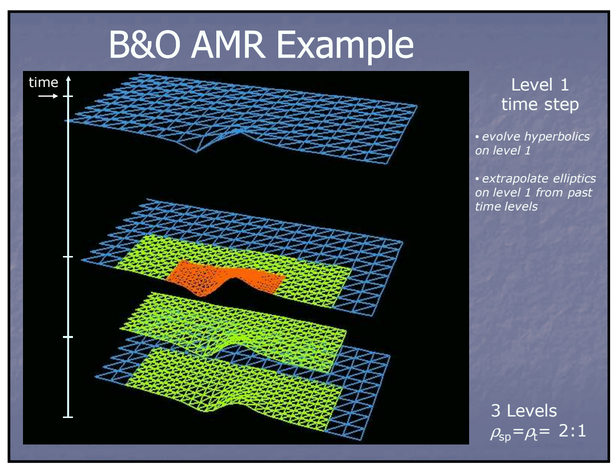
2 Levels

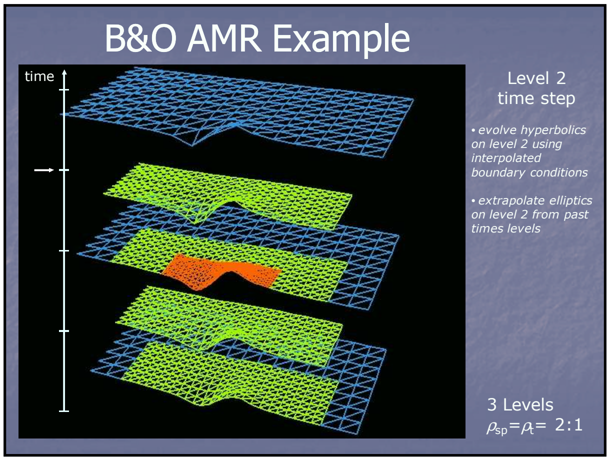
$\rho_{sp} = \rho_t = 2:1$

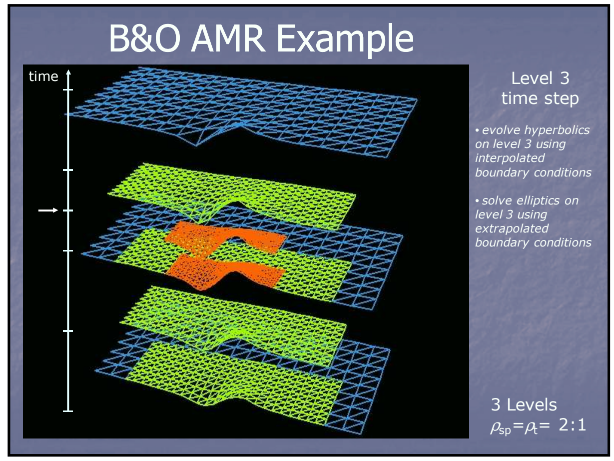

B&O AMR Example
time
Level 2 time step
• evolve hyperbolics on level 2 using interpolated boundary conditions
• solve elliptics on level 2 using extrapolated boundary conditions
NOTE: at this moment we have all the information we need to compute a truncation error estimate for the solution at level 2
2 Levels
$\rho_{sp}=\rho_t$= 2:1

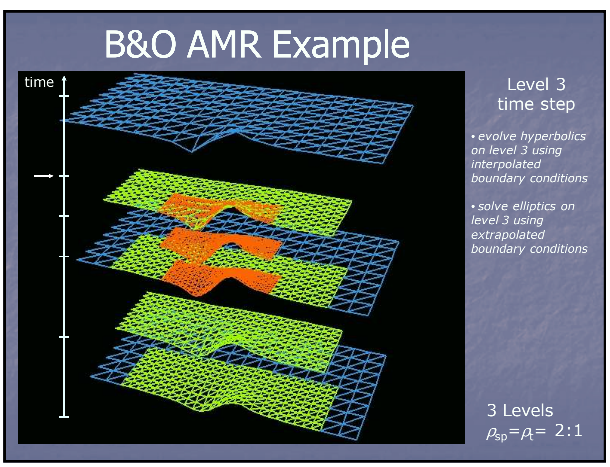

B&O AMR Example
time
Inject from level 2 to 1
• re-solve elliptics over the levels 2 & 1
2 Levels
$\rho_{sp}=\rho_t$= 2:1

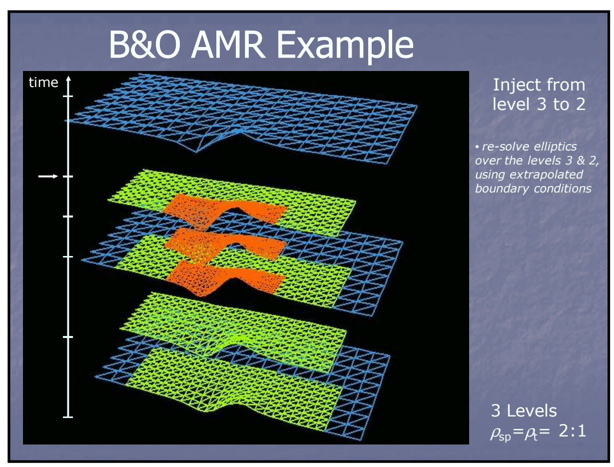

B&O AMR Example
time
Regrid
3 Levels
$\rho_{sp}=\rho_t$= 2:1

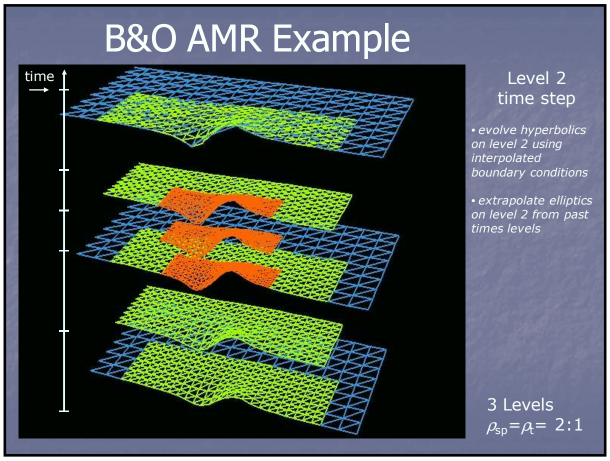

B&O AMR Example
time
Level 1 time step
• evolve hyperbolics on level 1
• extrapolate elliptics on level 1 from past time levels
3 Levels
$\rho_{sp}=\rho_t$= 2:1

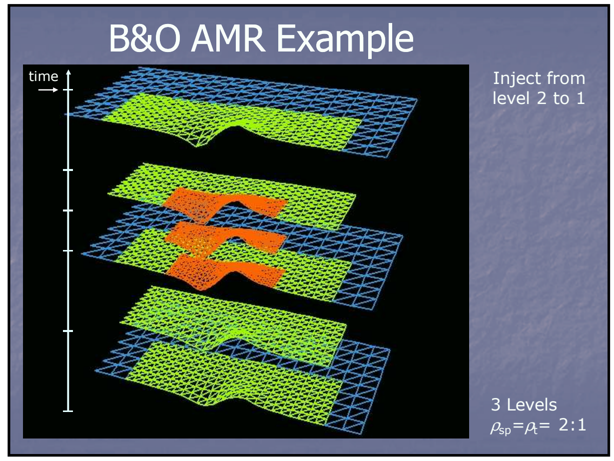

B&O AMR Example
time
Level 2 time step
• evolve hyperbolics on level 2 using interpolated boundary conditions
• extrapolate elliptics on level 2 from past times levels
3 Levels
$\rho_{sp}=\rho_t$= 2:1

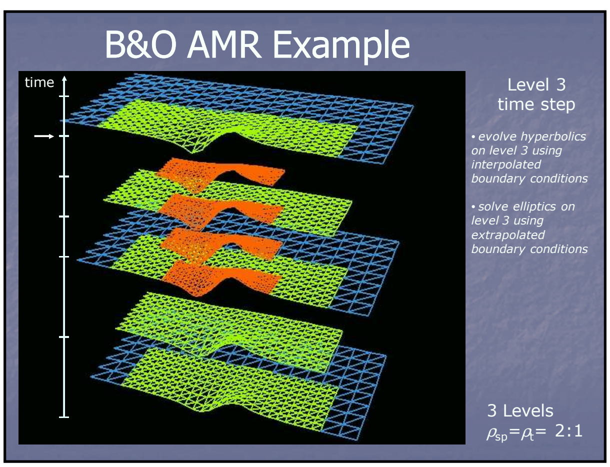

B&O AMR Example
time
Level 3 time step
• evolve hyperbolics on level 3 using interpolated boundary conditions
• solve elliptics on level 3 using extrapolated boundary conditions
3 Levels
$\rho_{sp}=\rho_t$= 2:1

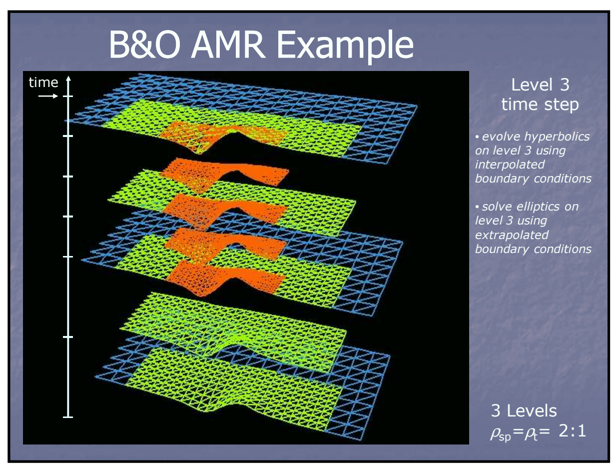

B&O AMR Example
time
Level 3 time step
• evolve hyperbolics on level 3 using interpolated boundary conditions
• solve elliptics on level 3 using extrapolated boundary conditions
3 Levels
$\rho_{sp}=\rho_t$= 2:1


B&O AMR Example
time
Inject from level 3 to 2
• re-solve elliptics over the levels 3 & 2, using extrapolated boundary conditions
3 Levels
$\rho_{sp}=\rho_t$= 2:1

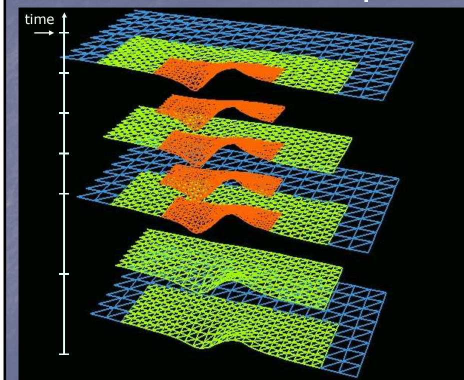

B&O AMR Example
time
Level 2
time step
• evolve hyperbolics on level 2 using interpolated boundary conditions
• extrapolate elliptics on level 2 from past times levels
3 Levels
$\rho_{sp}=\rho_t$= 2:1


B&O AMR Example
time
Inject from
level 2 to 1
3 Levels
$\rho_{sp}=\rho_t$= 2:1


B&O AMR Example
time
Level 3
time step
• evolve hyperbolics on level 3 using interpolated boundary conditions
• solve elliptics on level 3 using extrapolated boundary conditions
3 Levels
$\rho_{sp}=\rho_t$= 2:1


B&O AMR Example
time
Level 3
time step
• evolve hyperbolics on level 3 using interpolated boundary conditions
• solve elliptics on level 3 using extrapolated boundary conditions
3 Levels
$\rho_{sp}=\rho_t$= 2:1

B&O AMR Example



Inject from level 3 to 2 to 1

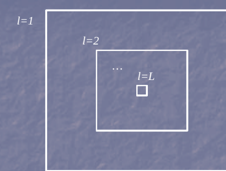
- *re-solve elliptics over the levels 3, 2 and 1*

3 Levels

$\rho_{sp} = \rho_t = 2:1$

Optimistically, what kind of speedup can we expect?

- Imagine
  - $d+1$ dimensional evolution
  - the coarsest level has N^d points
  - 2:1 spatial and temporal refinement ratio
  - L levels of refinement, with $l=1$ the coarsest level, and $l=L$ the finest
  - take N steps on the coarsest level; hence will need $N2^{(l-1)}$ on the level l
  - linear filling factor of ½
  - the total run-time is proportional to the total number of grid points in space and time (i.e. an optimal evolution scheme is used), and the overhead in the AMR algorithm is negligible
  - compare to a unigrid run at the resolution of the finest AMR level



$$T_{AMR} = C \sum_{l=1}^{L} N^d N 2^{(l-1)} < C N^{d+1} 2^L$$

$$T_{UNIGRID} = C\left[N 2^{(L-1)}\right]^{d+1}$$

$$\frac{T_{UNIGRID}}{T_{AMR}} > 2^{d(L-1)-1}$$

Example: Critical phenomena in gravitational collapse

- Discovered in 1993 by Choptuik, critical phenomena refers to interesting behavior observed at the *threshold of black hole formation* in gravitational collapse
- The question Choptuik was trying to answer was, "can one form black holes of arbitrarily small mass in scalar field collapse?" *(yes!)*
- In the process he discovered behavior that bears striking resemblance to critical phenomena observed at phase transitions in statistical mechanical systems:
  - power law scaling of order parameters (such as the black hole mass M) near threshold
  - universality of the threshold solution
  - scale invariance of the threshold solution
- The exact nature of the black hole threshold depends upon the kind of matter/energy undergoing collapse, and on the spacetime dimensionality
  - whether of relevance in astrophysical settings is unclear … need a natural fine tuning mechanism, otherwise occurrence would be exceedingly rare

Scalar field critical collapse

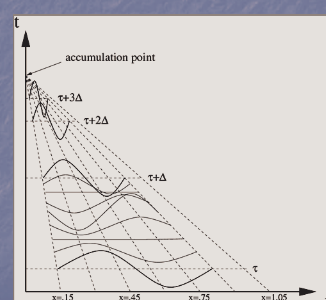
- the critical solution (scalar field and spacetime geometry) is spherically symmetric and scale invariant — specifically it is *discretely self-similar*

- example of a discretely self similar function $f(x,\tau)$
  - $f(x,\tau)$ is periodic in time τ with *echoing period* Δ
  - τ is related to the proper time t measured by a central observer (at radius $r=0$) via

    $\tau = -\ln(-t)$

  - x is a *dimensionless* variable, related to r and t via

    $x = r/t$

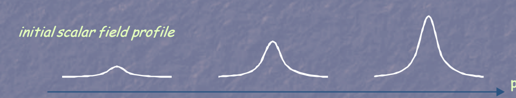


Scalar field critical collapse

- In other words:
  - the solution has a periodic component but collapses to small spatial-temporal scales exponentially fast
    - each "echo" of the field occurs on a scale 1/30th the previous, and in 1/30th the time that of the previous echo
    - the number of echo's observed is exponentially sensitive to the initial data … at threshold there are infinitely many
  - *ideal* problem for B&O style AMR, and this was essential for Choptuik's discovery
    - interestingly, this is also one of the few cases in science where a qualitatively new aspect of a theory was discovered using purely numerical methods

Finding the threshold of black hole formation

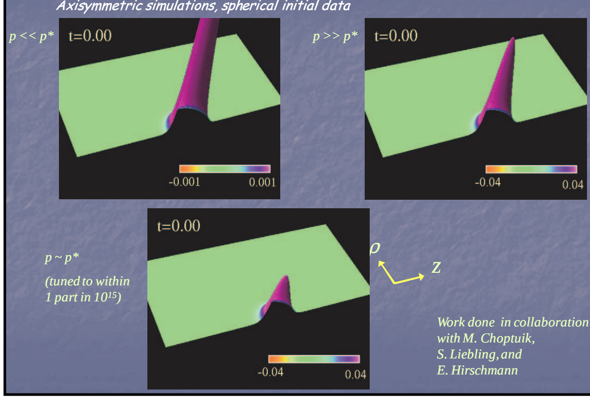
- Consider a smooth, one parameter (p) family of initial data, where p is, in some sense, related to the energy density of the initial configuration



- Then for $p<p^*$, evolution will lead to dispersal, while for $p>p^*$ a black hole will form — p^* labels the critical solution for this family
- In a numerical "experiment", p^* can be found via a bisection search

Scalar field gravitational collapse

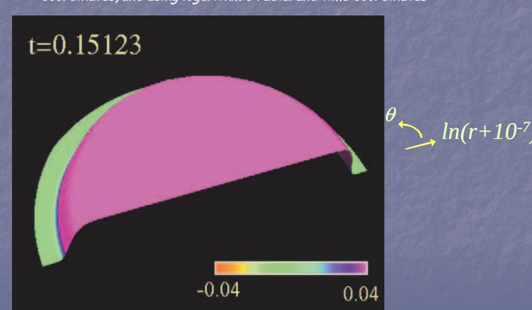
Axisymmetric simulations, spherical initial data



Work done in collaboration with M. Choptuik, S. Liebling, and E. Hirschmann

The scalar field threshold solution

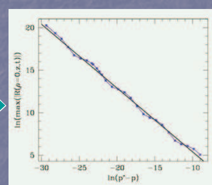
Same near critical solution, transformed to spherical polar coordinates, and using logarithmic radial and time coordinates



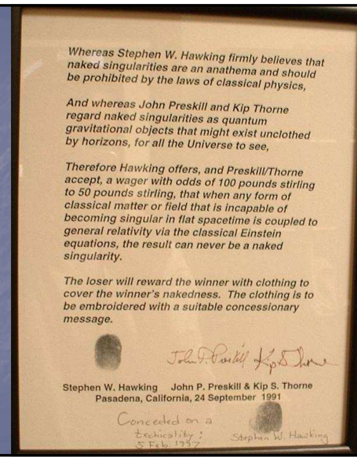
Properties of scalar field critical collapse

- Δ is equal to ~ 3.44
- The critical solution is (apparently) universal
  - the *same* solution is approached at threshold regardless of the initial conditions
- *Near* threshold, any length scale arising in the solution satisfies a universal power law relationship (to leading order)

$$M \propto (p - p^*)^{\gamma}, p > p^*$$
$$|R|_{\infty} \propto (p^* - p)^{-2\gamma}, p < p^*$$



- γ is called the scaling exponent, and is equal to ~ 0.37
- The critical solution is a naked singularity


Whereas Stephen W. Hawking firmly believes that naked singularities are an anathema and should be prohibited by the laws of classical physics,

And whereas John Preskill and Kip Thorne regard naked singularities as quantum gravitational objects that might exist unclothed by horizons, for all the Universe to see,

Therefore Hawking offers, and Preskill/Thorne accept, a wager with odds of 100 pounds stirling to 50 pounds stirling, that when any form of classical matter or field that is incapable of becoming singular in flat spacetime is coupled to general relativity via the classical Einstein equations, the result can never be a naked singularity.

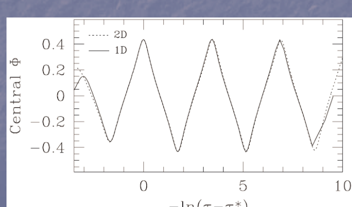
The loser will reward the winner with clothing to cover the winner's nakedness. The clothing is to be embroidered with a suitable concessionary message.

Stephen W. Hawking John P. Preskill & Kip S. Thorne
Pasadena, California, 24 September 1991

Conceded on a techicality: Stephen W. Hawking

Testing the universality hypothesis

- If the solution is indeed universal, then, regardless of the initial conditions, when fine-tuned to threshold, the scalar field should exhibit the behavior below at the center of collapse in logarithmic time
- Surely then, if we choose scalar field initial data that is reflection anti-symmetric about a plane passing through the origin (z=0), i.e. $\Phi(z)=\Phi(-z)$, this *cannot* have the same solution, as then $\Phi(z=0)=0$ is a conserved symmetry of the equations; i.e. this must "break" the universality
  - Also, recall that since it's the gradients of the field that gravitate, this symmetry does not prevent scalar field *energy* from reaching the origin

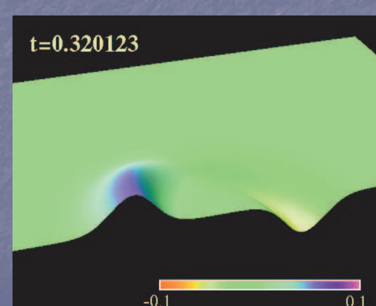


Evolution of plane anti-symmetric initial data

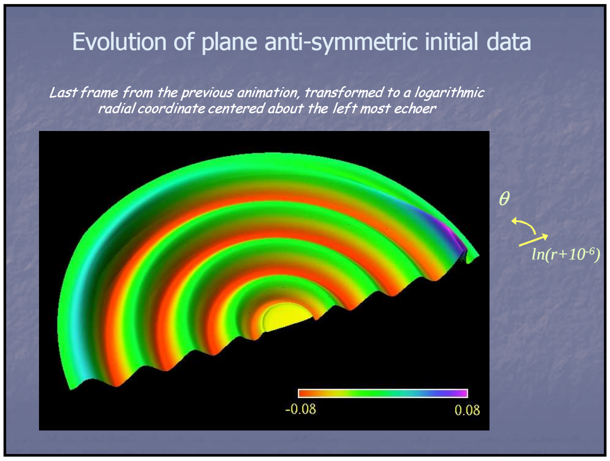
Initial data that is reflection anti-symmetric about z=0 (a conserved symmetry)

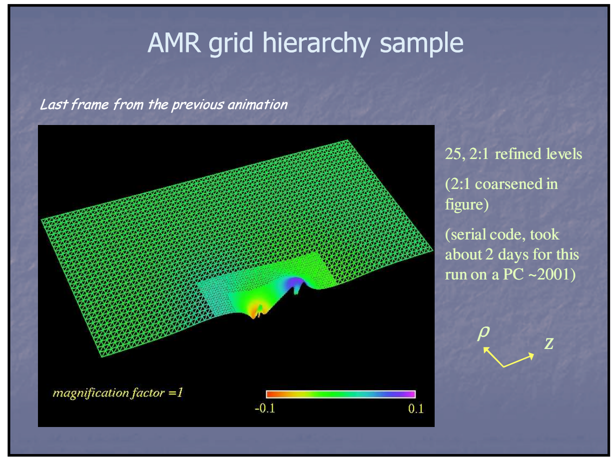
$p \sim p^*$

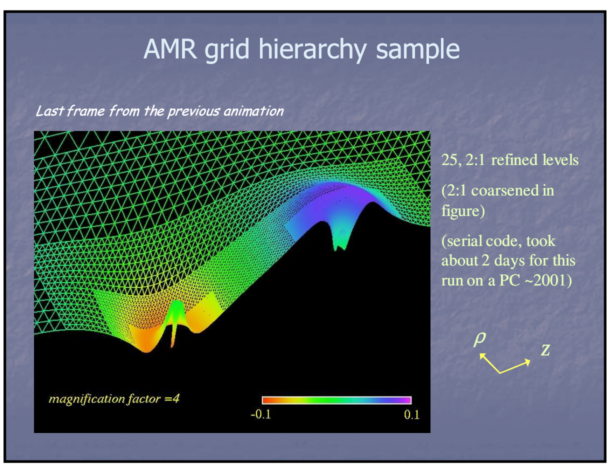
(tuned to within 1 part in 10^{15})

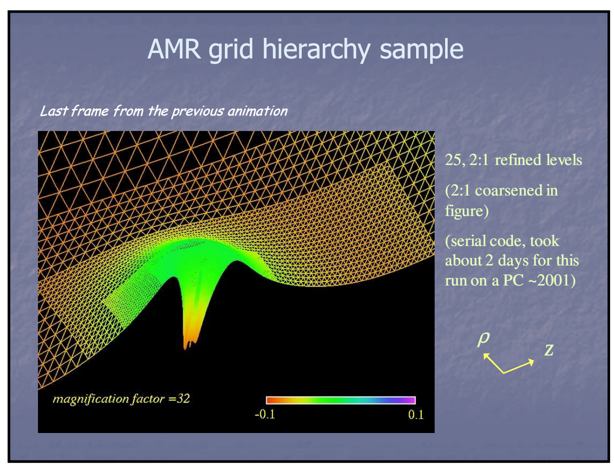


ρ z


Evolution of plane anti-symmetric initial data
Last frame from the previous animation, transformed to a logarithmic radial coordinate centered about the left most echoer
θ
ln(r+10^-6)
-0.08
0.08

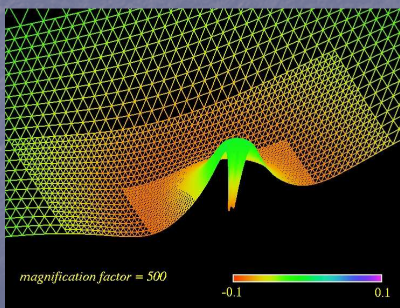

AMR grid hierarchy sample
Last frame from the previous animation
25, 2:1 refined levels
(2:1 coarsened in figure)
(serial code, took about 2 days for this run on a PC ~2001)
ρ
z
magnification factor =1
-0.1
0.1


AMR grid hierarchy sample
Last frame from the previous animation
25, 2:1 refined levels
(2:1 coarsened in figure)
(serial code, took about 2 days for this run on a PC ~2001)
ρ
z
magnification factor =4
-0.1
0.1


AMR grid hierarchy sample
Last frame from the previous animation
25, 2:1 refined levels
(2:1 coarsened in figure)
(serial code, took about 2 days for this run on a PC ~2001)
ρ
z
magnification factor =32
-0.1
0.1

AMR grid hierarchy sample

Last frame from the previous animation



25, 2:1 refined levels

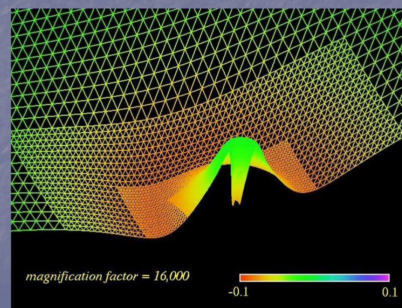
(2:1 coarsened in figure)

(serial code, took about 2 days for this run on a PC ~2001)

ρ z

AMR grid hierarchy sample

Last frame from the previous animation



25, 2:1 refined levels

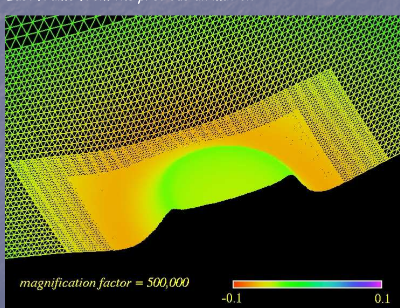
(2:1 coarsened in figure)

(serial code, took about 2 days for this run on a PC ~2001)

ρ z

AMR grid hierarchy sample

Last frame from the previous animation



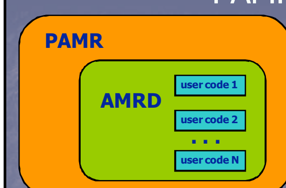
25, 2:1 refined levels

(2:1 coarsened in figure)

(serial code, took about 2 days for this run on a PC ~2001)

ρ z

PAMR/AMRD



- *PAMR* (parallel adaptive mesh refinement) manages distributed B&O style grid hierarchies
- *AMRD* (adaptive mesh refinement driver) implements the just-described version of B&O AMR, utilizing PAMR for hierarchy management
- *User codes* designed as (in-principle) standalone unigrid/serial numerical solvers, and supply AMRD with a series of "hook functions" to incorporate them into the B&O algorithm
- Reasons for this separation of functionality
  - from the point of view of a user writing a code to numerically solve a particular system of PDEs, AMR and parallel distribution are largely extraneous details
    - all the user should be aware of is the possibility that the code *could* be run in a parallel/adaptive environment, meaning grid boundaries could either be at the physical boundaries of the problem, or interior to the domain
      - in the latter case the user leaves the boundaries alone
  - The AMR driver does not need to know the details of how grids will be distributed in parallel, nor what equations the user will be solving on those grids
  - PAMR handles the non-local aspects of parallel grid distribution, and does not care what the underlying programs will do with the grids

PAMR

- Takes care of most parallel grid distribution issues
  - support for 1,2 and 3D grids, with or without periodic boundaries
  - support for interwoven AMR/multigrid hierarchies
  - simple base application program interface (API)
    - `PAMR_compose_hierarchy()` : regrid function
    - `PAMR_sync()` : synchronize data across ghost zones
    - `PAMR_inject()` : fine-to-coarse level injection
    - `PAMR_interp()` : coarse-to-fine level interpolation
  - a complete set of data structure management API's, so that it can be called from fortran programs
  - current version only supports vertex centered arrays; support for cell-centered arrays in the works (pretty much done, thanks to Branson Stephens)

AMRD

- Built on top of PAMR, hence "parallel ready"
- Implements a Berger and Oliger AMR algorithm, modified to support integrated solution of elliptic equations
- Provides a standard full approximation storage (FAS) adaptive multigrid algorithm
- User supplies a set of "hook functions" that are called by AMRD to perform the problem specific numerics
- Berger and Colella algorithm for conservative hyrdodynamics in the works (pretty much done, again thanks to Branson)

A few final remarks

- For the Einstein equations, time taken to evaluate expressions dominates over other tasks, which helps guide coding priorities
  - 'locality' of the data less of an issue in load-balancing a parallel code (strategies designed to guarantee locality, such as space filling curves, may even have a negative impact on the performance)
  - algorithmic tasks (truncation error estimation, regridding, interpolation, injection, etc.) are essentially "free"
- Solving elliptic equations solved using FAS multigrid is optimal and *fast*
  - at *worst* a constant factor of 2-3 times slower per equation compared to a typical hyperbolic equation
  - for example, 2D axisymetric gravitational collapse code solves 4 (3) hyperbolic equations and 3 (4) elliptic equations per time step; profiling indicated roughly 25-45% of the time is spent solving hyperbolics, 50-70% solving elliptics, with the remainder (usually ~ 5-10%) spent on miscellaneous functions in a typical simulation
- Code, including reference manuals and a couple of examples, can be downloaded from Matt Choptuik's web-page (google "Matt Choptuik", or see links from my web-page) [example next lecture]